Physics 3041 (Spring 2021) Solutions to Homework Set 1

1. Problem 1.6.1. (20 points)

Let's first try the most straightforward way:

$$f(x)=\frac{\sin x}{\cosh x+2}\Rightarrow f(0)=0$$

$$f'(x)=\frac{\cos x}{\cosh x+2}-\frac{\sin x\sinh x}{(\cosh x+2)^2}\Rightarrow f'(0)=\frac{1}{3}$$

$$\begin{aligned}
f''(x) &= -\frac{\sin x}{\cosh x+2}-\frac{\cos x\sinh x}{(\cosh x+2)^2}-\frac{\cos x\sinh x+\sin x\cosh x}{(\cosh x+2)^2}+\frac{2\sin x\sinh^2 x}{(\cosh x+2)^3}\\
&= -\frac{\sin x}{\cosh x+2}-\frac{2\cos x\sinh x+\sin x\cosh x}{(\cosh x+2)^2}+\frac{2\sin x\sinh^2 x}{(\cosh x+2)^3}\Rightarrow f''(0)=0
\end{aligned}$$

$$\begin{aligned}
f'''(x) &= -\frac{\cos x}{\cosh x+2}+\frac{\sin x\sinh x}{(\cosh x+2)^2}-\frac{2(-\sin x\sinh x+\cos x\cosh x)+\cos x\cosh x+\sin x\sinh x}{(\cosh x+2)^2}\\
&+ \frac{2(2\cos x\sinh x+\sin x\cosh x)\sinh x}{(\cosh x+2)^3}+\frac{2(\cos x\sinh^2 x+2\sin x\sinh x\cosh x)}{(\cosh x+2)^3}-\frac{6\sin x\sinh^3 x}{(\cosh x+2)^4}\\
&= -\frac{\cos x}{\cosh x+2}+\frac{\sin x\sinh x}{(\cosh x+2)^2}-\frac{-\sin x\sinh x+3\cos x\cosh x}{(\cosh x+2)^2}\\
&+ \frac{6(\cos x\sinh^2 x+\sin x\cosh x\sinh x)}{(\cosh x+2)^3}-\frac{6\sin x\sinh^3 x}{(\cosh x+2)^4}\\
&= -\frac{\cos x}{\cosh x+2}+\frac{2\sin x\sinh x-3\cos x\cosh x}{(\cosh x+2)^2}\\
&+ \frac{6(\cos x\sinh^2 x+\sin x\cosh x\sinh x)}{(\cosh x+2)^3}-\frac{6\sin x\sinh^3 x}{(\cosh x+2)^4}\Rightarrow f'''(0)=-\frac{1}{3}-\frac{3}{9}=-\frac{2}{3}
\end{aligned}$$

$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\cdots=\frac{x}{3}-\frac{2x^3}{3\times 6}+\cdots=\frac{x}{3}-\frac{x^3}{9}+\cdots$$

$$f(0.1)\approx\frac{0.1}{3}-\frac{0.1^3}{9}\approx 0.0332222 \text{ to be compared with } f(0.1)=\frac{\sin 0.1}{\cosh 0.1+2}=0.0332224$$

Now consider a simpler way starting with

$$\sin x=x-\frac{x^3}{6}+\cdots,\ \cosh x=1+\frac{x^2}{2}+\cdots,$$

where we have ignored terms of orders higher than x^3.

$$\begin{aligned}
f(x) &= \frac{\sin x}{\cosh x+2}=\frac{x-x^3/6+\cdots}{(1+x^2/2+\cdots)+2}=\frac{x-x^3/6+\cdots}{3+x^2/2+\cdots}=\frac{x-x^3/6+\cdots}{3(1+x^2/6+\cdots)}\\
&= \frac{x-x^3/6+\cdots}{3}(1-x^2/6+\cdots)=\frac{1}{3}(x-x^3/6-x^3/6+\cdots)=\frac{x}{3}-\frac{x^3}{9}+\cdots,
\end{aligned}$$

where we have used $(1+y)^{-1}=1-y+\cdots$ with $y=x^2/6+\cdots$.

2. Consider $f(x) = (1+x)^p$ for (a) $p = 1/3$ and (b) $p = -2$, respectively.

(1) Find the Taylor series of $f(x)$ around $x = 0$. (10 points)

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2}x^2 + \frac{p(p-1)(p-2)}{6}x^3 + \cdots + \frac{p(p-1)\cdots(p-n+1)}{n!}x^n + \cdots$$

For $p = 1/3$,

$$(1+x)^{1/3} = 1 + \frac{x}{3} - \frac{x^2}{9} + \frac{5x^3}{81} - \frac{10x^4}{243} \cdots + (-1)^{n-1}\frac{2\cdot 5\cdot 8\cdot (3n-4)}{3^n n!}x^n + \cdots,$$

and for $p = -2$,

$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 + \cdots + (-1)^n(n+1)x^n + \cdots.$$

(2) From the form of the general term, find the interval of convergence for the series. (10 points)

For $p = 1/3$, the general term is

$$a_n x^n = (-1)^{n-1}\frac{2\cdot 5\cdot 8\cdot (3n-4)}{3^n n!}x^n.$$

So the interval of convergence is

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\frac{2\cdot 5\cdot 8\cdot (3n-4)}{2\cdot 5\cdot 8\cdot (3n-4)(3n-1)} \times \frac{3^{n+1}(n+1)!}{3^n n!} = \lim_{n\to\infty}\frac{3(n+1)}{3n-1} = 1.$$

For $p = -2$, the general term is

$$a_n x^n = (-1)^n(n+1)x^n \Rightarrow R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\frac{n+1}{n+2} = 1.$$

(3) How many terms in the series do you need to estimate $f(0.1)$ to within 1%? Check that the difference between your estimate and the actual result has approximately the same magnitude as the next term in the series. (10 points)

For $p = 1/3$, the second term is $0.1/3 \approx 0.033$ and the third term is $-0.1^2/9 \approx -1.1\times 10^{-3}$. So within 1% we only need to keep the first two terms: $1.1^{1/3} \approx 1 + 0.1/3 \approx 1.0333$. The difference between the actual result and this estimate is $1.1^{1/3} - 1.0333 \approx 1.0323 - 1.0333 = -10^{-3}$, which indeed has the same magnitude and sign as the third term.

For $p = -2$, the third term is $3\times 0.1^2 = 0.03$ and the fourth term is $-4\times 0.1^3 = -4\times 10^{-3}$. So within 1% we only need to keep the first three terms: $1.1^{-2} \approx 1 - 2\cdot 0.1 + 3\cdot 0.1^2 = 0.83$. The difference between the actual result and this estimate is $1.1^{-2} - 0.83 \approx 0.82645 - 0.83 = -3.55\times 10^{-3}$, which indeed has the same magnitude and sign as the fourth term.

3. Expand $f(x) = \tan x^2$ to oder x^6 using (a) direct Taylor expansion of $\tan x$ with a substitution (20 points), and (b) the Taylor series of $\sin x$ and $\cos x$ along with appropriate substitutions (20 points).

(a) Direct Taylor expansion of $g(y) = \tan y$

$$g(0) = 0,\ g'(y) = \frac{1}{\cos^2 y} \Rightarrow g'(0) = 1$$

$$g''(y) = \frac{2\sin y}{\cos^3 y} \Rightarrow g''(0) = 0$$

$$g'''(y) = \frac{2\cos y}{\cos^3 y} + \frac{2\sin y(3\sin y)}{\cos^4 y} = \frac{2}{\cos^2 y} + \frac{6\sin^2 y}{\cos^4 y} \Rightarrow g'''(0) = 2$$

$$\begin{aligned} g(y) &= g(0) + g'(0)y + \frac{g''(0)}{2!}y^2 + \frac{g'''(0)}{3!}y^3 + \cdots = y + \frac{2}{6}y^3 + \cdots = y + \frac{y^3}{3} + \cdots \\ &\Rightarrow f(x) = g(x^2) = x^2 + \frac{x^6}{3} + \cdots \end{aligned}$$

(b) Use Taylor series of $\sin y$ and $\cos y$

$$\sin y = y - \frac{y^3}{6} + \cdots,\ \cos y = 1 - \frac{y^2}{2} + \cdots,$$

where we have ignored terms of orders higher than y^3.

$$\tan y = \frac{\sin y}{\cos y} = \frac{y - y^3/6 + \cdots}{1 - y^2/2 + \cdots} = (y - y^3/6 + \cdots)(1 + y^2/2 + \cdots) = y - \frac{y^3}{6} + \frac{y^3}{2} + \cdots = y + \frac{y^3}{3} + \cdots,$$

where we have used $(1+z)^{-1} = 1 - z + \cdots$ with $z = -y^2/2 + \cdots$.

$$y = x^2 \Rightarrow \tan x^2 = x^2 + \frac{x^6}{3} + \cdots$$

4. A particle of mass m moves along the $+x$-axis (i.e., $x > 0$) with a potential energy

$$V(x) = \frac{a}{2x^2} - \frac{b}{x},$$

where a and b are positive parameters.

(a) Find the equilibrium position x_0. (3 points)

$$V'(x) = -\frac{a}{x^3} + \frac{b}{x^2}, \ V'(x_0) = -\frac{a}{x_0^3} + \frac{b}{x_0^2} = 0 \Rightarrow x_0 = \frac{a}{b}$$

(b) Show that the particle executes harmonic oscillations near $x = x_0$. (5 points)

$$\begin{aligned}
V''(x) &= \frac{3a}{x^4} - \frac{2b}{x^3} \Rightarrow V''(x_0) = \frac{3a}{x_0^4} - \frac{2b}{x_0^3} = \frac{b^4}{a^3} > 0 \\
V(x) &\approx V(x_0) + V'(x_0)(x - x_0) + \frac{V''(x_0)}{2}(x - x_0)^2 = -\frac{b^2}{2a} + \frac{b^4}{2a^3}(x - x_0)^2 \\
F &= -V'(x) = -\frac{b^4}{a^3}(x - x_0) = m\ddot{x}(t), \ y \equiv x - x_0 \Rightarrow m\ddot{y} = -\frac{b^4}{a^3}y \\
\ddot{y} &= -\frac{b^4}{ma^3}y \equiv -\omega^2 y \Rightarrow y(t) = A\sin(\omega t + \phi_0), \ x(t) = x_0 + A\sin(\omega t + \phi_0)
\end{aligned}$$

(c) Find the angular frequency of oscillations. (2 points)

$$\omega^2 \equiv \frac{b^4}{ma^3} \Rightarrow \omega = \sqrt{\frac{b^4}{ma^3}}$$