HW 1

Physics 3041
Mathematical Methods for Physicists

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Contents

1 Problem 1.6.1

Expand the function $f(x) = \frac{\sin(x)}{\cosh(x)+2}$ in Taylor series around the origin going up to x^3. Calculate $f(0.1)$ from this series and compare to the exact answer obtained by using a calculator

Solution

The Taylor series of function $f(x)$ around origin is given by (1.3.16) ($\approx$ is used throughout this HW to mean that the left side is the Taylor series approximation of $f(x)$).

$$f(x) \approx \sum_{n=0}^{\infty} \frac{x^n}{n!} f^{(n)}(0)$$

Where $f^{(n)}(0)$ is the n^{th} derivative of $f(x)$ evaluated at $x = 0$.

For $n = 0$, $f^{(0)}(x) = f(x) = \frac{\sin(x)}{\cosh(x)+2}$, therefore $f(0) = 0$.

For $n = 1$

$$\begin{aligned}
f^{(1)}(x) &= \frac{d}{dx}\left(\frac{\sin(x)}{\cosh(x)+2}\right) \\
&= \frac{\cos(x)(\cosh(x)+2) - \sin(x)\sinh(x)}{(\cosh(x)+2)^2} \\
&= \frac{\cos(x)(\cosh(x)+2)}{(\cosh(x)+2)^2} - \frac{\sin(x)\sinh(x)}{(\cosh(x)+2)^2} \\
&= \frac{\cos(x)}{\cosh(x)+2} - \frac{\sin(x)\sinh(x)}{(\cosh(x)+2)^2}
\end{aligned}$$

The above evaluated at $x = 0$ becomes

$$\begin{aligned}
f^{(1)}(0) &= \frac{1}{1+2} - \frac{0}{(1+2)^2} \\
&= \frac{1}{3}
\end{aligned}$$

For $n = 2$

$$
\begin{aligned}
f^{(2)}(x) &= \frac{d}{dx}\left(\frac{d}{dx}\left(\frac{\sin(x)}{\cosh(x)+2}\right)\right) \\
&= \frac{d}{dx}\left(\frac{\cos(x)}{\cosh(x)+2} - \frac{\sin(x)\sinh(x)}{(\cosh(x)+2)^2}\right) \\
&= \frac{-\sin(x)(\cosh(x)+2) - \cos(x)\sinh(x)}{(\cosh(x)+2)^2} \\
&\quad - \frac{(\cos(x)\sinh(x)+\sin(x)\cosh(x))(\cosh(x)+2)^2 - \sin(x)\sinh(x)(2(\cosh(x)+2)\sinh(x))}{(\cosh(x)+2)^4} \\
&= \frac{-\sin(x)(\cosh(x)+2)}{(\cosh(x)+2)^2} - \frac{\cos(x)\sinh(x)}{(\cosh(x)+2)^2} - \frac{\cos(x)\sinh(x)(\cosh(x)+2)^2}{(\cosh(x)+2)^4} \\
&\quad - \frac{\sin(x)\cosh(x)(\cosh(x)+2)^2}{(\cosh(x)+2)^4} + \frac{\sin(x)\sinh(x)(2(\cosh(x)+2)\sinh(x))}{(\cosh(x)+2)^4} \\
&= \frac{-\sin(x)}{\cosh(x)+2} - \frac{\cos(x)\sinh(x)}{(\cosh(x)+2)^2} - \frac{\cos(x)\sinh(x)}{(\cosh(x)+2)^2} - \frac{\sin(x)\cosh(x)}{(\cosh(x)+2)^2} + \frac{2\sin(x)\sinh(x)\sinh(x)}{(\cosh(x)+2)^3} \\
&= \frac{-\sin(x)}{\cosh(x)+2} - 2\frac{\cos(x)\sinh(x)}{(\cosh(x)+2)^2} - \frac{\sin(x)\cosh(x)}{(\cosh(x)+2)^2} + \frac{2\sin(x)\sinh^2(x)}{(\cosh(x)+2)^3}
\end{aligned}
$$

The above evaluated at $x = 0$ becomes

$$
\begin{aligned}
f^{(2)}(0) &= \frac{-0}{1+2} - 2\frac{0}{(1+2)^2} - \frac{0}{(1+2)^2} + \frac{0}{(1+2)^3} \\
&= 0
\end{aligned}
$$

For $n = 3$

$$
\begin{aligned}
f^{(3)}(x) &= \frac{d}{dx}\left(\frac{d^2}{dx^2}\left(\frac{\sin(x)}{\cosh(x)+2}\right)\right) \\
&= \frac{d}{dx}\left(\frac{-\sin(x)}{\cosh(x)+2} - 2\frac{\cos(x)\sinh(x)}{(\cosh(x)+2)^2} - \frac{\sin(x)\cosh(x)}{(\cosh(x)+2)^2} + \frac{2\sin(x)\sinh^2(x)}{(\cosh(x)+2)^3}\right) \\
&= \frac{-\cos(x)(\cosh(x)+2) + \sin(x)\sinh(x)}{(\cosh(x)+2)^2} \\
&\quad - 2\frac{(-\sin(x)\sinh(x)+\cos(x)\cosh(x))(\cosh(x)+2)^2 - \cos(x)\sinh(x)(2(\cosh(x)+2)\sinh(x))}{(\cosh(x)+2)^4} \\
&\quad - \frac{(\cos(x)\cosh(x)+\sin(x)\sinh(x))(\cosh(x)+2)^2 - \sin(x)\cosh(x)(2(\cosh(x)+2)\sinh(x))}{(\cosh(x)+2)^4} \\
&\quad + 2\frac{\left(\cos(x)\sinh^2(x) + 2\sin(x)\cosh(x)\right)(\cosh(x)+2)^3 - \left(\sin(x)\sinh^2(x)\right)\left(3(\cosh(x)+2)^2\sinh(x)\right)}{(\cosh(x)+2)^6}
\end{aligned}
$$

The above evaluated at $x = 0$ gives

$$
\begin{aligned}
f^{(3)}(0) &= \frac{-1(1+2)+0}{(1+2)^2} - 2\frac{(-0+1)(1+2)^2 - 0}{(1+2)^4} - \frac{(1+0)(1+2)^2 - 0}{(1+2)^4} + 2\frac{(0+0)(1+2)^3 - (0)\left(3(1+2)^2 0\right)}{(1+2)^6} \\
&= \frac{-(3)}{(3)^2} - 2\frac{(1)(3)^2}{(3)^4} - \frac{(1)(3)^2}{(3)^4} + 2\frac{0}{(3)^6} \\
&= \frac{-1}{3} - 2\frac{1}{3^2} - \frac{1}{3^2} \\
&= -\frac{2}{3}
\end{aligned}
$$

The process stops here, because the problem is asking for $n = 3$. Substituting all the derivatives $f^{(n)}(0)$ values above into

$$
f(x) \approx \sum_{n=0}^{\infty} \frac{x^n}{n!} f^{(n)}(0)
$$

For up to $n = 3$ gives the following

$$
\begin{aligned}
f(x) &\approx f(0) + x f^{(1)}(0) + \frac{x^2}{2} f^{(2)}(0) + \frac{x^3}{3!} f^{(3)}(0) + \cdots \\
&\approx 0 + x\frac{1}{3} + \frac{x^2}{2}(0) + \frac{x^3}{3!}\left(-\frac{2}{3}\right) \\
&\approx x\frac{1}{3} - \frac{2}{3}\frac{x^3}{6} \\
&\approx \frac{x}{3} - \frac{x^3}{9}
\end{aligned}
$$

When $x = \frac{1}{10}$ the above becomes

$$
\begin{aligned}
f_{n=3}\left(\frac{1}{10}\right) &\approx \frac{1}{30} - \frac{1}{(1000)9} \\
&\approx \frac{1}{30} - \frac{1}{9000} \\
&\approx \frac{300-1}{9000} \\
&\approx \frac{299}{9000}
\end{aligned}
$$

From the calculator

$$
\frac{299}{9000} \approx 0.0332222
$$

And from the exact expression

$$
\begin{aligned}
\frac{\sin(x)}{\cosh(x)+2} &= \frac{\sin(0.1)}{\cosh(0.1)+2} \\
&= 0.0332224
\end{aligned}
$$

The error is about 1.67×10^{-7}.

2 Problem 2

Consider $f(x) = (1+x)^p$ for (a) $p = \frac{1}{3}$ and (b) $p = -2$, respectively. (1) Find the Taylor series of $f(x)$ around $x = 0$. (2) From the form of the general term, find the interval of convergence of the series. (3) How many terms in the series do you need to estimate $f(0.1)$ to within 1% ? Check that the difference between your estimate and the actual result has approximately the same magnitude as the next term in the series.

Solution

2.1 Case $p = \frac{1}{3}$

$$f(x) = (1+x)^{\frac{1}{3}}$$

Part (1) The Taylor series is given by

$$f(x) \approx f(0) + xf'(0) + \frac{x^2}{2!}f''(x) + \frac{x^3}{3!}f'''(x) + \cdots \tag{1}$$

Where $f(0) = 1$ and $f'(x) = \frac{1}{3}(1+x)^{-\frac{2}{3}}$. Hence $f'(0) = \frac{1}{3}$ and $f''(x) = \frac{1}{3}\left(-\frac{2}{3}\right)(1+x)^{-\frac{5}{3}}$. Hence $f''(0) = -\frac{(2)}{3^2}$, and $f'''(x) = \frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)(1+x)^{-\frac{8}{3}}$, hence $f'''(0) = \frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right) = \frac{(2)(5)}{3^3}$, and $f^{(4)}(x) = \frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\left(-\frac{8}{3}\right)(1+x)^{-\frac{11}{3}}$, hence $f^{(4)}(0) = \frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\left(-\frac{8}{3}\right) = -\frac{1}{3^4}((2)(5)(8))$ and on. The series in (1) becomes

$$\begin{aligned}
f(x) &\approx 1 + \frac{1}{3}x - \frac{(2)}{3^2}\frac{x^2}{2!} + \frac{(2)(5)}{3^3}\frac{x^3}{3!} - \frac{(2)(5)(8)}{3^4}\frac{x^4}{4!} + \frac{(2)(5)(8)(11)}{3^5}\frac{x^5}{5!} - \frac{(2)(5)(8)(11)(14)}{3^6}\frac{x^6}{6!} - \cdots \\
&\approx 1 + \frac{1}{3}x - \frac{1}{3^2}x^2 + \frac{5}{3^4}x^3 - \frac{10}{3^5}x^4 + \frac{22}{3^6}x^5 - \frac{154}{3^8}x^6 + \cdots
\end{aligned} \tag{2}$$

The general term is found by comparing the above to the general term obtained from binomial expansion. Since

$$(1+x)^p = \binom{p}{0}x^0 + \binom{p}{1}x + \binom{p}{2}x^2 + \cdots \tag{3}$$

Comparing (2,3) shows that the general term is the binomial coefficient $\binom{\frac{1}{3}}{n}$. Therefore the Taylor series for $(1+x)^{\frac{1}{3}}$ can be written as

$$f(x) \approx \sum_{n=0}^{\infty} \binom{p}{n} x^n$$

For $p = \frac{1}{3}$ the above becomes

$$f(x) \approx \sum_{n=0}^{\infty} \binom{\frac{1}{3}}{n} x^n$$

<u>Part(2)</u>

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$
$$= \lim_{n\to\infty} \left| \frac{\binom{p}{n}}{\binom{p}{n+1}} \right|$$

The Binomial coefficient $\binom{p}{n} = \frac{p!}{n!(p-n)!}$, for when p is integer. This is not the case here.

For non-integer p The Binomial coefficient becomes $\binom{p}{n} = \frac{\Gamma(p+1)}{\Gamma(n+1)\Gamma(p-n+1)}$ where $\Gamma(p)$ is the Gamma function. The above ratio now becomes

$$R = \lim_{n\to\infty} \left| \frac{\frac{\Gamma(p+1)}{\Gamma(n+1)\Gamma(p-n+1)}}{\frac{\Gamma(p+1)}{\Gamma(n+2)\Gamma(p-n)}} \right|$$
$$= \lim_{n\to\infty} \left| \frac{\Gamma(n+2)\Gamma(p-n)}{\Gamma(n+1)\Gamma(p-n+1)} \right|$$
$$= \lim_{n\to\infty} \left| \frac{n\Gamma(p-n)}{\Gamma(p-n+1)} \right|$$
$$= \lim_{n\to\infty} \left| \frac{n}{p-n} \right|$$

But $p = \frac{1}{3}$, hence the above becomes

$$
\begin{aligned}
R &= \lim_{n\to\infty} \left| \frac{n}{\frac{1}{3} - n} \right| \\
&= \lim_{n\to\infty} \left| \frac{n}{n - \frac{1}{3}} \right| \\
&= 1
\end{aligned}
$$

Therefore the radius of convergence is 1. This means the Taylor series found above converges to $f(x)$ for $|x| < 1$.

Part 3

$$f(x) = (1 + x)^{\frac{1}{3}}$$

When $x = 0.1$

$$
\begin{aligned}
f(0.1) &= (1.1)^{\frac{1}{3}} \\
&= 1.032280115
\end{aligned}
$$

one percent of the above is

$$\frac{1}{100}(1.032280115) = 0.01032280115$$

The value n is now found such that

$$|R_n(x)| \leq M \frac{(0.1)^{n+1}}{(n+1)!} \leq 0.01032280115$$

Where $R_n(x)$ is the Taylor series remainder using n terms. M is the upper bound for the $n+1$ derivative of $f(x)$ any where between $[0, 0.1]$. Instead of trying to find M, few calculations are used to find how many terms are needed.

For $n = 0$, $\tilde{f}(0.1) = 1$ and the error is $1.032280115 - 1 = 032280115$.

For $n = 1$, $\tilde{f}(0.1) = 1 + \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} 0.1 = 1.0333333$, and the error is $|1.032280115 - 1.0333333| = 0.001053218$. Because this is smaller than $R_n(x)$ then only two terms are needed in the Taylor series to obtained the required accuracy. Therefore

$$f(x) \approx 1 + \frac{1}{3}x$$

2.2 Case $p = -2$

$$f(x) = (1+x)^{-2}$$

Part (1) The Taylor series is

$$f(x) \approx f(0) + xf'(0) + \frac{x^2}{2!}f''(x) + \frac{x^3}{3!}f'''(x) + \cdots$$

But $f(0) = 1$ and $f'(x) = (-2)(1+x)^{-3}$. Hence $f'(0) = -2$ and $f''(x) = (-2)(-3)(1+x)^{-4}$. Hence $f''(0) = (-2)(-3)$, and $f'''(x) = -2(-3)(-4)(1+x)^{-5}$, hence $f'''(0) = (-2)(-3)(-4)(-5)$ and so on. The above becomes

$$\begin{aligned} f(x) &\approx 1 + (-2)x - (-2)(-3)\frac{x^2}{2!} + (-2)(-3)(-4)\frac{x^3}{3!} + \cdots \\ &\approx 1 - 2x + (2)(3)\frac{x^2}{2!} - (2)(3)(4)\frac{x^3}{3!} + \cdots \\ &\approx 1 - 2x + 3x^2 - 4x^3 + \cdots \end{aligned}$$

The general term is therefore

$$f(x) \approx \sum_{n=0}^{\infty} (-1)^n (n+1) x^n$$

Part(2)

$$\begin{aligned} R &= \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| \\ &= \lim_{n\to\infty} \left| \frac{(n+1)}{(n+2)} \right| \\ &= 1 \end{aligned}$$

Hence the series converges to $f(x)$ for $|x| < 1$.

Part 3

$$f(x) = (1+x)^{-2}$$

For $x = 0.1$

$$\begin{aligned} f(0.1) &= (1.1)^{-2} \\ &= \frac{1}{1.1^2} \\ &= 0.82644628 \end{aligned}$$

One percent of the above is

$$\frac{1}{100}(0.8264462810) = 0.0082644628$$

The value n is now found such that

$$|R_n(x)| \leq M\frac{(0.1)^{n+1}}{(n+1)!} \leq 0.0082644628$$

Where $R_n(x)$ is the Taylor series remainder using n terms. M is the upper bound for the $n+1$ derivative of $f(x)$ any where between $[0, 0.1]$. Doing few calculations gives

For $n = 0$, $\tilde{f}(0.1) = 1$, the error is $|0.82644628 - 1| = 0.1735537190$.

For $n = 1$, $\tilde{f}(0.1) = 0.8$, the error is $|0.82644628 - 0.8| = 0.02644628$.

For $n = 2$, $\tilde{f}(0.1) = 0.83$, the error is $|0.82644628 - 0.83| = 0.0035537190$. Because this is within 1% then only <u>three terms</u> are needed. Therefore

$$f(x) \approx 1 - 2x + 3x^2$$

3 Problem 3

Expand $f(x) = \tan\left(x^2\right)$ to order x^6 using (a) direct Taylor expansion. (b) The Taylor series for $\sin(x)$ and $\cos x$ with appropriate substitution.

Solution

3.1 Part a

Using Taylor series

$$f(x) \approx \sum_{n=0}^{\infty} \frac{x^n}{n!} f^{(n)}(0)$$

Where $f(x) = \tan\left(x^2\right)$ and the expansion is around $x = 0$. The Taylor series for $f(u) = \tan(u)$ is found instead of $\tan\left(x^2\right)$, and then at the end u is replaced by x^2 . This is called the substitution method. This simplifies the derivations. Therefore $f(0) = 0$. The first derivative is

$$\begin{aligned}
f'(u) &= \frac{d}{du} \tan(u) \\
&= \frac{d}{du}\left(\frac{\sin u}{\cos u}\right) \\
&= \frac{\cos^2 u + \cos^2 u}{\cos^2 u} \\
&= \frac{1}{\cos^2 u}
\end{aligned}$$

At $u = 0$ this gives $f'(0) = 1$.

The next derivative using the above result gives

$$\begin{aligned}
f''(u) &= \frac{d}{du}\left(\frac{1}{\cos^2 u}\right) \\
&= \frac{2 \cos u \sin u}{\cos^4 u} \\
&= \frac{2 \sin u}{\cos^3 u}
\end{aligned}$$

At $u=0$ this gives $f^{(2)}(0)=0$. The next derivative gives

$$\begin{aligned}
f^{(3)}(u) &= 2\frac{d}{du}\left(\frac{\sin u}{\cos^3 u}\right)\\
&= 2\frac{\cos u\cos^3 u - \sin u\left(3\cos^2 u(-\sin u)\right)}{\cos^6 u}\\
&= 2\frac{\cos^4 u + 3\sin^2 u\cos^2 u}{\cos^6 u}\\
&= \frac{2\cos^4 u}{\cos^6 u} + \frac{6\sin^2 u\cos^2 u}{\cos^6 u}\\
&= \frac{2}{\cos^2 u} + \frac{6\sin^2 u}{\cos^4 u}\\
&= \frac{2}{\cos^2 u} + \frac{6\left(1-\cos^2 u\right)}{\cos^4 u}\\
&= \frac{2}{\cos^2 u} + \frac{6}{\cos^4 u} - \frac{6}{\cos^2 u}\\
&= -\frac{4}{\cos^2 u} + \frac{6}{\cos^4 u}
\end{aligned}$$

At $u=0$ this gives $f^{(3)}(0) = -\frac{4}{1}+\frac{6}{1} = 2$. Since the problem is asking for order x^6 the process stops here, as this is the same as order u^3 when u is replaced by x^2.

Therefore the Taylor series for $\tan(u)$ is (for up to $n=3$)

$$\begin{aligned}
f(u) &\approx f(0) + uf'(0) + \frac{u^2}{2!}f^{(2)}(0) + \frac{u^3}{3!}f^{(3)}(0) + \cdots\\
&\approx 0 + u + 0 + 2\frac{u^3}{3!}\\
&\approx u + \frac{1}{3}u^3
\end{aligned}$$

Replacing $u=x^2$, gives the Taylor series for $\tan\left(x^2\right)$ for up to x^6 term as

$$\tan\left(x^2\right) \approx x^2 + \frac{1}{3}x^6$$

3.2 Part b

To obtain the above result using the Taylor series for $\sin\left(x^2\right), \cos\left(x^2\right)$, the Taylor series for $\sin\left(x^2\right)$ and $\cos\left(x^2\right)$ is found, and long division is applied using the definition of $\tan\left(x^2\right) = \frac{\sin\left(x^2\right)}{\cos\left(x^2\right)}$. Terms with order larger than x^6 are ignored. The Taylor series for $\sin(x)$ is

$$\sin(x) \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

Using the substitution method, the Taylor series for $\sin\left(x^2\right)$ becomes

$$\begin{aligned}\sin\left(x^2\right) &\approx x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \cdots \\ &\approx x^2 - \frac{x^6}{6} + \frac{x^{10}}{120} - \cdots \end{aligned} \tag{1}$$

The Taylor series for $\cos(x)$ is

$$\cos(x) \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

Using the substitution method, the Taylor series for $\cos\left(x^2\right)$ becomes

$$\begin{aligned}\cos\left(x^2\right) &\approx 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \cdots \\ &\approx 1 - \frac{x^4}{2} + \frac{x^8}{24} - \cdots \end{aligned} \tag{2}$$

Since $\tan\left(x^2\right) = \frac{\sin\left(x^2\right)}{\cos\left(x^2\right)}$ then the Taylor series for $\tan\left(x^2\right)$ is

$$\tan\left(x^2\right) \approx \frac{x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \cdots}{1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \cdots}$$

Performing long division and stopping when the remainder has powers larger than x^6 gives

$$\tan\left(x^2\right) \approx x^2 + \frac{1}{3}x^6 + \cdots$$

Which is same result as part(a).

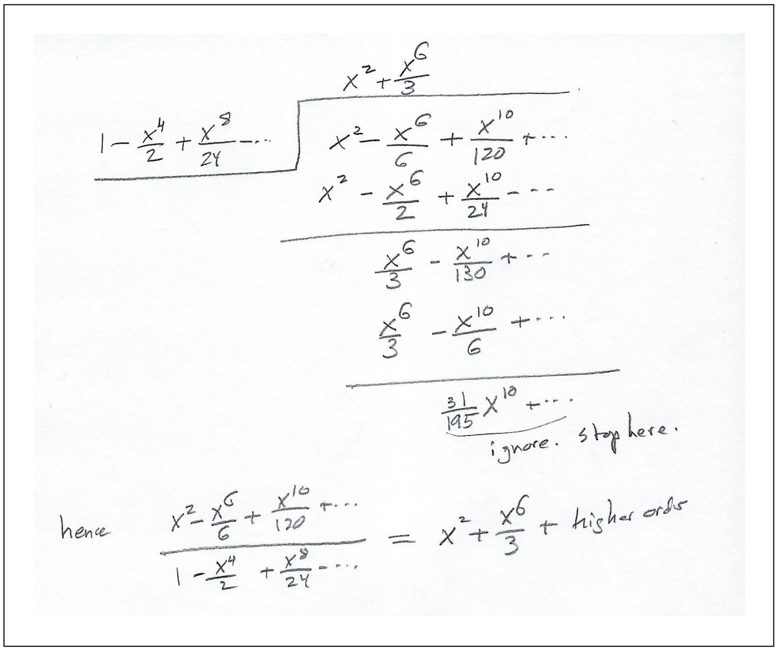

Figure 1: Polynomals long division

4 Problem 4

A particle of mass m moves along the $+x$ axis (i.e. $x > 0$) with potential energy

$$V(x) = \frac{a}{2x^2} - \frac{b}{x}$$

Where a and b are positive parameters. (a) Find the equilibrium position x_0. (b) Show that the particle executes harmonic oscillations near $x = x_0$. (c) Find the angular frequency of oscillations.

Solution

4.1 Part a

Equilibrium position is where the slope of the potential energy is zero. This position x_0 is found by solving for x from

$$\frac{dV}{dx} = 0$$

But

$$\begin{aligned} \frac{dV}{dx} &= \frac{a}{2}\left(-2x^{-3}\right) - b\left(-x^{-2}\right) \\ &= \frac{-a}{x^3} + \frac{b}{x^2} \\ &= \frac{-a + bx}{x^3} \end{aligned}$$

Hence

$$\begin{aligned} \frac{-a + bx}{x^3} &= 0 \\ bx &= a \end{aligned}$$

Therefore

$$x_0 = \frac{a}{b}$$

4.2 Part b

Approximating $V(x)$ around x_0 using Taylor series gives

$$V(x) \approx V(x_0) + (x - x_0)V'(x_0) + \frac{(x - x_0)^2}{2!}V''(x_0) + \cdots$$

But $\frac{dV}{dx}$ evaluated at x_0 is zero, since this the equilibrium point. The above simplifies to

$$V(x) \approx V(x_0) + \frac{(x - x_0)^2}{2!}V''(x_0) + \cdots \tag{A}$$

Higher terms are ignored, because $(x - x_0)$ is assumed small and mass remain close to x_0. But

$$V(x_0) = \frac{a}{2x_0^2} - \frac{b}{x_0}$$

And since $x_0 = \frac{a}{b}$ from part (a), the above simplifies to

$$\begin{aligned} V(x_0) &= \frac{a}{2\left(\frac{a}{b}\right)^2} - \frac{b}{\left(\frac{a}{b}\right)} \\ &= \frac{ab^2}{2a^2} - \frac{b^2}{a} \\ &= \frac{b^2}{2a} - \frac{b^2}{a} \\ &= -\frac{1}{2}\frac{b^2}{a} \end{aligned} \tag{A1}$$

And

$$\begin{aligned} \frac{d^2V}{dx^2} &= \frac{d}{dx}\left(\frac{-a}{x^3} + \frac{b}{x^2}\right) \\ &= \frac{3a}{x^4} - \frac{b}{x^3} \end{aligned}$$

At $x = x_0$ the above becomes

$$\begin{aligned} V''(x_0) &= \frac{3a}{\left(\frac{a}{b}\right)^4} - \frac{b}{\left(\frac{a}{b}\right)^3} \\ &= \frac{b^4}{a^3} \end{aligned} \tag{A2}$$

Using (A1,A2) into A gives

$$\begin{aligned} V(x) &\approx -\frac{1}{2}\frac{b^2}{a} + \frac{(x-x_0)^2}{2!}\frac{b^4}{a^3} + \cdots \\ &\approx -\frac{1}{2}\frac{b^2}{a} + \frac{\left(x-\frac{a}{b}\right)^2}{2}\frac{b^4}{a^3} + \cdots \\ &\approx -\frac{1}{2}\frac{b^2}{a} + \frac{1}{2}\left(x^2 + \frac{a^2}{b^2} - 2x\frac{a}{b}\right)\frac{b^4}{a^3} + \cdots \\ &\approx -\frac{1}{2}\frac{b^2}{a} + \frac{1}{2a}b^2 + \frac{1}{2a^3}b^4x^2 - \frac{1}{a^2}b^3x + \cdots \\ &\approx \frac{b^4}{2a^3}x^2 - \frac{b^3}{a^2}x + \cdots \end{aligned}$$

Therefore near x_0 the potential energy is approximated as

$$V(x) \approx \frac{b^4}{2a^3}x^2 - \frac{b^3}{a^2}x \tag{1}$$

The force on the mass is given by

$$F = -\frac{dV}{dx}$$

Using $V(x)$ in (1) the force becomes

$$F = -\frac{b^4}{a^3}x - \frac{b^3}{a^2}$$

But $F = m\frac{d^2x}{dt^2}$. Hence we obtain the equation of motion as

$$\begin{aligned} m\frac{d^2x}{dt^2} &= F \\ &= -\frac{b^4}{a^3}x - \frac{b^3}{a^2} \end{aligned}$$

Therefore

$$\begin{aligned} m\frac{d^2x(t)}{dt^2} + \frac{b^4}{a^3}x(t) &= -\frac{b^3}{a^2} \\ \frac{d^2x(t)}{dt^2} + \left(\frac{b^4}{ma^3}\right)x(t) &= -\frac{b^3}{ma^2} \end{aligned} \tag{B}$$

Let

$$\frac{b^4}{ma^3} = \omega^2$$

The equation of motion (B) becomes

$$\frac{d^2x(t)}{dt^2} + \omega^2 x(t) = -\frac{b^3}{ma^2}$$

But this is standard second order ode whose solution is

$$x(t) = A\cos(\omega t) + B\sin(\omega t) + x_p(t)$$

Where $x_p(t)$ is the particular solution due to the forcing function $-\frac{b^3}{ma^2}$ and A, B are constants of integrations found from initial conditions. Since the forcing function is just constant, and not function function of time, the above becomes

$$\begin{aligned} x(t) &= A\cos(\omega t) + B\sin(\omega t) + F_p \\ &= A\cos\left(\omega t + \phi\right) + F_p \end{aligned}$$

Therefore the motion is simple harmonic motion since $\cos\left(\omega t + \phi\right)$ is harmonic. The forcing function F_p has no effect on the nature of the harmonic motion, other than adding an extra constant displacement shift to $x(t)$ for all time. Since there is no damping, the particle will continue this motion forever.

The following is a plot of the solution for 10 seconds using arbitrary values for a, b, m and with initial conditions $x(0) = 1, x'(0) = 0$. The solution shows the motion is harmonic as expected.

```
In[]:= ClearAll[x, t, w, a, b, m];
       ode = x''[t] + w x[t] == -b^3/(m a^2);
       ic = {x[0] == 0, x'[0] == 0};
       sol = x[t] /. First@DSolve[{ode, ic}, x[t], t]
Out[]= (-b^3 + b^3 Cos[t Sqrt[w]])/(a^2 m w)

In[]:= parms = {m -> 1, a -> 1, b -> 3, w -> Sqrt[b^4/(m a^3)]};
       sol //. parms
Out[]= 1/9 (-27 + 27 Cos[3 t])

In[]:= Plot[sol //. parms, {t, 0, 10}, AxesLabel -> {"t", "x(t)"}, BaseStyle -> 14, PlotStyle -> Blue]
Out[]=
```

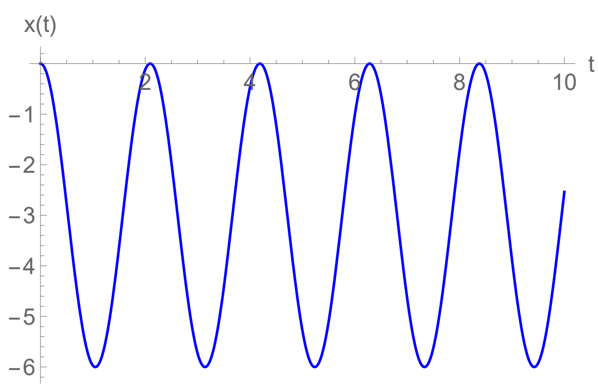

Figure 2: Plot of solution

4.3 Part c

The angular frequency of oscillation is

$$\omega = \sqrt{\frac{b^4}{ma^3}}$$

In radians per second. The quantity $\frac{b^4}{a^3}$ can be called the stiffness k (Newton per meter). Hence $\omega = \sqrt{\frac{k}{m}}$.

4.4 Appendix

An easier way to do part b, is to keep $(x - x_0)$ intact and replace this with y at the end. Like this

Using (A1,A2) into A gives

$$V(x) \approx -\frac{1}{2}\frac{b^2}{a} + \frac{(x - x_0)^2}{2!}\frac{b^4}{a^3} + \cdots$$

The force on the mass is given by

$$\begin{aligned} F &= -\frac{dV}{dx} \\ &= -(x - x_0)\frac{b^4}{a^3} \end{aligned}$$

But $F = m\frac{d^2x}{dt^2}$. Hence we obtain the equation of motion as

$$\begin{aligned} m\frac{d^2x}{dt^2} &= F \\ &= -(x - x_0)\frac{b^4}{a^3} \end{aligned}$$

Now let $y = x - x_0$. the above becomes

$$\begin{aligned} m\frac{d^2y}{dt^2} &= -y\frac{b^4}{a^3} \\ m\frac{d^2y}{dt^2} + y\frac{b^4}{a^3} &= 0 \\ \frac{d^2y}{dt^2} + y\frac{b^4}{ma^3} &= 0 \end{aligned}$$

Which is SHM. Using this method, it is faster to show.