Exam 2

Math 5525
Introduction to Ordinary Differential Equations

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Contents

0.1 Problem 1

1. State the Bendixon criterion of non-existence of periodic orbits of a differential equation.

2. Consider the differential equation

$$\ddot{x} + f(x)\dot{x} + x = 0$$

Where $f(x) = x^2 + x + a,\ a \in \mathbb{R}$. Determine the range of values of a for which the equation does not have any periodic orbits

solution

Part 1 The Bendixon criterion gives a condition to check if a system of first order ODE's defined in region $D \subset \mathbb{R}^2$ has only periodic solutions. It uses the state space form of the system given as $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \boldsymbol{F} = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix}$. The domain D has to be simply connected and f_1, f_2 are continuously differentiable functions in D. Given the above, then if the divergence of $\boldsymbol{F}$ is never zero at any point in the domain D then the system has no periodic solutions . In other words, Bendixon criterion says that

if $\nabla \cdot F = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \neq 0$ at any point in D, then system has no periodic solutions.

No periodic solution is the same as saying phase plot contains no closed orbits.

This criterion can be also stated in another way. The condition given above that $\nabla \cdot F \neq 0$ anywhere, just means that the sign of the divergence do not change in D. (For the sign to change, $\nabla \cdot F$ has to cross the value zero somewhere. This is from calculus). Therefore we can also say the criterion as follows. If $\nabla \cdot F$ changes sign somewhere in D or if it attains the value zero in D then system has only periodic solutions.

Part 2

$$\ddot{x} + f(x)\dot{x} + x = 0$$

We first observe immediately that the above ODE is a harmonic oscillator with damping coefficient $f(x)$ present, so we can make some observations before applying Bendixon criterion. We know from Physics that if the damping coefficient is positive, then the solution is stable and will have the form of exponentially decaying damped oscillations (this is because damping now takes energy away). But if the damping is negative then the solution becomes unstable and has the form of exponentially increasing damped oscillations (because now damping adds energy to the system).

If the damping coefficient is zero, then the ODE becomes pure harmonic oscillator $\ddot{x} + x = 0$ and the solution is pure oscillatory (periodic) that lasts for all time (given non-zero initial conditions).

Therefore, just from Physics considerations only, we expect that only if $f(x) = 0$ everywhere then closed trajectory (or periodic solution) will result. Or if $f(x) \neq 0$ then no periodic solutions exist. In real physical mechanical systems the damping coefficient, if present, is always positive.

Now we can apply Bendixon criterion. Let $x_1 = x, x_2 = \dot{x}$. Taking time derivatives gives $\dot{x}_1 = x_2, \dot{x}_2 = -f(x)\dot{x} - x = -f(x_1)x_2 - x_1$. In state space the system becomes

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \boldsymbol{F} = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix} = \begin{pmatrix} x_2 \\ -f(x_1)x_2 - x_1 \end{pmatrix}$$

Therefore the gradient is

$$\begin{aligned} \nabla \cdot F &= \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \\ &= 0 - f(x_1) \\ &= -f(x_1) \end{aligned}$$

Using Bendixon criterion which says that if $\nabla \cdot F \neq 0$ at every point, then no periodic solutions exist, shows right away that the condition for no periodic solution is that $f(x_1) \neq 0$ for all x_1 domain, which is what Physics tells us. This means that if (where now we write x_1 as x since they are the same and for simplicity)

$$x^2 + x + a \neq 0 \qquad \text{For all } x$$

Or if

$$a \neq -\left(x^2 + x\right)$$

Then no periodic solutions exist.

0.2 Problem 2

Consider the system of differential equations that models the growth of two competing species with populations $x \geq 0$ and $y \geq 0$

$$\dot{x} = x\left(2 - x - y\right)$$
$$\dot{y} = y\left(3 - 2x - y\right)$$

1. Find all equilibrium points and determine their stability type.
2. Determine the nullclines of the system.
3. Find the invariant regions of the xy-plane.
4. Draw the phase-plane using your favorite software.
5. Explain why these equations make it mathematically possible, but extremely unlikely, for both species to survive.

solution

Part 1

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \boldsymbol{F} = \begin{pmatrix} f_1\left(x,y\right) \\ f_2\left(x,y\right) \end{pmatrix} = \begin{pmatrix} x\left(2-x-y\right) \\ y\left(3-2x-y\right) \end{pmatrix}$$

Equilibrium points are solutions of $f_1 = 0, f_2 = 0$. When $f_1 = 0$ then $x = 0$ or $x = 2 - y$. Now, when $x = 0$, then $f_2 = 0 = y\left(3-y\right)$ which has solution as $y = 0$ and $y = 3$. Therefore the critical points found so far are $\{0,0\},\{0,3\}$.

Now when $x = 2 - y$ then $f_2 = 0 = y\left(3 - 2\left(2-y\right) - y\right) = y\left(y-1\right)$, which has solution as $y = 0$ and $y = 1$. This means that $x = 2$ and $x = 1$ respectively. This adds the points $\{2,0\},\{1,1\}$ to what was found above. Therefore the list of critical points are

$$\left(x_i, y_i\right) = \{0,0\},\{0,3\},\{2,0\},\{1,1\}$$

The Jacobian matrix for the system is given by gradient of $\boldsymbol{F}$

$$\begin{aligned} \nabla\boldsymbol{F} = J &= \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} \\ &= \begin{pmatrix} \frac{\partial}{\partial x}\left(2x - x^2 - xy\right) & \frac{\partial}{\partial y}\left(2x - x^2 - xy\right) \\ \frac{\partial}{\partial x}\left(3y - 2xy - y^2\right) & \frac{\partial}{\partial y}\left(3y - 2xy - y^2\right) \end{pmatrix} \\ &= \begin{pmatrix} 2 - 2x - y & -x \\ -2y & 3 - 2x - 2y \end{pmatrix} \end{aligned} \tag{1}$$

At point $(0,0)$ the linearized system A matrix is the Jacobian above evaluated at this point, which gives

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

Hence $|A - \lambda I| = 0$ gives

$$\begin{vmatrix} 2-\lambda & 0 \\ 0 & 3-\lambda \end{vmatrix} = 0$$
$$(2-\lambda)(3-\lambda) = 0$$

Therefore $\lambda_1 = 2, \lambda_2 = 3$. Since both eigenvalues are positive, then this is unstable critical point. It is a negative attractor.

At point $(0,3)$ the linearized system A matrix is the Jacobian from (1) evaluated at this point, which gives

$$A = \begin{pmatrix} -1 & 0 \\ -6 & -3 \end{pmatrix}$$

Hence $|A - \lambda I| = 0$ gives

$$\begin{vmatrix} -1-\lambda & 0 \\ -6 & -3-\lambda \end{vmatrix} = 0$$
$$(-1-\lambda)(-3-\lambda) = 0$$

Therefore $\lambda_1 = -1, \lambda_2 = -3$. Since both eigenvalues are negative, then this is a stable critical point. It is a positive attractor.

At point $(2,0)$ the linearized system A matrix is the Jacobian from (1) evaluated at this point, which gives

$$A = \begin{pmatrix} -2 & -2 \\ 0 & -1 \end{pmatrix}$$

Hence $|A - \lambda I| = 0$ gives

$$\begin{vmatrix} -2-\lambda & -2 \\ 0 & -1-\lambda \end{vmatrix} = 0$$
$$(-2-\lambda)(-1-\lambda) = 0$$

Therefore $\lambda_1 = -2, \lambda_2 = -1$. Since both eigenvalues are negative, then this is a stable critical point. It is a positive attractor.

At point $(1,1)$ the linearized system A matrix is the Jacobian from (1) evaluated at this point, which gives

$$A = \begin{pmatrix} -1 & -1 \\ -2 & -1 \end{pmatrix}$$

Hence $|A - \lambda I| = 0$ gives

$$\begin{vmatrix} -1-\lambda & -1 \\ -2 & -1-\lambda \end{vmatrix} = 0$$
$$(-1-\lambda)^2 - 2 = 0$$
$$\lambda^2 + 2\lambda - 1 = 0$$

Using quadratic formula, $\lambda = -\frac{b}{2a} \pm \frac{1}{2}\sqrt{b^2 - 4ac} = -1 \pm \frac{1}{2}\sqrt{4+4} = -1 \pm \sqrt{2}$. Hence $\lambda_1 = -1 + \sqrt{2}, \lambda_2 = -1 - \sqrt{2}$. Since $\sqrt{2} > 1$ then $\lambda_1 > 0$ and $\lambda_2 < 0$. Because one eigenvalue is positive and one is negative, then this is a saddle point (unstable).

The following table is a summary of the above results

critical point	eigenvalues	type of equilibrium
$(0,0)$	$\lambda_1 = 2, \lambda_2 = 3$	negative attraction, unstable
$(0,3)$	$\lambda_1 = -1, \lambda_2 = -3$	positive attraction, stable
$(2,0)$	$\lambda_1 = -2, \lambda_2 = -1$	positive attraction, stable
$(1,1)$	$\lambda = -1 \pm \sqrt{2}$	Saddle, unstable

Part 2 Since the system is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \boldsymbol{F} = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix} = \begin{pmatrix} x(2-x-y) \\ y(3-2x-y) \end{pmatrix} \tag{1}$$

Then the x nullclines are the solution of $x(2-x-y) = 0$ and the y nullclines are solutions of $y(3-2x-y)$. This shows that the x nullclines are given by $x = 0$ (the y axis) line and by $y = 2 - x$ line. Similarly the y nullclines are $y = 0$ line (the x axis) and $y = 3 - 2x$ line. This is plot of nullclines for $x \geq 0, y \geq 0$

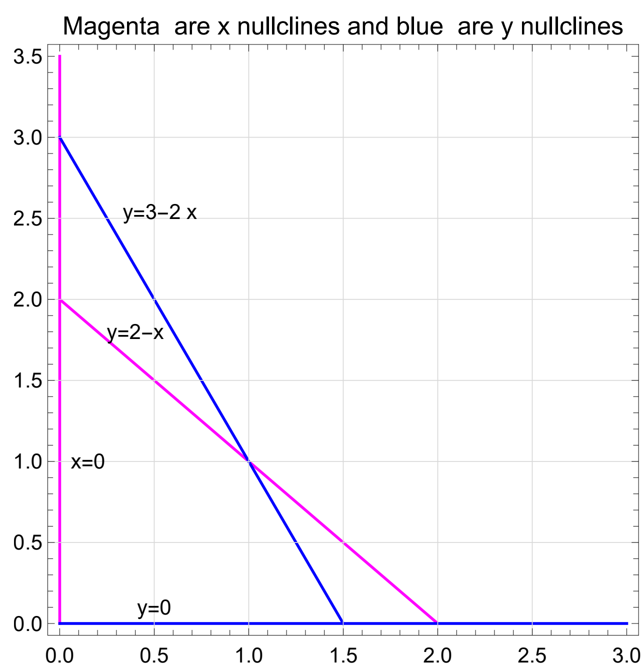

Figure 1: nullclines lines

This plot adds the critical points on the above plot to make it more clear. Red points are unstable and Blue points are stable. The location of the critical points are from part 1.

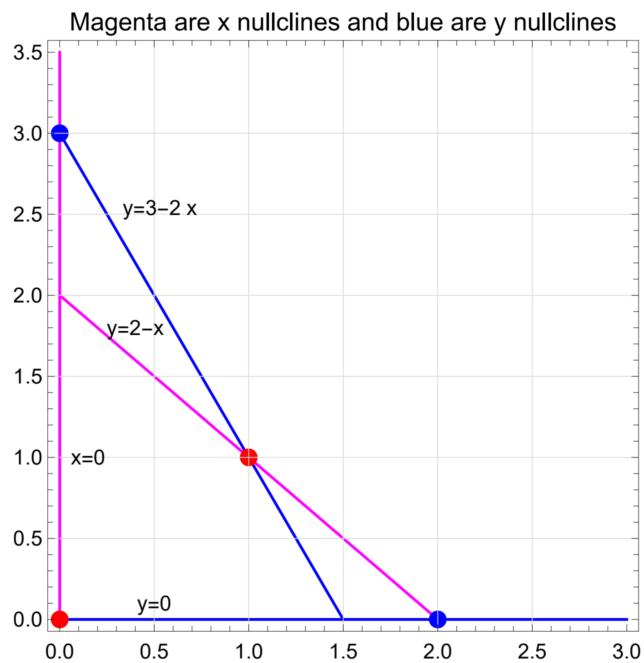

Figure 2: nullclines lines with critical points added

Part 3 invariant regions are those regions where solutions always remain inside the region for all time, given that initial conditions are inside the region. We know that $x = 0$ is invariant (which means solutions that start on this line remain on this line). This is because when $x = 0$ then $\dot{x} = 0$ and $\dot{y} = 3y - 3y^2$. So solution remain on $x = 0$ line. We also know that $y = 0$ is invariant line. This is because when $y = 0$ then $\dot{y} = 0$ and $\dot{x} = 2x - x^2$. So solution remain on $y = 0$ line. This means orbits can not cross these two lines. There are four main regions. These are shown in this plot. We also know that orbits can not cross over invariant lines.

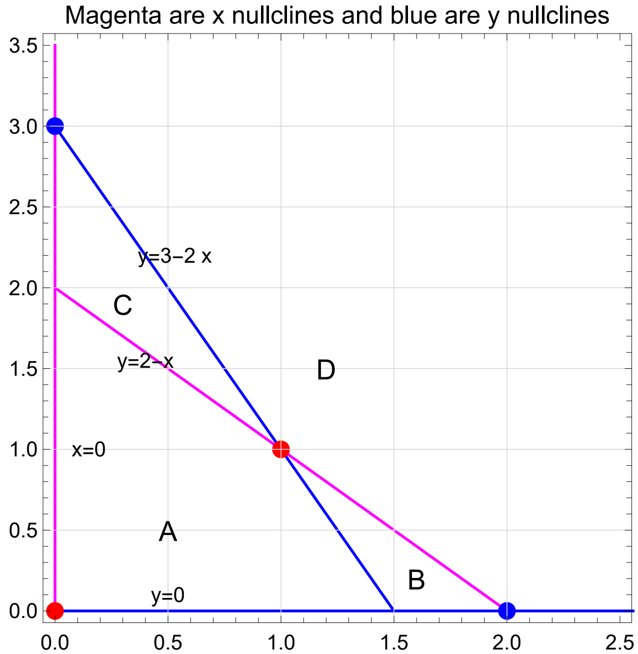

Figure 3: Four regions to examin

Starting with x nullcline $(x = 2 - y)$ where $\dot{x} = 0$, substituting this in the second equation in (1) gives $\dot{y} = y\left(3 - 2\left(2 - y\right) - y\right) = y\left(y - 1\right)$. When $0 < y < 1$ then we see that $\dot{y} < 0$. Hence vector field is pointing downwards. When $1 < y < 3$ then $\dot{y} > 0$. Hence vector field is pointing upwards. The above plot is now updated with this new information.

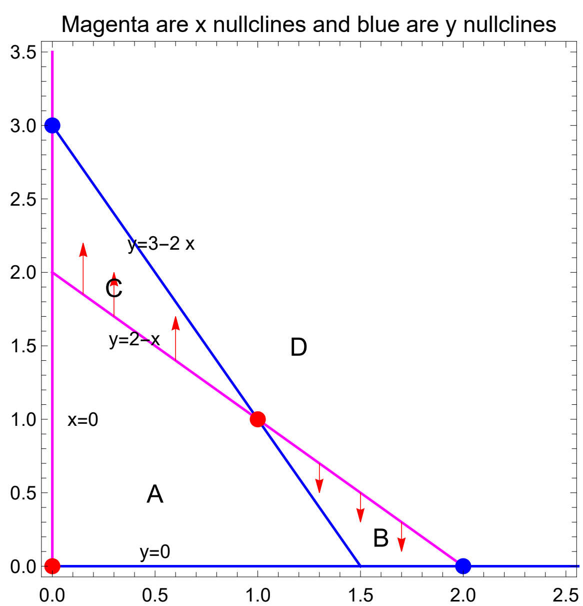

Figure 4: Direction fields based on x nullcline analysis

Now we do the same for the y nullcline line $(y = 3 - 2x)$. substituting this in the first equation in (1) gives $\dot{x} = x(2 - x - (3 - 2x)) = x(x - 1)$. When $x > 1$ then $\dot{x} > 1$ hence vector field is pointing to the right. When $0 < x < 1$ then $\dot{x} < 0$ and vector field is pointing to the left. The above plot is now updated with this new information.

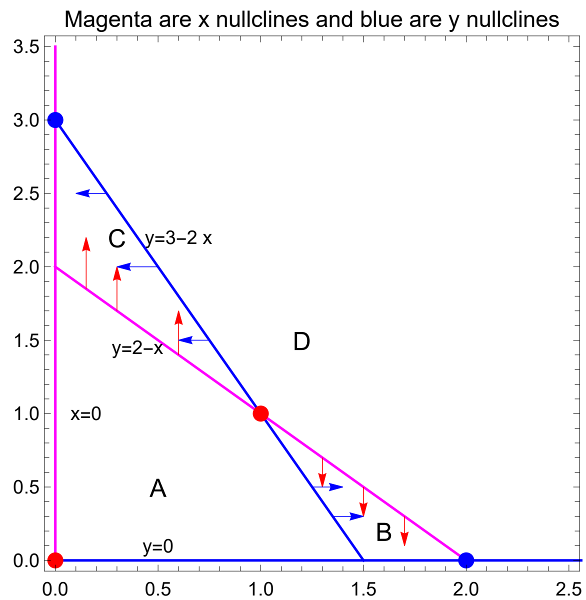

Figure 5: Direction fields based on y nullcline analysis

We see from the above that solutions that starts in region B will remain in B and eventually as $t \to \infty$ reach the stable point $(2, 0)$. Hence region B is an invariant region.

Also solutions that start in C remains in C and eventually as $t \to \infty$ reach the stable point $(0, 3)$. Hence C is invariant region.

A solution that starts in region A can either go to the critical stable point $(2, 0)$ or towards the critical stable point $(0, 3)$ or enter regions C or B first and eventually reach $(2, 0)$ or $(0, 3)$.[1]

We could also consider the union of regions A, C, B as new region say E. Then region E is invariant, since any solution that starts in E remains in E. Similarly, we could also consider the union of regions D, C, B as new region say F. Then region F is also invariant.

Part 4 The phase plot was generated numerically on the computer. The following is the result

[1]If solution starts in region A on exactly the stable eigenvector associated with saddle point $(1, 1)$ then it will remain in A and reach $(1, 1)$ at $t \to \infty$ but this is very unlikely to happen. More on this in part 5 below

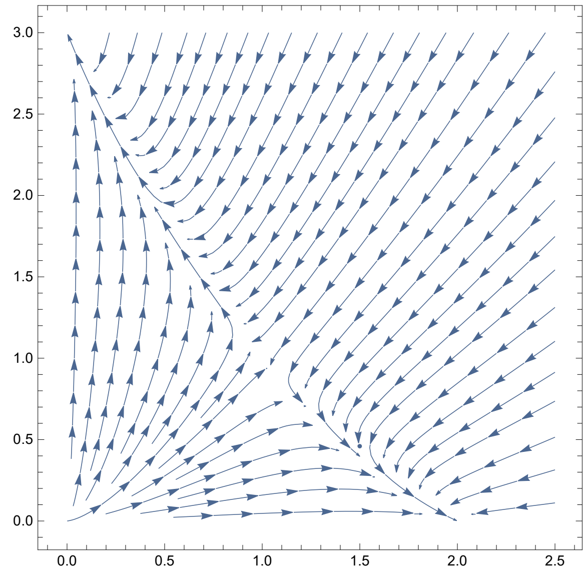

Figure 6: Phase plot

In the following plot, the nullclines are plotted on top of the phase plot to better see the invariant sets found in part 3.

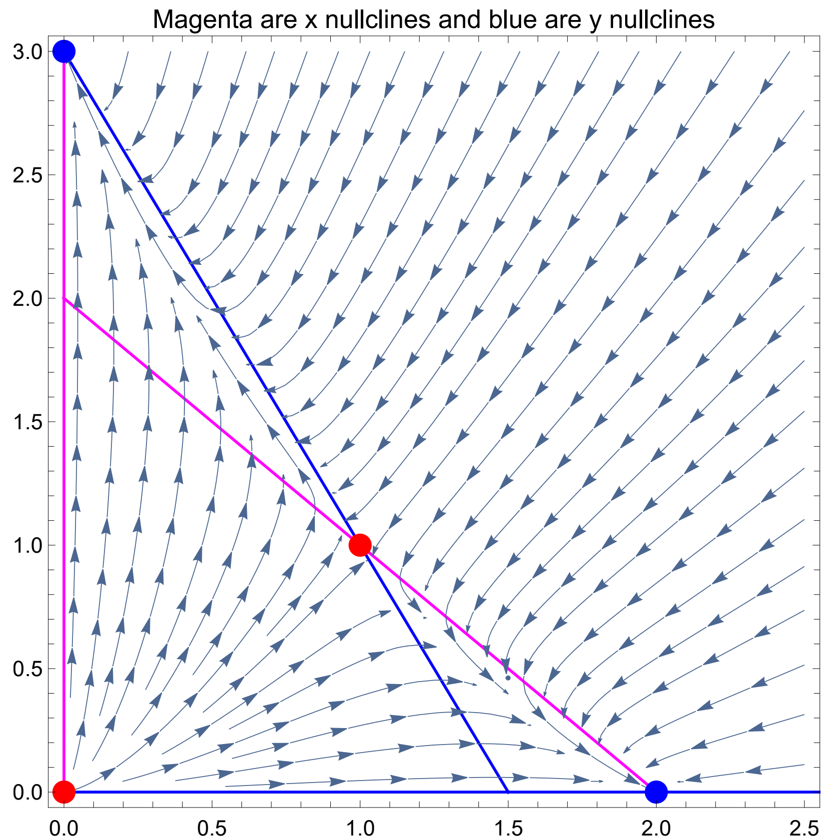

Figure 7: Phase plot with nullclines

Part 5 The point $(1,1)$ is the only saddle point in the system. With the right exact initial conditions, it is possible for the solution to reach this saddle point instead of the other two stable points, if the initial conditions are on the stable eigenvector associated with this saddle point. This saddle equilibrium point have one stable eigenvector and one unstable eigenvector.

This is the only possibility for both species to survive, since at this saddle point both x, y are not zero and remain so for all time. But the probability of the initial conditions being exactly on the stable eigenvector direction for this saddle point is very low compared to having initial conditions being any where else in $\mathbb{R}^2$ where $x \geq 0, y \geq 0$ as any very small deviation in initial conditions will make the solution go to one of the two stable points.

Any other initial conditions location anywhere else in the first quadrant, the solution will reach as $t \to \infty$ either the stable point $(2,0)$ where x only survives or will reach the other stable point $(0,3)$ where now y only survives. This shows that it is extremely unlikely, for both species to survive.