MATH 5525- MIDTERM EXAMINATION II

April 4, 2020

Problem 1.

1. State the Bendixon criterion of non-existence of periodic orbits of a differential equation.

2. Consider the differential equation

$$\ddot{x} + f(x)\dot{x} + x = 0,$$

where $f(x) = x^2 + x + a,\ a \in \mathbb{R}$. Determine the range of values of a for which the equation does not have any periodic orbits.

Solution 1. Let

$$\dot{x} = f(x, y),\ \dot{y} = g(x, y), \quad (x, y) \in D \subset \mathbb{R}^2.$$

1. *Criterion of Bendixon.* Suppose that D is simply connected and (f, g) continuously differentiable in D. The equation can only have periodic solutions if $\nabla \cdot (f, g)$ changes sign in D or if $\nabla \cdot (f, g) = 0$ in D.

2. For the given second order equation, which written as a system takes the form $\dot{x} = y, \quad \dot{y} = -f(x)y - x,$

the divergence of the vector field is

$$\nabla \cdot (y,\, -f(x)y - x) = -f(x) = -(x^2 + x + a).$$

Note that the zeros of the quadratic polynomial $f(x)$ are $-\frac{1}{2} \pm \frac{1}{2}\sqrt{1 - 4a}$. Therefore, for $f(x)$ not to change sign, we must require its zeros to be complex, that is,

$$1 - 4a < 0, \quad a > \frac{1}{4}.$$

Problem 2. Consider the system of differential equations that models the growth of two competing species with populations $x \geq 0$ and $y \geq 0$:

$$\dot{x} = x(2 - x - y), \quad \dot{y} = y(3 - 2x - y).$$

1. Find all equilibrium points and determine their stability type.
2. Determine the nullclines of the system.
3. Find the invariant regions of the xy-plane.
4. Draw the phase-plane using your favorite software (Matlab, Mathematica, ...).
5. Explain why these equations make it mathematically possible, but extremely unlikely, for both species to survive.

Solution 2.

1. $(0,0)$, $(0,3)$, $(2,0)$, $(1,1)$.
   - $(0,0)$: associated matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$; eigenvalues: $(2,3)$. *Unstable node.*
   - $(0,3)$: associated matrix $A = \begin{bmatrix} -1 & 0 \\ -6 & -3 \end{bmatrix}$; eigenvalues: $(-3,-1)$. *Stable node.*
   - $(2,0)$: associated matrix $A = \begin{bmatrix} -2 & -2 \\ 0 & -1 \end{bmatrix}$; eigenvalues: $(-2,-1)$. *Stable node.*
   - $(1,1)$: associated matrix $A = \begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix}$; eigenvalues: $(0.4142, -2.4142)$. *Saddle point.*
2. The nullclines of the system are the lines

$$x = 0; \quad \text{on this line} \quad \dot{y} = y(3 - y). \tag{1}$$
$$x + y = 2; \quad \text{on this line} \quad \dot{y} = (-1 + y). \tag{2}$$
$$y = 0; \quad \text{on this line} \quad \dot{x} = x(2 - x). \tag{3}$$
$$2x + y = 3; \quad \text{on this line} \quad \dot{x} = (-1 + x). \tag{4}$$

The vector field of the system is $\mathbf{f} := (x(2 - x - y), y(3 - 2x - y))$.

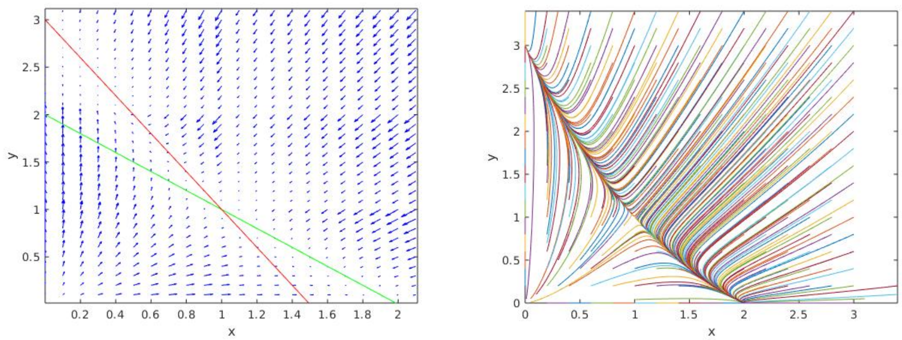

Figure 1: Part 4. The left figure shows vector field and the nullclines (2 green) and (4 red). The equilibrium points are $(0,0),(3,0),\ (1,1),(2,0)$. The right figure shows some orbit plots.

- On the points of the nullcline (2), we have $\mathbf{f}=(0,y(1-y))$. Therefore, the vector field is vertical and points *up* for $0<y<1$, and *down* for $y>1$.
- On the points of the nullcline (4), we have $\mathbf{f}=(-1+3x,0)$. Therefore, the vector field is *horizontal* and points to the *left* for $0<x<1$, and to the *right* for $x>1$.

3. Note that the lines $x=0$ and $y=0$, that is the axes, are *invariant.*

   The first quadrant is also *invariant*, since, $\dot{x}<0$ for $x+y>2$ and $\dot{y}<0$ for $2x+y>3$. That is, the vector field points down and towards the left above the nullcline (4), for $0<x<1$; the vector field also points down and towards the left above the nullcline (2), for $1<x$.

   Within the first quadrant there are two *invariant* regions, the triangle with vertices $(1,1),(0,2)$ and $(0,3)$, and the triangle $(2,0),(\frac{3}{2},0)$ and $(1,1)$. (You only need to verify that the vector field on the sides of the triangles always points towards the interior.)

   The consequence of the previous statements is that, for sufficiently large times, the solutions will either enter the *top* triangle or the *lower* one. Moreover, at $t\to\infty$, the former of will tend to the stable node $(0,3)$ and the latter tend to the other stable node $(2,0)$.

4. Phase-plane, nullclines and vector field. Figure1.

5. We have shown that, as $t\to\infty$, the solutions either converge to $(0,3)$ or $(2,0)$. So, in each case, only one of the two species survives.