MATH 5525- Test 1-Solutions

March 2, 2020

Problem 1. (50 points). Consider the system of differential equations

$$\dot{y} = v, \quad \dot{v} = f(y),$$

where f is a continuous function $f : \mathbf{R} \to \mathbf{R}$.

1. *Find a first integral of the system.* Let us rewrite the system as a single, second order ordinary differential equation:

$$\frac{d^2y}{dt^2} = f(y).$$

Now, multiply both sides by $\dot{y}$ to get:

$$\dot{y}\frac{d^2y}{dt^2} = \dot{y}f(y), \quad \text{or, equivalently} \quad \frac{1}{2}\frac{d\dot{y}^2}{dt} - \frac{dF(y)}{dt} = 0,$$

where $F(y)$ is the antiderivative of $f(y)$, that is, it satisfies the relation

$$F'(y) = f(y).$$

Hence, the first integral of the system is

$$\frac{1}{2}(\dot{y})^2 - F(y) = E,$$

where E is an arbitrary constant.

2. *Find the equilibrium points of the system in the case that $f(y) = \sin y$.* From now on, consider $f(y) = \sin(y)$. The equilibrium points satisfy the equations $\sin y = 0$ and $\dot{y} = 0$. That is,

$$y = 0, \pm\pi, \pm 2\pi, \ldots \quad \text{and} \quad \dot{y} = 0.$$

3. *Find the Jacobian matrix of the system at the equilibrium points.* (That is, write the linearized system about the equilibrium points).

We first express the function $f(y)$ in a Taylor series about $y = y_0$, an equilibrium value, taking into account that $f(y_0) = 0$:

$$f(y) = f'(y_0)(y - y_0) + o(|y - y_0|).$$

In particular, for $f(y) = \sin y$, the matrix of the linear system about $(y_0, 0)$ becomes:

$$A = \begin{bmatrix} 0 & 1 \\ \cos y_0 & 0 \end{bmatrix}.$$

A. For $y_0 = 0, \pm 2\pi, \pm 4\pi, \ldots, \quad A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$

B. For $y_0 = 0, \pm \pi, \pm 3\pi, \ldots, \quad A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$

4. *Determine the nature of the equilibrium points.*

The eigenvalues of $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ are $\lambda = \pm 1$. Hence the equilibrium points $(0,0), (\pm 2\pi, 0), (\pm 4\pi, 0), \ldots$ are saddle points.

The eigenvalues of $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ are $\lambda = \pm i$. Hence the equilibrium points $(0,0), (\pm \pi, 0), (\pm 3\pi, 0), \ldots$ are centers.

5. *Sketch the phase plane of the system in the interval $-\pi \le y \le \pi$.*

Problem 2. (40 points). Consider the predator-pray system governing the number of individuals x y of the two species at time $t > 0$:

$$\dot{x} = x(1 - x - y), \quad \dot{y} = y(-2 + x).$$

1. *Find the equilibrium points of the system.* For this, we need to solve the equations

$$x(1 - x - y) = 0 \quad \text{and} \quad y(-2 + x) = 0.$$

The solutions are given by

$$x = 0 \text{ and } y = 0; \quad x + y = 1 \text{ and } y = 0; \quad x + y = 1 \text{ and } \; -2 + x = 0.$$

Consequently, the equilibrium points (all of them) are

$$(0,0),\ (1,0),\ (2,-1).$$

2. *Find two invariant sets.*

- $x = 0$ is an invariant set. Note that a solution such that $x(0) = 0$ will satisfy $x(t) = 0$, for all $t \geq 0$. The variable y is obtained by solving the second equation, now given by $\dot{y} = -2y$.

- $y = 0$ is another invariant set. A solution such that $y(0) = 0$ will satisfy $y(t) = 0$, for all $t \geq 0$. The variable x is obtained by solving the equation $\dot{x} = x(1-x)$.

3. *Sketch the phase plane.* Let us classify the equilibrium points. For this, denote

$$f(x,y) = x - x^2 - xy, \quad g(x,y) = -2y + xy.$$

Calculate

$$\frac{\partial f}{\partial x} = 1 - 2x - y; \quad \frac{\partial f}{\partial y} = -x; \quad \frac{\partial g}{\partial x} = y; \quad \frac{\partial g}{\partial y} = -2 + x.$$

- The matrix of the linearized system about $(0,0)$ is $A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$. Its eigenvalues are 1 and -2; the corresponding eigenvectors are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, respectively. $(0,0)$ is a *saddle point.*

- The matrix of the linearized system about $(2,-1)$ is $A = \begin{bmatrix} -2 & -2 \\ -1 & 0 \end{bmatrix}$. Its eigenvalues are $\lambda = -1 \pm \sqrt{3}$; the corresponding eigenvectors are $\begin{bmatrix} 1 \pm \sqrt{3} \\ 1 \end{bmatrix}$. $(2,-1)$ is a *saddle point.*

- The matrix of the linearized system about $(1,0)$ is $A = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}$. It has a (double) eigenvalue $\lambda = -1$ with eigenvector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. $(1,0)$ is a degenerate stable node.

Phase plane sketches will be given in separate handout.