Final Exam for Dynamical Systems, Math 5525

May 10, 2020

Problem 1

Consider the following system of ordinary differential equations

$$\begin{aligned} \dot{x} =& -x + y + xy \\ \dot{y} =& x - y - x^2 - y^3. \end{aligned} \tag{1}$$

1. Find the (unique) equilibrium point (x^*, y^*) of the system (1).
2. Linearize the system about about (x^*, y^*) and write the corresponding Jacobian matrix A (that is, the matrix of the linear system.)
3. Find the eigenvalues and eigenvectors of A.
4. Can you reach any conclusions about the stability of (x^*, y^*)?
5. Write down the definition of Lyapunov stability of an equilibrium point.
6. Write down the (Lyapunov) theorem that gives sufficient conditions for the stability of an equilibrium point.
7. Apply the previous theorem to show that the equilibrium solution (x^*, y^*) is, indeed, stable. For this, choose $a \in \mathbb{R}$, so that $V(x, y) = ax^2 + 2y^2$, is a Lyapunov function of the system.
8. Determine an ω-limit set of the system.
9. Write down the Poincaré-Bendixon theorem for two dimensional systems.
10. Taking into account the Poincaré-Bendixon theorem, would you say that a limit cycle is possible for system (1)?

Problem 2.

This problem is about discrete dynamical systems/one-dimensional dynamics. (Class notes, pages 63-69, *Lecture notes, 5525-May4-2020* and section 14.4 of textbook.)

Consider the map

$$f(x) = \frac{x}{1+x^2} - ax, \quad a \in \mathbb{R}, \tag{2}$$

and the discrete *orbit* defined by the sequence

$$x_0,\ x_1 = f(x_0),\ x_2 = f(x_1) = f^2(x_0),\ \dots x_n = f(x_{n-1}) = f^n(x_0), \dots$$

11. Define a *fixed-point* of a map.
12. Find all the fixed points x^* of the map (2) and determine in which intervals of a they exist.
13. Determine the stability of the nonzero fixed point in the parameter interval $a \in (-1, 0)$. Hint: Use the proposition in page 68 of the notes.
14. For $x = \epsilon > 0$, ϵ very small, consider the approximate map $g(\cdot)$ given by

$$g(x) := (1-a)x.$$

Show that the map $g(\cdot)$ has a *two-cycle*, that is a discrete periodic orbit of period 2.

Guidelines:

1. All questions are equally weighted.
2. You may use books, notes and internet resources as you wish.
3. The class notes are posted on Canvas, with the last set of *lecture notes* labelled as *5525-May4-2020.pdf*.
4. The work has to be personal, that is, you may not consult with anyone or receive any help. (You may always email me, if you have questions or difficulties.)
5. The exam should be back tonight (Monday, May 11), by midnight.
6. Upload the complete work on canvas. If you experience difficulties, please email it directly to me.

Please, sign the following statement:
I hereby certify that I have not received help from anyone in the completion of this test.

Signature:

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Minneapolis, May 11, 2020.