MATH 5525: HOMEWORK ASSIGNMENT 3

Date (April 3, 2020)

Problem 1. A modification of the predator-prey system is given by

$$\dot{x} = x(1-x) - \frac{axy}{x+1}, \quad \dot{y} = y(1-y), \tag{1}$$

where $a > 0$ is a parameter.

1. Find all equilibrium points, in the following two cases: $0 < a < 1$ and $a > 1$. (You may select spefic values of a, if you wish.)
2. Classify the equilibrium points in each case.
3. Sketch the nullclines and the phase portraits of for different values of a.
4. What is special about the parameter value $a = 1$? (It is called a bifurcation value, why?)

Solution.

1. Equilibrium points:

- For $0 < a < 1$: $(0,0)$, $(1,0)$, $(0,1)$, $(\sqrt{1-a}, 1)$.
- For $a > 1$: $(0,0)$, $(1,0)$, $(0,1)$.

2. Classification of equilibrium points:

- Linearized system about $(0,0)$: $\dot{x} = x$, $\dot{y} = y$. So the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 1$. *Unstable (or negatively attractive) node critical point.*
- Linearized system about $(0,1)$: $\dot{x} = x(1-a)$, $\dot{y} = (1-y)$. To calculate the eigenvalues, we change variables to $v = y - 1$, with $\dot{v} = \dot{y}$, leave x unchanged. The new linear system is $\dot{x} = x(1-a)$, $\dot{v} = -v$. The eigenvalues are $\lambda_1 = 1 - a$, $\lambda_2 = -1$. Hence, the critical point $(0,1)$ is a:
  1. A *stable (or positively attractive) node* for $a > 1$.
  2. A *saddle point* for $a < 1$.
- Linearized system about $(1,0)$: To calculate the eigenvalues, we change variables to $u = x - 1$, with $\dot{u} = \dot{x}$, leave y unchanged. The new linear system is $\dot{u} = -u - \frac{ay}{2}$, $\dot{y} = y$. The eigenvalues are $\lambda_1 = 1$, $\lambda_2 = -1$. Hence, the critical point $(1,0)$ is a *saddle point.*
- Linearized system about $(\sqrt{1-a}, 1)$, in the case $a < 1$. Let us denote $x_0 = \sqrt{1-a}$ and $u = x - x_0$ and $v = y - 1$. The linear system about $(x = x_0, y_0 = 1)$ (equivalently, about $(u = 0, v = 0)$) in terms of the new variables (u, v) is:

  $$\dot{u} = x_0\big(-1 + \frac{a}{(1+x_0)^2}\big)u - \frac{ax_0}{1+x_0}v, \qquad \dot{v} = -v.$$

  The eigenvalues of the system are:

  $$\lambda_1 = x_0\big(-1 + \frac{a}{(1+x_0)^2}\big) \quad \text{and} \quad \lambda_2 = -1.$$

  Since $0 < a < 1$, it immediately follows that $\lambda_1 < 0$. Since both eigenvalues are real and negative, we conclude that the equilibrium point $(\sqrt{1-a}, 1)$ is a *stable (or positively attractive) node.*

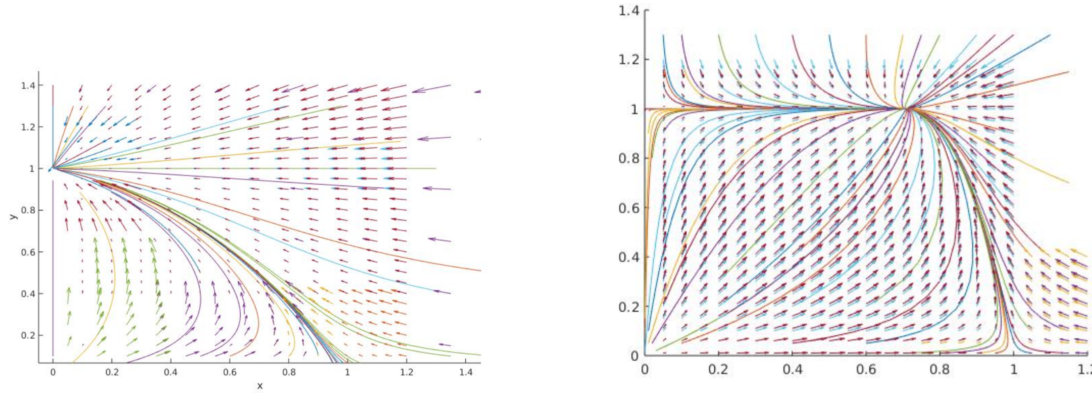

Figure 1: Left: phase plane for $a = 2$. Right: a=0.5. The parameter value $a = 1$ is called a bifurcation value due to the change of structure of the solutions as a increases from $a < 1$ to $a > 1$.

3. The nullclines are obtained by alternatively setting the right-hand sides of the governing equations equal to 0. These gives the following cases:

1. $x = 0, \quad \dot{y} = y(1 - y).$ *The y-axis is a nullcline.*
2. $\dot{x} = x(1 - x), \quad y = 0.$ *The x-axis is a nullcline.*
3. $\dot{x} = x(1 - x) - \frac{ax}{x+1}, \quad y = 1.$ *The line $y = 1$ is a nullcline.*

Problem 2. The spread of infectious diseases such as measles, malaria or corona virus may be modeled as nonlinear system of differential equations, the SIR model. In the simplest form of the model, we postulate three disjoint groups: $S = S(t)$, the population of susceptible individuals, $I = I(t)$, the infected population, and $R = R(t)$ the recovered population.

We assume for simplicity, that the total population is constant:

$$\frac{d}{dt}(S + I + R) = 0.$$

The SIR model, in its simplest form, is stated as

$$\begin{aligned} \dot{S} &= -\beta SI, \\ \dot{I} &= \beta SI - \nu I, \\ \dot{R} &= \nu I, \end{aligned}$$

where $\nu > 0$ and $\beta > 0$ are parameters.

1. Show that the line $I = 0$ is an equilibrium line.
2. Find the matrix that results from linearizing the system about $I = 0$.
3. Calculate the eigenvalues of the resulting matrix.
4. Find the nullclines of the system.
5. What can we infer about the prospects of full recovery of the population?

Solution.

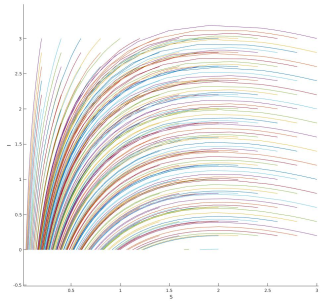

Figure 2: Phase plane for $\beta = 1$.and $\nu = 2$. Note that, in each orbit, the maximum of I is reached at $S = \frac{\nu}{\beta} = 2$.

First, note that the unknown fields $S, I, R \geq 0$, since they represent population figures. Let us consider the constraint $\frac{d}{dt}(S + I + R) = 0$ and its integrated form

$$S + I + R = C,$$

where $C \geq 0$ is a constant. We can solve it for $R = C - S - I$ and use it to calculate R, once we have found S and I. For this, we only need to consider the first two equations:

$$\dot{S} = -\beta S I, \quad \dot{I} = \beta S I - \nu I.$$

(In particular, note that they are independent of R.)

1. Note that the line $I = 0$ (the S-axis) is an equilibrium line since it makes $\dot{S} = 0$ and $\dot{I} = 0$, regardless the value of S.

2. Linearization at $S = 0$ yields the matrix $\begin{bmatrix} 0 & -\beta S \\ 0 & \beta S - \nu \end{bmatrix}$.

3. The eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = \beta S - \nu$. This second eigenvalue is negative if $0 < S < \frac{\nu}{\beta}$ and positive $S > \frac{\nu}{\beta}$.

4. The nullclines of the system are the lines $S = 0$, $I = 0$ and $S = \frac{\nu}{\beta}$.

5. From the graph, we can see that given an initial population (I_0, S_0) with $S_0 > \frac{\nu}{\beta}$ and $I_0 > 0$, the susceptible population decreases monotonically, while the infected population at first rises, but eventually reaches a maximum and then declines to 0.

   [Note: We can analytically reach such a conclusion by examining the equation $\frac{dI}{dS} = \frac{\dot{I}}{\dot{S}} = -1 + \frac{\nu}{\beta S}$.]

Problem 3. Solve exercise 4.1, page 57 (textbook, second edition).

Solution.

1. The critical points of the system are:

$$(0, 0, 0), \quad (\pm 1, 0, 1), \quad (\frac{1}{2}, \pm\frac{1}{\sqrt{2}}, 1).$$

2. Setting $x_3 = 0$ in the governing equations, we get

$$\dot{x}_1 = x_1 - x_1 x_2 - x_2^3, \quad \dot{x}_2 = x_1, \quad \dot{x}_3 = 0.$$

Consider a solution with initial data $(a, b, 0)$. The third equation implies that x_3 is constant. So, $x_3 = 0$ for all time, and the system reduces to the firs two equations above. Hence, a solution with initial data $x_1 = a$ and $x_2 = b$ will remain in the $x_1 - x_2$-plane for all time.

The system obtained from the original one setting $x_3 = 1$ is

$$\dot{x}_1 = x_1^2 + x_2^2 - 1, \quad \dot{x}_2 = x_2(1 - 2x_1), \quad \dot{x}_3 = 0.$$

Arguing as in the previous case $(x_3 = 0)$, we conclude that $x_3 = 1$ is an invariant set.

3. Let us consider the system

$$\dot{x}_1 = x_1^2 + x_2^2 - 1, \quad \dot{x}_2 = x_2(1 - 2x_1).$$

Calculate

$$\nabla \cdot \left(x_1^2 + x_2^2 - 1, x_2(1 - 2x_1)\right) = 2x_1 + 1 - 2x_1 = 1.$$

Since the divergence of the vector field does not change sign anywhere in the plane, by the Bendixon criterion, we conclude that the 2-d system does not have periodic solutions. That is, there are no periodic solutions in the plane $x_3 = 1$.