HW 3

Math 5525
Introduction to Ordinary Differential Equations

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Contents

1 Problem 1

A modification of the predator-prey system is given by

$$\begin{aligned}\dot{x} &= x(1-x) - \frac{axy}{x+1}\\ \dot{y} &= y\left(1-y\right)\end{aligned}$$

Where $a>0$ is a parameter.

1. Find all equilibrium points, in the following two cases: $0<a<1$ and $a>1$. (You may select specific values of a, if you wish.)
2. Classify the equilibrium points in each case.
3. Sketch the nullclines and the phase portraits for different values of a.
4. What is special about the parameter value $a=1$? (It is called a bifurcation value, why?)

solution

1.1 Part 1

case $0<a<1$

Equilibrium points are found by solving

$$x(1-x) - \frac{axy}{x+1} = 0 \tag{1}$$
$$y\left(1-y\right) = 0 \tag{2}$$

EQ (2) gives $y=0$ or $y=1$. When $y=0$ then EQ(1) becomes $x(1-x)=0$ which has solutions $x=0, x=1$. Hence the critical points found so far are $(0,0),(1,0)$.

When $y=1$ then EQ(1) becomes

$$\begin{aligned}x(1-x) - \frac{ax}{x+1} &= 0\\ x(1-x)(1+x) - ax &= 0\\ x - x^3 - ax &= 0\\ x\left(1-x^2-a\right) &= 0\end{aligned}$$

Hence $x=0$ or $1-x^2-a=0$ or $x^2-1+a=0$. Therefore $x^2=1-a$ or $x=\pm\sqrt{1-a}$. Since we are in the case $0<a<1$ then $\sqrt{1-a}$ is positive. Let $\sqrt{1-a}=n^2$. Hence $x=\pm n$. Therefore the critical points found for this case are $(0,1),\left(\sqrt{1-a},1\right),\left(-\sqrt{1-a},1\right)$.

Hence all the critical points for $0<a<1$ are

$$\left(x_i,y_i\right) = \left\{(0,0),(1,0),(0,1),\left(\sqrt{1-a},1\right),\left(-\sqrt{1-a},1\right)\right\}$$

For say $a=\frac{1}{2}$, these critical points become

$$\begin{aligned}\left(x_i,y_i\right) &= \left\{(0,0),(1,0),(0,1),\left(\sqrt{\frac{1}{2}},1\right),\left(-\sqrt{\frac{1}{2}},1\right)\right\}\\ &= \{(0,0),(1,0),(0,1),(0.707107,1),(-0.707107,1)\}\end{aligned}$$

case $a>1$

Equilibrium points are found by solving

$$x(1-x) - \frac{axy}{x+1} = 0 \tag{1}$$
$$y\left(1-y\right) = 0 \tag{2}$$

EQ (2) gives $y = 0$ or $y = 1$. When $y = 0$ then EQ(1) becomes $x(1-x) = 0$ which has solutions $x = 0, x = 1$. Hence the critical points found so far are $(0,0), (1,0)$.

When $y = 1$ then EQ(1) reduces to (as was done above)

$$x\left(1 - x^2 - a\right) = 0$$

Hence $x = 0$ or $x = \pm\sqrt{1-a}$. Since we are in the case $a > 1$ then $\sqrt{1-a}$ is negative. which means $\sqrt{1-a}$ is complex. We are assuming real domain, then these solutions are rejected. This leaves the critical points for $a > 1$ as only the following

$$\left(x_i, y_i\right) = \{(0,0), (1,0), (0,1)\}$$

1.2 Part 2

The Jacobian matrix for the system

$$\begin{aligned}\dot{x} &= x(1-x) - \frac{axy}{x+1}\\ \dot{y} &= y\left(1-y\right)\end{aligned}$$

Is given by the following, where the rule of derivative $\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{gf'-fg'}{g^2}$ is used

$$\begin{aligned}J = \begin{pmatrix}\frac{\partial\dot{x}}{\partial x} & \frac{\partial\dot{x}}{\partial y}\\ \frac{\partial\dot{y}}{\partial x} & \frac{\partial\dot{y}}{\partial y}\end{pmatrix} &= \begin{pmatrix}(1-x) - x - \frac{(x+1)ay-\left(axy\right)}{(1+x)^2} & -\frac{ax}{x+1}\\ 0 & \left(1-y\right) - y\end{pmatrix}\\ &= \begin{pmatrix}1-2x-\frac{axy+ay-axy}{(1+x)^2} & -\frac{ax}{x+1}\\ 0 & 1-2y\end{pmatrix}\\ &= \begin{pmatrix}1-2x-\frac{ay}{(1+x)^2} & -\frac{ax}{x+1}\\ 0 & 1-2y\end{pmatrix}\end{aligned}$$

case $0 < a < 1$

The critical points for this case from part (1) are

$$\left(x_i, y_i\right) = \left\{(0,0), (1,0), (0,1), \left(\sqrt{1-a}, 1\right), \left(-\sqrt{1-a}, 1\right)\right\}$$

At point $(0,0)$ the linearized system A matrix is the Jacobian above evaluated at this point, which gives

$$\begin{aligned}A &= \begin{pmatrix}1-2x-\frac{ay}{(1+x)^2} & -\frac{ax}{x+1}\\ 0 & 1-2y\end{pmatrix}\\ &= \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\end{aligned}$$

Hence $|A - \lambda I| = 0$ becomes

$$\begin{aligned}\begin{vmatrix}1-\lambda & 0\\ 0 & 1-\lambda\end{vmatrix} &= 0\\ (1-\lambda)^2 &= 0\\ \lambda &= 1 \text{ double root}\end{aligned}$$

Since the eigenvalues are positive, this is unstable critical point.

At point $(1,0)$ the linearized system A matrix is the Jacobian above evaluated at this point, which gives

$$
\begin{aligned}
A &= \begin{pmatrix} 1-2x-\frac{ay}{(1+x)^2} & -\frac{ax}{x+1} \\ 0 & 1-2y \end{pmatrix} \\
&= \begin{pmatrix} 1-2 & -\frac{a}{2} \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} -1 & -\frac{a}{2} \\ 0 & 1 \end{pmatrix}
\end{aligned}
$$

Hence $|A-\lambda I|=0$ becomes

$$
\begin{aligned}
\begin{vmatrix} -1-\lambda & -\frac{a}{2} \\ 0 & 1-\lambda \end{vmatrix} &= 0 \\
(-1-\lambda)(1-\lambda) &= 0 \\
\lambda^2-1 &= 0 \\
\lambda^2 &= 1 \\
\lambda &= \pm 1
\end{aligned}
$$

This means this critical points is a saddle point (unstable) since one eigenvalue is negative and one is negative.

At point $(0,1)$ the linearized system A matrix is the Jacobian above evaluated at this point, which gives

$$
\begin{aligned}
A &= \begin{pmatrix} 1-2x-\frac{ay}{(1+x)^2} & -\frac{ax}{x+1} \\ 0 & 1-2y \end{pmatrix} \\
&= \begin{pmatrix} 1-a & 0 \\ 0 & -1 \end{pmatrix}
\end{aligned}
$$

Hence $|A-\lambda I|=0$ becomes

$$
\begin{aligned}
\begin{vmatrix} (1-a)-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} &= 0 \\
((1-a)-\lambda)(-1-\lambda) &= 0 \\
\lambda^2+a\lambda+a-1 &= 0
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lambda &= -\frac{b}{2a} \pm \frac{1}{2a}\sqrt{b^2-4ac} \\
&= -\frac{a}{2} \pm \frac{1}{2}\sqrt{a^2-4(a-1)} \\
&= -\frac{a}{2} \pm \frac{1}{2}\sqrt{a^2-4a+4}
\end{aligned}
$$

Since $0<a<1$ then $-3<a^2-4a<0$ which means the term under the root will remain positive for all a values between 0 and 1. This means this critical points is a saddle point (unstable) since one eigenvalue will be negative and one is positive.

At point $\left(\sqrt{1-a},1\right)$ the linearized system A matrix is the Jacobian above evaluated at this point, which gives

$$
\begin{aligned}
A &= \begin{pmatrix} 1-2x-\frac{ay}{(1+x)^2} & -\frac{ax}{x+1} \\ 0 & 1-2y \end{pmatrix} \\
&= \begin{pmatrix} 1-2\sqrt{1-a}-\frac{a}{\left(1+\sqrt{1-a}\right)^2} & -\frac{a\sqrt{1-a}}{\sqrt{1-a}+1} \\ 0 & -1 \end{pmatrix}
\end{aligned}
$$

Hence $|A-\lambda I|=0$ becomes

$$\begin{vmatrix} \left(1-2\sqrt{1-a}-\frac{a}{\left(1+\sqrt{1-a}\right)^2}\right)-\lambda & -\frac{a\sqrt{1-a}}{\sqrt{1-a}+1} \\ 0 & -1-\lambda \end{vmatrix}=0$$

$$\left(\left(1-2\sqrt{1-a}-\frac{a}{\left(1+\sqrt{1-a}\right)^2}\right)-\lambda\right)(-1-\lambda)=0$$

To simplify this, let us pick $a=\frac{1}{2}$. Hence the point $\left(\sqrt{1-a},1\right)$ becomes $(0.707,1)$. The above becomes

$$\left(\left(1-2\sqrt{\frac{1}{2}}-\frac{\frac{1}{2}}{\left(1+\sqrt{\frac{1}{2}}\right)^2}\right)-\lambda\right)(-1-\lambda)=0$$

$$(\lambda+1)\left(\lambda-\sqrt{2}+2\right)=0$$

Hence $\lambda=-1$ and $\lambda=\sqrt{2}-2$. So one eigenvalue is negative and also the second is negative. This means this is stable point (positive attraction). Even though we used specific a value here, this result is value for all $0<a<1$.

At point $\left(-\sqrt{1-a},1\right)$ the linearized system A matrix is the Jacobian above evaluated at this point, which gives

$$\begin{aligned} A&=\begin{pmatrix} 1-2x-\frac{ay}{(1+x)^2} & -\frac{ax}{x+1} \\ 0 & 1-2y \end{pmatrix} \\ &=\begin{pmatrix} 1+2\sqrt{1-a}-\frac{a}{\left(1-\sqrt{1-a}\right)^2} & \frac{a\sqrt{1-a}}{-\sqrt{1-a}+1} \\ 0 & -1 \end{pmatrix} \end{aligned}$$

Hence $|A-\lambda I|=0$ becomes

$$\begin{vmatrix} \left(1+2\sqrt{1-a}-\frac{a}{\left(1-\sqrt{1-a}\right)^2}\right)-\lambda & -\frac{a\sqrt{1-a}}{\sqrt{1-a}+1} \\ 0 & -1-\lambda \end{vmatrix}=0$$

$$\left(\left(1+2\sqrt{1-a}-\frac{a}{\left(1-\sqrt{1-a}\right)^2}\right)-\lambda\right)(-1-\lambda)=0$$

To simplify this, let us pick $a=\frac{1}{2}$. Hence the point $\left(-\sqrt{1-a},1\right)$ becomes $(-0.707,1)$. The above becomes

$$\left(\left(1+2\sqrt{\frac{1}{2}}-\frac{\frac{1}{2}}{\left(1-\sqrt{\frac{1}{2}}\right)^2}\right)-\lambda\right)(-1-\lambda)=0$$

$$(\lambda+1)\left(\lambda+\sqrt{2}+2\right)=0$$

Hence $\lambda=-1$ and $\lambda=-\sqrt{2}-2$. So one eigenvalue is negative and also the second is negative. This means this is stable point (positive attraction). Even though we used specific a value here, this result is value for all $0<a<1$.

The following table is a summary of the above results, all for $0<a<1$. To obtain numerical values below for eigenvalues, $a=\frac{1}{2}$ was used as an example.

critical point	eigenvalues	type of equilibrium
$(0,0)$	$\lambda_1=1,\lambda_2=1$	negative attraction, unstable
$(1,0)$	$\lambda=\pm1$	saddle point, unstable.
$(0,1)$	$\lambda=-\frac{a}{2}\pm\frac{1}{2}\sqrt{a^2-4a+4}$	saddle point, unstable.
$\left(\sqrt{1-a},1\right)$	$\lambda_1=-1,\lambda_2=\sqrt{2}-2$	positive attraction, stable
$\left(-\sqrt{1-a},1\right)$	$\lambda_1=-1,\lambda_2=-\sqrt{2}-2$	positive attraction, stable

For specific value $a = \frac{1}{2}$ the above table becomes

critical point	eigenvalues	type of equilibrium
$(0,0)$	$\lambda_1 = 1, \lambda_2 = 1$	negative attraction, unstable
$(1,0)$	$\lambda = \pm 1$	saddle point, unstable.
$(0,1)$	$\lambda_1 = -1, \lambda_2 = \frac{1}{2}$	saddle point, unstable.
$(0.707,1)$	$\lambda_1 = -1, \lambda_2 = -0.5857$	positive attraction, stable
$(-0.707,1)$	$\lambda_1 = -1, \lambda_2 = -3.41421$	positive attraction, stable

case $a > 1$

The critical points for this case from part(1) are

$$\left(x_i, y_i\right) = \{(0,0), (1,0), (0,1)\}$$

At point $(0,0)$ the linearized system A matrix is the Jacobian above evaluated at this point, which gives

$$\begin{aligned} A &= \begin{pmatrix} 1 - 2x - \frac{ay}{(1+x)^2} & -\frac{ax}{x+1} \\ 0 & 1 - 2y \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

This is the same as with part 1. Which gives $\lambda = 1$ double root. Since the eigenvalues are positive, this is unstable critical point.

At point $(1,0)$ the linearized system A matrix is the Jacobian above evaluated at this point, which gives

$$\begin{aligned} A &= \begin{pmatrix} 1 - 2x - \frac{ay}{(1+x)^2} & -\frac{ax}{x+1} \\ 0 & 1 - 2y \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

This is the same as with part 1. Which gives $\lambda = 1$ double root. Since the eigenvalues are positive, this is unstable critical point.

At point $(1,0)$ the linearized system A matrix is the Jacobian above evaluated at this point, which gives

$$\begin{aligned} A &= \begin{pmatrix} 1 - 2x - \frac{ay}{(1+x)^2} & -\frac{ax}{x+1} \\ 0 & 1 - 2y \end{pmatrix} \\ &= \begin{pmatrix} -1 & -\frac{a}{2} \\ 0 & 1 \end{pmatrix} \end{aligned}$$

This is the same as $0 < a < 1$, since a is not involved and cancels out. Hence $\lambda = \pm 1$. This means this critical points is a saddle point (unstable) since one eigenvalue is negative and one is negative.

At point $(0,1)$ the linearized system A matrix is the Jacobian above evaluated at this point, which gives

$$\begin{aligned} A &= \begin{pmatrix} 1 - 2x - \frac{ay}{(1+x)^2} & -\frac{ax}{x+1} \\ 0 & 1 - 2y \end{pmatrix} \\ &= \begin{pmatrix} 1 - a & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}$$

From case $0 < a < 1$ we found

$$\lambda = -\frac{a}{2} \pm \frac{1}{2}\sqrt{a^2 - 4a + 4}$$

Let us pick $a = 3$. Hence

$$\begin{aligned}\lambda &= -\frac{3}{2} \pm \frac{1}{2}\sqrt{9 - 12 + 4} \\ &= -\frac{3}{2} \pm \frac{1}{2} \\ &= -2, -1\end{aligned}$$

Therefore this is stable. This is different from case $0 < a < 1$ where this point was unstable.

The following table is a summary of the above results, all for $a > 1$. To obtain numerical values below for eigenvalues, $a = 3$ was used as an example.

critical point	eigenvalues	type of equilibrium
$(0, 0)$	$\lambda_1 = 1, \lambda_2 = 1$ (same as $0 < a < 1$)	negative attraction, unstable
$(1, 0)$	$\lambda = \pm 1$ (same as $0 < a < 1$)	saddle point, unstable.
$(0, 1)$	$\lambda = -\frac{a}{2} \pm \frac{1}{2}\sqrt{a^2 - 4a + 4}$ or $\lambda_1 = -1, \lambda_2 = -2$	positive attraction, stable

From the above, we notice that when a changes from $0 < a < 1$ to $a > 1$ then one critical point $(0, 1)$ switches from being unstable to stable. This implies solution encountered bifurcation value.

1.3 Part 3

$$\begin{aligned}\dot{x} &= x(1 - x) - \frac{axy}{x + 1} \\ \dot{y} &= y\left(1 - y\right)\end{aligned}$$

The x nullclines are the solution of $x(1 - x) - \frac{axy}{x+1} = 0$ and the y nullclines are solutions of $y\left(1 - y\right) = 0$. Therefore y nullclines are $y = 0$ (which is the x axis) and $y = 1$. These are both straight lines. To find the x nullclines

$$\begin{aligned}x(1 - x) - \frac{axy}{x + 1} &= 0 \\ x(1 - x)(x + 1) - axy &= 0 \\ x - x^3 - axy &= 0 \\ x\left(1 - x^2 - ay\right) &= 0\end{aligned}$$

Hence $x = 0$ (which is the y axis) and $x^2 = 1 - ay$ are the x nullclines.

x nullclines	y nullclines
$x = 0$	$y = 0$
$x^2 = 1 - ay$	$y = 1$

The following is a plot of the nullclines for the case of $0 < a < 1$, using $a = \frac{1}{2}$

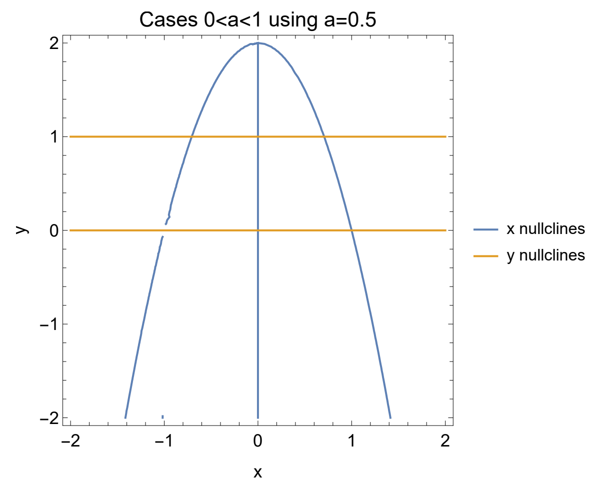

Figure 1: nullclines for case $0 < a < 1$

```
ClearAll[x, y, a]
a = 1/2;
f1 = x (1 - x) - a x y/(x + 1);
f2 = y (1 - y);
p = ContourPlot[{f1 == 0, f2 == 0}, {x, -2, 2}, {y, -2, 2},
        PlotLegends → {"x nullclines", "y nullclines"},
        PlotLabel → "Cases 0<a<1 using a=0.5",
        BaseStyle → 14, FrameLabel → {{"y", None}, {"x", None}}];
```

Figure 2: code used for the above plot

The following is a plot of the nullclines for the case of $a > 1$, using $a = 3$

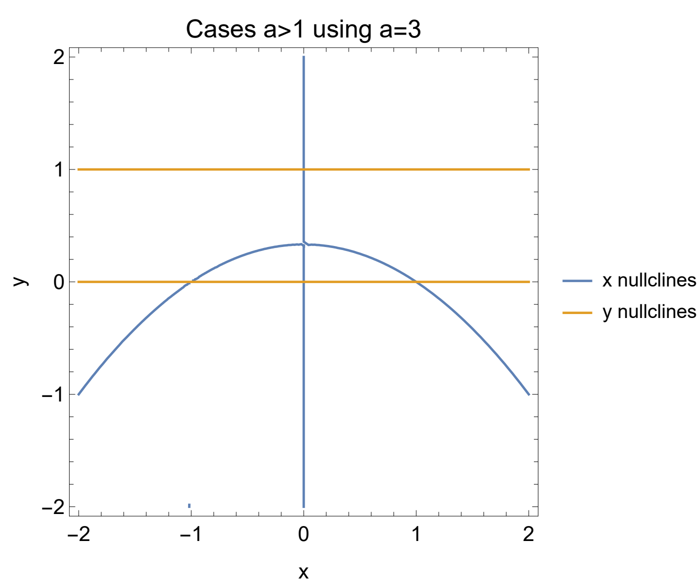

Figure 3: nullclines for case $a > 1$

We notice that the case of $a = 1$ is the following

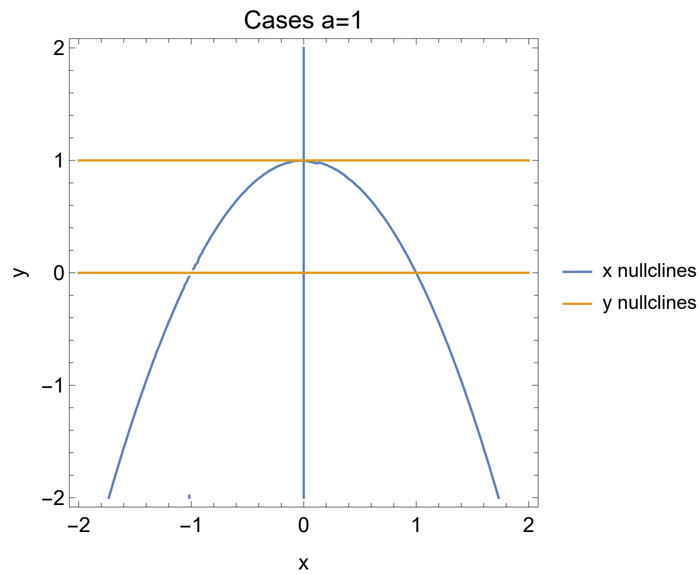

Figure 4: nullclines for case $a > 1$

The following is a phase plot for the case of $0 < a < 1$, using $a = \frac{1}{2}$ with the critical points highlighted. Red dot indicates unstable point and green dot color indicates stable point.

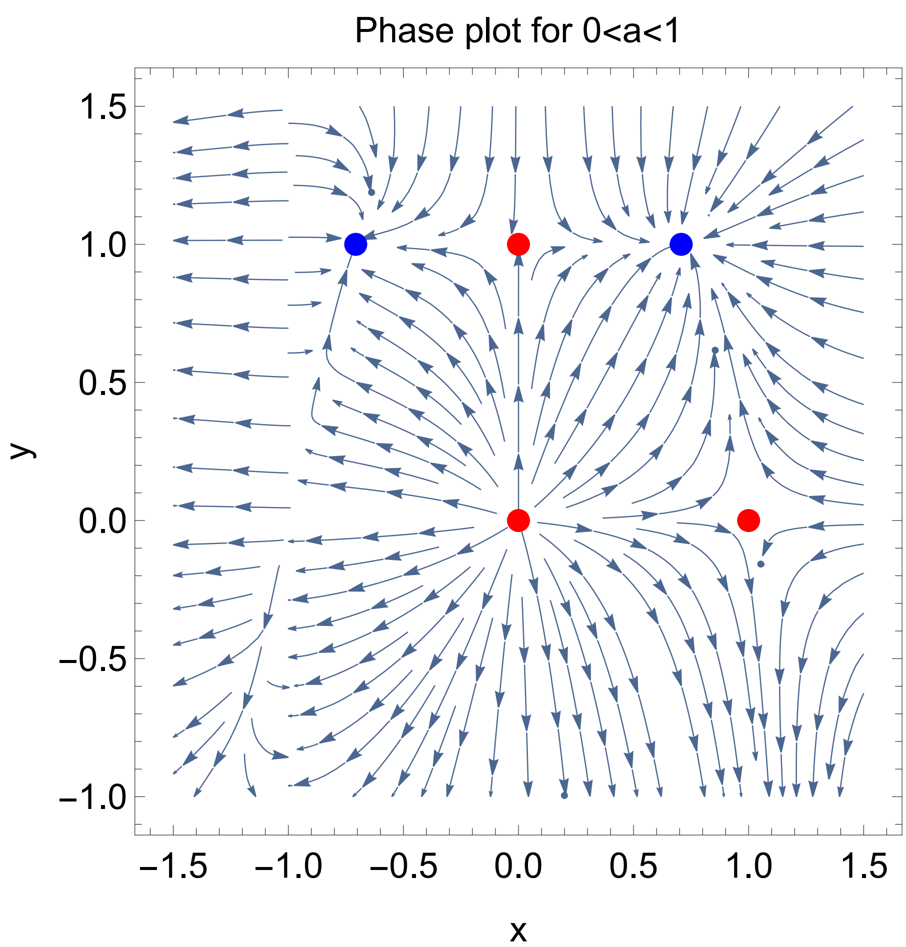

Figure 5: Phase plot $0 < a < 1$ using $a = 0.5$

```
ClearAll[x, y, a]
a = 1/2;
f1 = x (1 - x) - a x y/ (x + 1);
f2 = y (1 - y);
p1 = {Red, PointSize[0.03], Point[{0, 0}]};
p2 = {Red, PointSize[0.03], Point[{1, 0}]};
p3 = {Red, PointSize[0.03], Point[{0, 1}]};
p4 = {Blue, PointSize[0.03], Point[{Sqrt[1 - a], 1}]};
p5 = {Blue, PointSize[0.03], Point[{-Sqrt[1 - a], 1}]};
ps = StreamPlot[{f1, f2}, {x, -1.5, 1.5}, {y, -1, 1.5}, Epilog → {p1, p2, p3, p4, p5},
        FrameLabel → {{"y", None}, {"x", "Phase plot for 0<a<1"}}, BaseStyle → 14];
```

Figure 6: Code for the above plot

The following is a phase plot for the case of $a > 1$, using $a = 3$.

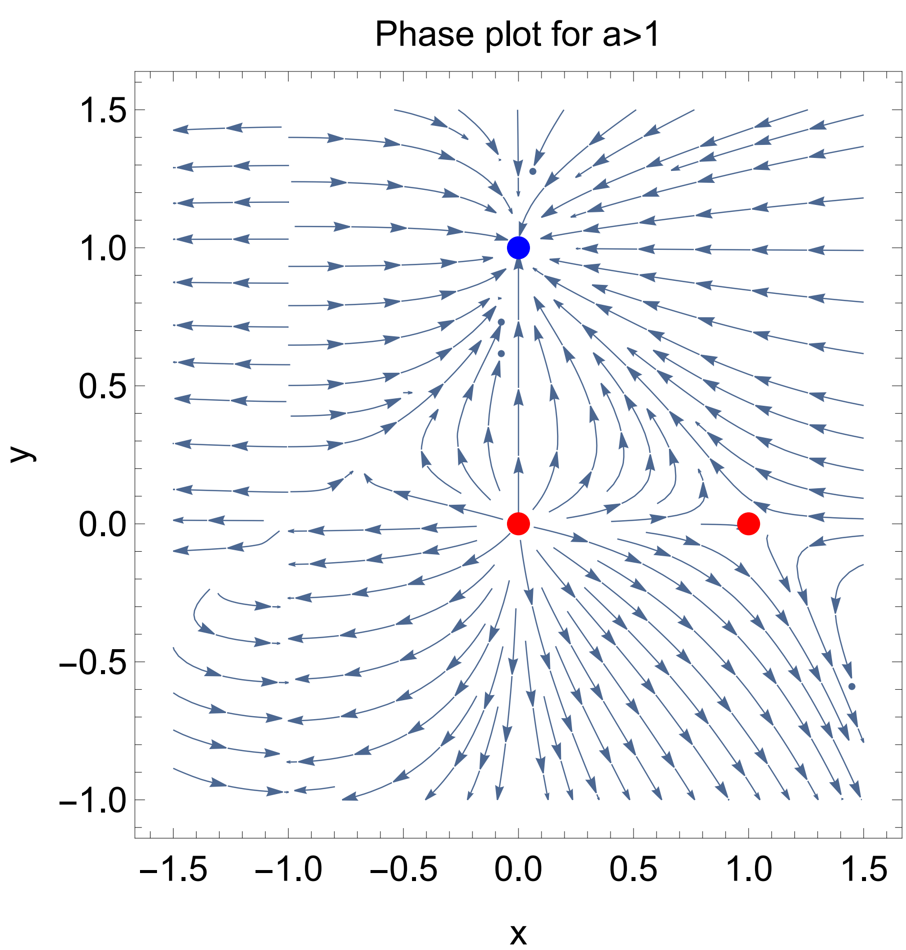

Figure 7: Phase plot $a > 1$ using $a = 3$

1.4 Part 4

When $a = 1$ the solution of the system changes abruptly. To see this, the following is the two phase plots about side by side, one for $a = 0.99$ and one for $a = 1.01$.

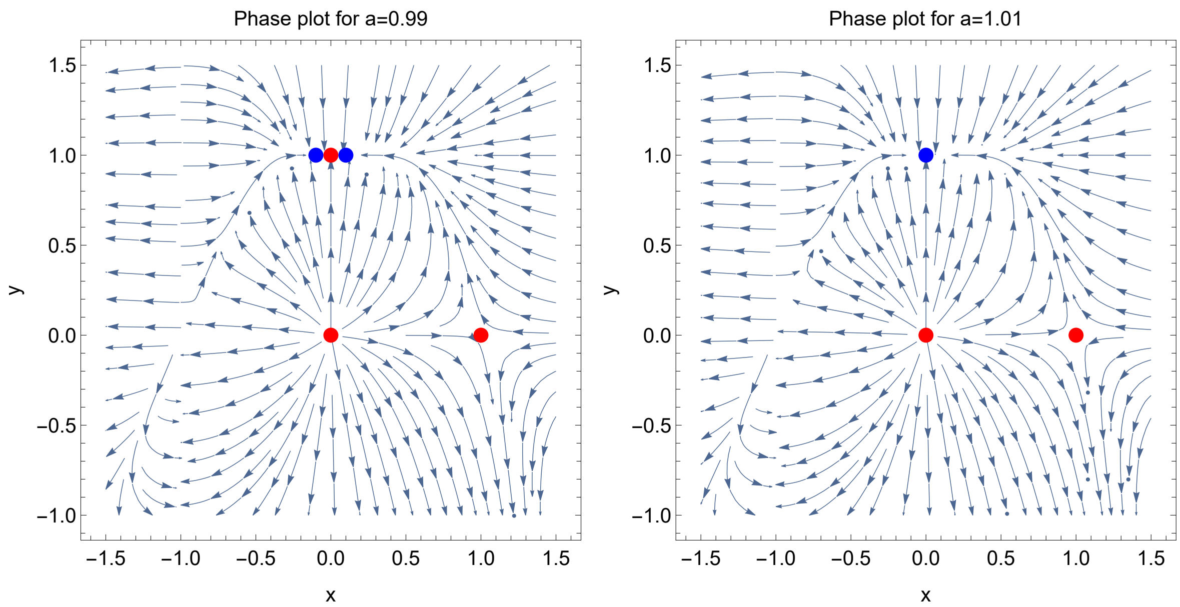

Figure 8: Phase plots changes as a cross over 1

We notice the following. As a changes from $0 < a < 1$ to $a > 1$, the equilibrium point $(0, 1)$ changes from being unstable to stable (this is in addition to now having 3 equilibrium points instead of 5). This is exactly what bifurcation is. So $a = 1$ is a bifurcation value. It is a parameter in the system, which cause sudden change in the solution trajectories when its value crosses over some specific value, which is 1 in this problem.

2 Problem 2

The spread of infectious diseases such as measles, malaria or corona virus may be modeled as nonlinear system of differential equations, the SIR model. In the simplest form of the model, we postulate three disjoint groups: $S = S(t)$, the population of susceptible individuals, $I = I(t)$, the infected population, and $R = R(t)$ the recovered population. We assume for simplicity, that the total population is constant:

$$\frac{d}{dt}(S + I + R) = 0$$

The SIR model, in its simplest form, is stated as

$$\dot{S} = -\beta SI \tag{1}$$
$$\dot{I} = \beta SI - \upsilon I \tag{2}$$
$$\dot{R} = \upsilon I \tag{3}$$

Where $\upsilon > 0$ and $\beta > 0$ are parameters.

1. Show that the line $I = 0$ is an equilibrium line.
2. Find the matrix that results from linearizing the system about $I = 0$.
3. Calculate the eigenvalues of the resulting matrix.
4. Find the nullclines of the system.
5. What can we infer about the prospects of full recovery of the population?

<u>solution</u>

This is a diagram of the SIR model

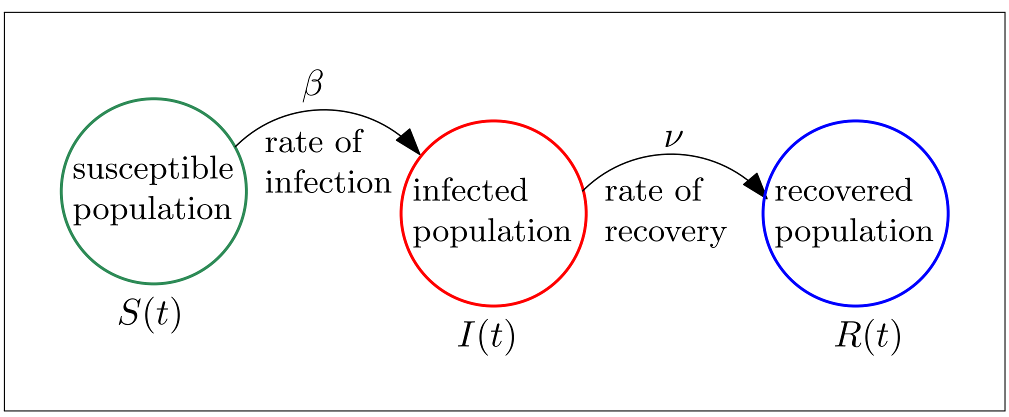

Figure 9: SIR model of infection

Hence total population is

$$N = S + I + R$$

Which is a constant (This assumes no death occurs, only infection and recovery). In this model, it is assumed that recovered population $R(t)$ can not become infected again.

2.1 Part 1

Critical points are solutions to

$$0 = -\beta SI \tag{1A}$$
$$0 = \beta SI - \upsilon I \tag{2A}$$
$$0 = \upsilon I \tag{3A}$$

Eq. (3) shows that, since υ can not be zero, then $I = 0$ is equilibrium. This says that if there are no infected individuals, then the population N do not change which is to be expected since $N = S + I + R$ and hence if $I = 0$ then this implies also that $R = 0$ and hence N remains constant.

2.2 Part 2

The Jacobian Matrix is

$$J = \begin{pmatrix} \frac{\partial \dot{S}}{\partial S} & \frac{\partial \dot{S}}{\partial I} & \frac{\partial \dot{S}}{\partial R} \\ \frac{\partial \dot{I}}{\partial S} & \frac{\partial \dot{I}}{\partial I} & \frac{\partial \dot{I}}{\partial R} \\ \frac{\partial \dot{R}}{\partial S} & \frac{\partial \dot{R}}{\partial I} & \frac{\partial \dot{R}}{\partial R} \end{pmatrix} = \begin{pmatrix} -\beta I & -\beta & 0 \\ \beta I & \beta S - v & 0 \\ 0 & v & 0 \end{pmatrix}$$

Evaluating the above at $I = 0$ gives

$$A = \begin{pmatrix} 0 & -\beta & 0 \\ 0 & \beta S - v & 0 \\ 0 & v & 0 \end{pmatrix}$$

2.3 Part 3

To find the eigenvalues

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} -\lambda & -\beta & 0 \\ 0 & \left(\beta S - v\right) - \lambda & 0 \\ 0 & v & -\lambda \end{vmatrix} = 0$$

$$-\lambda \begin{vmatrix} \left(\beta S - v\right) - \lambda & 0 \\ v & -\lambda \end{vmatrix} + \beta \begin{vmatrix} 0 & 0 \\ 0 & -\lambda \end{vmatrix} + 0 = 0$$

$$\lambda^2 \left(\beta S - v - \lambda\right) = 0$$

Hence $\lambda = 0$ (double root) and $\lambda = \beta S - v$ are the eigenvalues.

2.4 Part 4

The S nullclines are the solution of $-\beta SI = 0$ and the I nullclines are solutions of $\beta SI - vI = 0$ and R nullclines are solutions of vI

<u>S nullclines</u> are therefore $I = 0$ and $S = 0$.

<u>I nullclines</u> are therefore $I = 0$ and $S = \frac{v}{\beta}$.

<u>R nullclines</u> are $I = 0$.

2.5 Part 5

To answer this, we need to assume that $I(0)$ is not zero. This means initial condition such that some infection exist, otherwise $S(t)$ will not change.

Since $\dot{I} = \beta SI - vI$ then as I increases (infected population increases) and because $v > 0$ then the term vI becomes more negative. Since $S(t)$ also at the same time becomes smaller during this process (because more people are infected), then we see that $\dot{I}$ will eventually starts to decrease as I increases and this happens when vI becomes larger than βSI. (This is the peak infection).

This means infected population size eventually decreases as $I(t)$ passes some peak value. Since population is assumed constant, this implies the recovered population size will also increase and eventually all susceptible population that became infected will recover and infected population will go to zero with time.

3 Problem 3 (exercise 4.1, page 57)

In exercise 2.3 of chapter 2 we analyzed the existence of periodic solutions in an invariant set of a three-dimensional system. Obtain this result in a more straightforward manner. The following is 2.3 of chapter 2

We are studying the three-dimensional system

$$\begin{aligned}
\dot{x}_1 &= x_1 - x_1x_2 - x_2^3 + x_3\left(x_1^2 + x_2^2 - 1 - x_1 + x_1x_2 + x_2^3\right)\\
\dot{x}_2 &= x_1 - x_3\left(x_1 - x_2 + 2x_1x_2\right)\\
\dot{x}_3 &= \left(x_3 - 1\right)\left(x_3 + 2x_3x_2^2 + x_3^3\right)
\end{aligned}$$

Consider the invariant set $x_3 = 1$. Does this set contain periodic solutions?

solution

Writing the above as

$$\begin{aligned}
\dot{\boldsymbol{x}} &= \boldsymbol{F}(x_1, x_2, x_2)\\
\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{pmatrix} = \begin{pmatrix}f_1(x_1,x_2,x_3)\\ f_2(x_1,x_2,x_3)\\ f_3(x_1,x_2,x_3)\end{pmatrix} &= \begin{pmatrix}x_1 - x_1x_2 - x_2^3 + x_3\left(x_1^2 + x_2^2 - 1 - x_1 + x_1x_2 + x_2^3\right)\\ x_1 - x_3\left(x_1 - x_2 + 2x_1x_2\right)\\ \left(x_3 - 1\right)\left(x_3 + 2x_3x_2^2 + x_3^3\right)\end{pmatrix}
\end{aligned}$$

We now need to set $x_3 = 1$ before taking the divergence. The above simplifies to

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{pmatrix} = \begin{pmatrix}f_1(x_1,x_2,x_3)\\ f_2(x_1,x_2,x_3)\\ f_3(x_1,x_2,x_3)\end{pmatrix} = \begin{pmatrix}x_1 - x_1x_2 - x_2^3 + \left(x_1^2 + x_2^2 - 1 - x_1 + x_1x_2 + 1\right)\\ x_1 - \left(x_1 - x_2 + 2x_1x_2\right)\\ 0\end{pmatrix}$$

Then using Bendixson's criterion, periodic solution exist only if divergence of $\boldsymbol{F}(x_1, x_2, x_2)$ changes sign in the domain D or if the divergence is identically zero. D is taken as the whole of $\mathbb{R}^3$.

$$\nabla \cdot \boldsymbol{F}(x_1, x_2, x_2) = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3} \tag{1}$$

But

$$\begin{aligned}
\frac{\partial f_1}{\partial x_1} &= 1 - x_2 + \left(2x_1 - 1 + x_2\right) = 2x_1\\
\frac{\partial f_2}{\partial x_2} &= -\left(-1 + 2x_1\right) = 1 - 2x_1\\
\frac{\partial f_3}{\partial x_3} &= 0
\end{aligned}$$

Hence (1) now becomes

$$\begin{aligned}
\nabla \cdot \boldsymbol{F}(x_1, x_2, x_2) &= 2x_1 + 1 - 2x_1\\
&= 1
\end{aligned}$$

Therefore $\nabla \cdot \boldsymbol{F}(x_1, x_2, x_2)$ does not change sign. Therefore no periodic solution exist.