MATH 5525: HOMEWORK ASSIGNMENT 2

Date (March 4, 2020)

3.1, page 36, textbook. Consider the system of ODEs

$$\dot{x} = y(1 + x - y^2) := f(x, y), \quad \dot{y} = x(1 + y - x^2) := g(x, y). \tag{1}$$

Find the critical points and characterize their stability properties.

1. To find the critical points, we solve the algebraic equations

$$y(1 + x - y^2) = 0, \quad \text{and} \quad x(1 + y - x^2) = 0.$$

The solutions satisfy the relations:

$$y = 0 \quad \text{or} \quad 1 + x - y^2 = 0, \quad \text{and} \quad x = 0 \quad \text{or} \quad 1 + y - x^2 = 0 \tag{2}$$

This results in the following pairs:

$$(0, 0),\ (\pm 1, 0),\ (0, \pm 1),\ (\frac{1}{2} \pm \frac{\sqrt{5}}{2}, \frac{1}{2} \pm \frac{\sqrt{5}}{2}).$$

The two last solution pairs, result from searching solutions such that $x = y$, in which case $1 + x - x^2 = 0$ must hold.

2. The Jacobian matrix (that is, the matrix of the partial derivatives of $f(x, y)$ and $g(x, y)$) is given by

$$\begin{bmatrix} y & 1 + x - 3y^2 \\ 1 + y - 3x^2 & x \end{bmatrix}.$$

The corresponding matrix near the equilibrium points is:

$(0, 0) : \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Eigenvalues ± 1. Saddle point.

$(0, 1) : \quad \begin{bmatrix} 1 & -2 \\ 2 & 0 \end{bmatrix}$. Eigenvalues $\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$. Focus, negative attractor.

$(0, -1) : \quad \begin{bmatrix} -1 & -2 \\ 0 & 0 \end{bmatrix}$. Eigenvalues $0, -1$. Degenerate.

$(1, 0) : \quad \begin{bmatrix} 0 & 2 \\ -3 & 1 \end{bmatrix}$. Eigenvalues $\frac{1}{2}(1 \pm \sqrt{23} i)$. Focus, negative atttractor.

$(-1, 0) : \quad \begin{bmatrix} 0 & 0 \\ 3 & -1 \end{bmatrix}$. Eigenvalues $0, -1$. Degenerate.

$(\frac{1}{2} + \frac{\sqrt{5}}{2}, \frac{1}{2} + \frac{\sqrt{5}}{2}) :$ saddle point

$(\frac{1}{2} - \frac{\sqrt{5}}{2}, \frac{1}{2} - \frac{\sqrt{5}}{2}) :$ saddle point.

3.3, page 36, textbook. Consider the system

$$\dot{x} = 16x^2 + 9y^2 - 25 := f(x,y), \quad \dot{y} = 16x^2 - 16y^2 := g(x,y).$$

1. To find the critical points, we solve the algebraic equations

$$16x^2 + 9y^2 - 25 = 0 \quad \text{and} \quad 16x^2 - 16y^2 = 0.$$

Solutions, equilibrium points, are

$$(1,1),\ (1,-1),\ (-1,1),\ (-1,-1).$$

2. The Jacobian matrix is given by

$$\begin{bmatrix} 32x & 18y \\ 32x & -32y \end{bmatrix}.$$

The corresponding matrix near the equilibrium point is:

$(1,1)$: $\begin{bmatrix} 32 & 18 \\ 32 & -32 \end{bmatrix}$. Eigenvalues ± 40. Saddle point.

$(1,-1)$: $\begin{bmatrix} 32 & -18 \\ 32 & 32 \end{bmatrix}$. Eigenvalues $32 \pm 24i$. Unstable spiral (or negative focus).

$(-1,1)$: $\begin{bmatrix} -32 & 18 \\ -32 & 32 \end{bmatrix}$. Eigenvalues $-32 \pm 24i$. Stable spiral (or positive focus).

$(-1,-1)$: $\begin{bmatrix} -32 & 18 \\ -32 & 32 \end{bmatrix}$. Eigenvalues ± 40. Saddle point.

3. Sketch the phase-plane of the system.

3.5, page 36, textbook. Consider the second order ODE:

$$x'' + cx' - x(1-x) = 0.$$

The limiting conditions imply that $x = 0, x' = 0$ and $x = 1, x' = 0$ are critical points of the equation. Moreover, $x = 0$ is unstable and $x = 1$ is stable (because the limits correspond to $t \to -\infty$ and $t \to \infty$, respectively.)

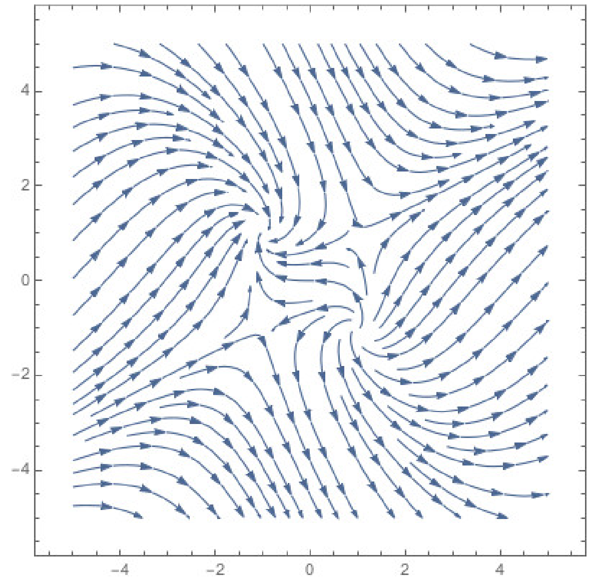

Figure 1: Phase-plane exercise 3.3.

The matrix of the linear system about $(0,0)$ is $\begin{bmatrix} 0 & 1 \\ 1 & -c \end{bmatrix}$. Its eigenvalues are

$$2\lambda = -c \pm \sqrt{c^2+4}$$

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The matrix of the linear system about $(1,0)$ is $\begin{bmatrix} 0 & 1 \\ 1 & -c \end{bmatrix}$. Its eigenvalues are

$$2\lambda = -c \pm \sqrt{c^2-4}.$$

Note that the stability of $x=1$ requires $c>0$. Moreover, if $c<2$ the eigenvalues are complex, in which case the equilibrium point is a stable focus. It is easy to see that, in such as case, $x'<0$ for some values of t. Hence, we require $c \geqslant 2$, so that both eigenvalues are real and negative and $x=1$ is a stable node.

With $c \geqslant 2$, then $x=0$ is a saddle point.

3.5, page 36, textbook. Consider the system

$$\dot{x} = x(1-x^2-6y^2), \quad \dot{y} = y(1-3x^2-3y^2).$$

1. To find the critical points, we solve the algebraic equations

$$x(1-x^2-6y^2) = 0 \quad \text{and} \quad y(1-3x^2-3y^2) = 0.$$

The critical points are:

- $(0,0)$. Unstable node.
- $(0, \pm\frac{1}{\sqrt{3}})$. 2 stable nodes.
- $(\pm 1, 0)$. 2 stable nodes.
- $(\pm\frac{1}{\sqrt{5}}, \pm\frac{\sqrt{2}}{\sqrt{15}})$. These are 4 saddle points