HW 2

Math 5525
Introduction to Ordinary Differential Equations

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Contents

1 Problem 3.1

Consider the two-dimensional system in R^2

$$\dot{x} = y\left(1 + x - y^2\right)$$
$$\dot{y} = x\left(1 + y - x^2\right)$$

Determine the critical points and characterize the linearized flow in a neighborhood of these points.

<u>solution</u>

Let

$$\dot{x} = y\left(1 + x - y^2\right) = f_1\left(x, y\right) \tag{1}$$
$$\dot{y} = x\left(1 + y - x^2\right) = f_2\left(x, y\right) \tag{2}$$

The critical points are found by solving $f_1 = 0, f_2 = 0$. Equation $f_1 = 0$ gives the following solutions

$$y = 0 \tag{3}$$
$$1 + x - y^2 = 0 \tag{4}$$

Starting with (3). Substituting in (2) the solution $y = 0$ gives

$$x\left(1 - x^2\right) = 0$$

This gives solutions $x = 0$ or $x = \pm 1$. The first set of critical points generated from (3) is $(0,0), (1,0), (-1,0)$. Now we do the same starting from (4). Solving (4) for x gives

$$x = y^2 - 1 \tag{5}$$

Substituting for x from above back into (2) gives

$$\left(y^2 - 1\right)\left(1 + y - \left(y^2 - 1^2\right)\right) = 0$$

This gives solutions $y^2 - 1 = 0$ or $\left(1 + y - \left(y^2 - 1^2\right)\right) = 0$. Starting $y^2 - 1 = 0$. This gives $y = \pm 1$. From (5) this gives $x = 0$ for both cases. So now we can add the next set of critical points found so far $(0,1), (0,-1)$.

When $\left(1 + y - \left(y^2 - 1^2\right)\right) = 0$, or $1 + y - y^4 - 1 + 2y^2 = 0$ or $y^4 - 2y^2 - y = 0$ or $y\left(y^3 - 2y - 1\right) = 0$. Hence $y = 0$ which from (5) gives another critical point $x = -1$. Hence $(-1,0)$. This critical point is one already found earlier. For $y^3 - 2y - 1 = 0$, this gives solutions $y = -1, y = \frac{1}{2}\left(1 - \sqrt{5}\right), y = \frac{1}{2}\left(1 + \sqrt{5}\right)$. From each one of these solutions, using EQ. (5) gives x. When $y = -1$, then (5) gives $x = 0$ and when $y = \frac{1}{2}\left(1 - \sqrt{5}\right)$ then (5) gives

$$x = \left(\frac{1}{2}\left(1 - \sqrt{5}\right)\right)^2 - 1 = \frac{1}{2} - \frac{1}{2}\sqrt{5}$$

And when $y=\frac{1}{2}\left(1+\sqrt{5}\right)$ then (5) gives

$$x=\left(\frac{1}{2}\left(1+\sqrt{5}\right)\right)^2-1=\frac{1}{2}\sqrt{5}+\frac{1}{2}$$

Therefore we have found the following 3 extra critical points

$$(0,-1)\,,\left(\frac{1}{2}-\frac{1}{2}\sqrt{5},\frac{1}{2}\left(1-\sqrt{5}\right)\right),\left(\frac{1}{2}\sqrt{5}+\frac{1}{2},\frac{1}{2}\left(1+\sqrt{5}\right)\right)$$

In summary, the following are all the critical points found. There are 7 of them

$$\begin{aligned}\left(x,y\right)^* &= (0,0)\\ &= (1,0)\\ &= (-1,0)\\ &= (0,1)\\ &= (0,-1)\\ &= \left(\frac{1}{2}\left(1-\sqrt{5}\right),\frac{1}{2}\left(1-\sqrt{5}\right)\right)\\ &= \left(\frac{1}{2}\left(1+\sqrt{5}\right),\frac{1}{2}\left(1+\sqrt{5}\right)\right)\end{aligned}$$

To characterize the linearized flow in a neighborhood of these points, the Jacobian matrix is evaluated at each of the critical points and from its eigenvalues, the type of critical point is determined.

$$\begin{aligned}J &= \begin{pmatrix}\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y}\\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y}\end{pmatrix}\\ &= \begin{pmatrix} y & \left(1+x-y^2\right)+y\left(-2y\right)\\ \left(1+y-x^2\right)+x\left(-2x\right) & x\end{pmatrix}\\ &= \begin{pmatrix} y & -3y^2+x+1\\ -3x^2+y+1 & x\end{pmatrix}\end{aligned}$$

At Point $(0,0)$ the Jacobian matrix becomes

$$J=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

Hence $|J-\lambda I|=0$ gives

$$\begin{aligned}\begin{vmatrix}-\lambda & 1\\ 1 & -\lambda\end{vmatrix} &= 0\\ \lambda^2-1 &= 0\\ \lambda &= \pm 1\end{aligned}$$

This is saddle point because one eigenvalue is positive (not stable) and one is negative (stable).

At Point $(1,0)$ the Jacobian is

$$J=\begin{pmatrix}0 & 2\\ -2 & 1\end{pmatrix}$$

Hence $|J-\lambda I|=0$ gives

$$\begin{aligned}\begin{vmatrix}-\lambda & 2\\ -2 & 1-\lambda\end{vmatrix}&=0\\ -\lambda(1-\lambda)+4&=0\\ \lambda^2-\lambda+4&=0\\ \lambda&=\frac{1}{2}\pm\frac{1}{2}i\sqrt{15}\end{aligned}$$

Since the real part is positive, then this is unstable point. Spiral out. (book calls this focus with negative attraction).

At Point $(-1,0)$ the Jacobian is

$$J=\begin{pmatrix}0 & 0\\ -2 & -1\end{pmatrix}$$

Hence $|J-\lambda I|=0$ gives

$$\begin{aligned}\begin{vmatrix}-\lambda & 0\\ -2 & -1-\lambda\end{vmatrix}&=0\\ -\lambda(-1-\lambda)&=0\\ \lambda(\lambda+1)&=0\\ \lambda&=0,-1\end{aligned}$$

Since this is nonlinear system, and one eigenvalue is zero, then unable to decide on stability of this critical point.

Note: In back of text book, it says that this degenerate. But it is not clear why that is. Because to determine if a critical point is degenerate, the determinant of Hessian $\det\left(\nabla^2F\left(x,y\right)\right)$ must be zero at that point, where $F\left(x,y\right)$ is the first integral (or energy of system). I could not find $F\left(x,y\right)$ for this system, and so I could not check if this was the case. Will follow the book for now and call this point degenerate, but it will be useful to find out how or why the book calls this degenerate.

At Point $(0,1)$ the Jacobian is

$$J=\begin{pmatrix}1 & -2\\ 2 & 0\end{pmatrix}$$

Hence $|J - \lambda I| = 0$ gives

$$
\begin{aligned}
\begin{vmatrix} 1-\lambda & -2 \\ 2 & -\lambda \end{vmatrix} &= 0 \\
-\lambda(1-\lambda) + 4 &= 0 \\
\lambda^2 - \lambda + 4 &= 0 \\
\lambda &= \frac{1}{2} \pm \frac{1}{2} i\sqrt{15}
\end{aligned}
$$

This is the same $(1,0)$. Since the real part is positive, then this is unstable point. Spiral out. (focus with negative attraction).

At Point $(0,-1)$ the Jacobian is

$$
J = \begin{pmatrix} -1 & -2 \\ 0 & 0 \end{pmatrix}
$$

Hence $|J - \lambda I| = 0$ gives

$$
\begin{aligned}
\begin{vmatrix} -1-\lambda & -2 \\ 0 & -\lambda \end{vmatrix} &= 0 \\
-\lambda(-1-\lambda) &= 0 \\
\lambda(1+\lambda) &= 0 \\
\lambda &= 0, -1
\end{aligned}
$$

This is the same as point $(-1,0)$ above. Since this is nonlinear system, and one eigenvalue is zero, then unable to decide on stability of this critical point. degenerate.

Point $\left(\frac{1}{2}\left(1-\sqrt{5}\right), \frac{1}{2}\left(1-\sqrt{5}\right)\right)$

At this point Jacobian becomes

$$
\begin{aligned}
J &= \begin{pmatrix} y & -3y^2 + x + 1 \\ -3x^2 + y + 1 & x \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2}\left(1-\sqrt{5}\right) & -3\left(\frac{1}{2}\left(1-\sqrt{5}\right)\right)^2 + \frac{1}{2}\left(1-\sqrt{5}\right) + 1 \\ -3\left(\frac{1}{2}\left(1-\sqrt{5}\right)\right)^2 + \frac{1}{2}\left(1-\sqrt{5}\right) + 1 & \frac{1}{2}\left(1-\sqrt{5}\right) \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2}\left(1-\sqrt{5}\right) & \sqrt{5}-3 \\ \sqrt{5}-3 & \frac{1}{2}\left(1-\sqrt{5}\right) \end{pmatrix}
\end{aligned}
$$

Hence $|J-\lambda I|=0$ gives

$$
\begin{aligned}
\begin{vmatrix} \frac{1}{2}\left(1-\sqrt{5}\right)-\lambda & \sqrt{5}-3 \\ \sqrt{5}-3 & \frac{1}{2}\left(1-\sqrt{5}\right)-\lambda \end{vmatrix} &= 0 \\
\left(\frac{1}{2}\left(1-\sqrt{5}\right)-\lambda\right)\left(\frac{1}{2}\left(1-\sqrt{5}\right)-\lambda\right)-\left(\sqrt{5}-3\right)^2 &= 0 \\
\lambda^2+\lambda\left(\sqrt{5}-1\right)+\frac{11}{2}\sqrt{5}-\frac{25}{2} &= 0 \\
\lambda &= \frac{7}{2}-\frac{3}{2}\sqrt{5}, \frac{1}{2}\sqrt{5}-\frac{5}{2} \\
&= 0.146, -1.382
\end{aligned}
$$

This is <u>saddle</u> point because one eigenvalue is positive (not stable) and one is negative (stable).

Point $\left(\frac{1}{2}\left(1+\sqrt{5}\right), \frac{1}{2}\left(1+\sqrt{5}\right)\right)$

At this point Jacobian becomes

$$
\begin{aligned}
J &= \begin{pmatrix} y & -3y^2+x+1 \\ -3x^2+y+1 & x \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2}\left(1+\sqrt{5}\right) & -3\left(\frac{1}{2}\left(1+\sqrt{5}\right)\right)^2+\frac{1}{2}\left(1+\sqrt{5}\right)+1 \\ -3\left(\frac{1}{2}\left(1+\sqrt{5}\right)\right)^2+\frac{1}{2}\left(1+\sqrt{5}\right)+1 & \frac{1}{2}\left(1+\sqrt{5}\right) \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2}\left(1+\sqrt{5}\right) & -\sqrt{5}-3 \\ -\sqrt{5}-3 & \frac{1}{2}\left(1+\sqrt{5}\right) \end{pmatrix}
\end{aligned}
$$

Hence $|J-\lambda I|=0$ gives

$$
\begin{aligned}
\begin{vmatrix} \frac{1}{2}\left(1+\sqrt{5}\right)-\lambda & -\sqrt{5}-3 \\ -\sqrt{5}-3 & \frac{1}{2}\left(1+\sqrt{5}\right) \end{vmatrix} &= 0 \\
\left(\frac{1}{2}\left(1+\sqrt{5}\right)-\lambda\right)\left(\frac{1}{2}\left(1+\sqrt{5}\right)-\lambda\right)-\left(-\sqrt{5}-3\right)^2 &= 0 \\
\lambda^2-\lambda\left(\sqrt{5}+1\right)-\frac{11}{2}\sqrt{5}-\frac{25}{2} &= 0 \\
\lambda &= \frac{3}{2}\sqrt{5}+\frac{7}{2}, -\frac{1}{2}\sqrt{5}-\frac{5}{2} \\
&= 6.854, -3.618
\end{aligned}
$$

This is <u>saddle</u> point because one eigenvalue is positive (not stable) and one is negative (stable).

The following is summary of result of the above

critical point	stable/unstable	λ_1, λ_2	type
$(0,0)$	unstable	$1, -1$	Saddle
$(1,0)$	unstable	$\frac{1}{2} \pm \frac{1}{2} i \sqrt{15}$	Spiral out (focus, negative attraction)
$(-1,0)$	unable to decide	$0, -1$	Degenerate
$(0,1)$	unstable	$\frac{1}{2} \pm \frac{1}{2} i \sqrt{15}$	Spiral out (focus, negative attraction)
$(0,-1)$	unable to decide	$0, -1$	Degenerate
$\left(\frac{1}{2}\left(1-\sqrt{5}\right), \frac{1}{2}\left(1-\sqrt{5}\right)\right)$	unstable	$0.146, -1.382$	Saddle
$\left(\frac{1}{2}\left(1+\sqrt{5}\right), \frac{1}{2}\left(1+\sqrt{5}\right)\right)$	unstable	$6.854, -3.618$	Saddle

The following is phase plot, generated from the nonlinear system numerically using the computer. Red dots are the unstable points. Blue points are the degenerate points.

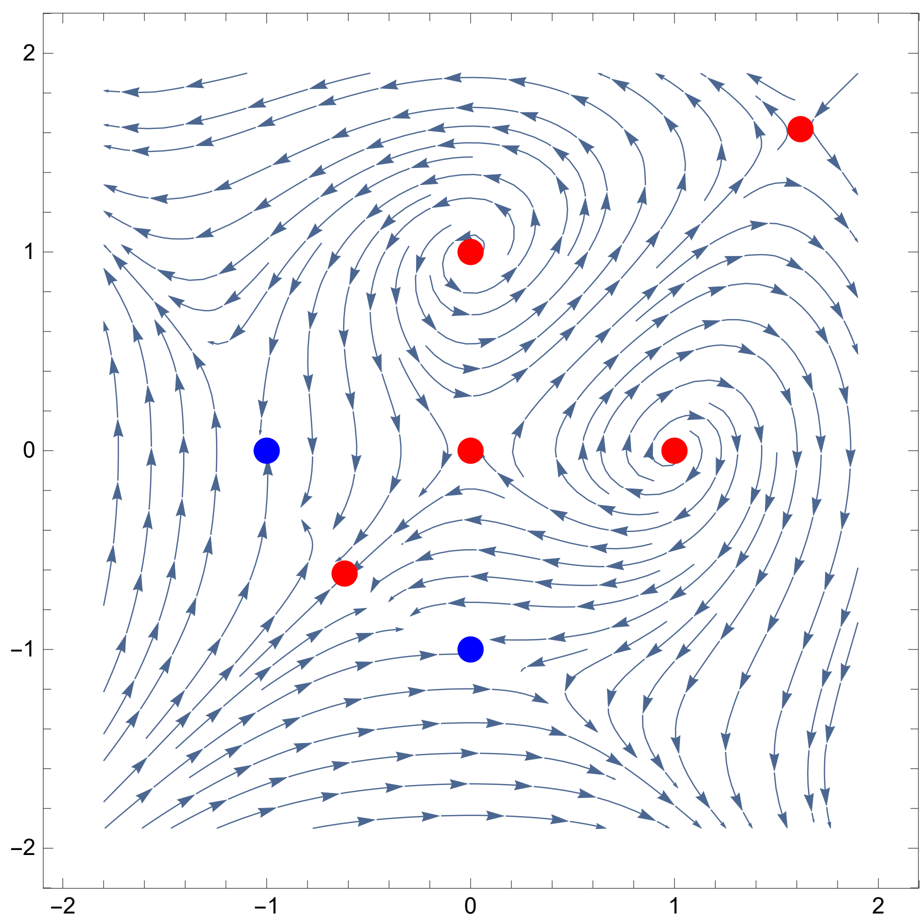

Figure 1: Phase plot of the nonlinear system

```
f1 = y (1 + x - y^2);
f2 = x (1 + y - x^2);
p1 = {Red, PointSize[0.03], Point[{0, 0}]};
p2 = {Red, PointSize[0.03], Point[{1, 0}]};
p3 = {Blue, PointSize[0.03], Point[{-1, 0}]};
p4 = {Red, PointSize[0.03], Point[{0, 1}]};
p5 = {Blue, PointSize[0.03], Point[{0, -1}]};
p6 = {Red, PointSize[0.03], Point[{1/2 (1 - Sqrt[5]), 1/2 (1 - Sqrt[5])}]};
p7 = {Red, PointSize[0.03], Point[{1/2 (1 + Sqrt[5]), 1/2 (1 + Sqrt[5])}]};
p = StreamPlot[{f1, f2}, {x, -1.8, 1.9}, {y, -1.9, 1.9}, Epilog → {p1, p2, p3, p4, p5, p6, p7}];
```

Figure 2: Code used for the above plot

2 Problem 3.3

Consider the system

$$\dot{x} = 16x^2 + 9y^2 - 25$$
$$\dot{y} = 16x^2 - 16y^2$$

(a) Determine the critical points and characterize them by linearization. (b) Sketch the phase-flow.

solution

2.1 Part a

Let

$$\dot{x} = 16x^2 + 9y^2 - 25 = f_1\left(x,y\right) \tag{1}$$
$$\dot{y} = 16x^2 - 16y^2 = f_2\left(x,y\right) \tag{2}$$

The critical points are found by solving $f_1 = 0, f_2 = 0$. The equation $f_2 = 0$ gives solutions

$$\begin{aligned} 16x^2 - 16y^2 &= 0 \\ y &= \pm x \end{aligned} \tag{3}$$

When $y = x$, substitution into $f_1 = 0$ gives

$$\begin{aligned} 16x^2 + 9x^2 - 25 &= 0 \\ x &= \pm 1 \end{aligned}$$

Hence the first set of critical points is $(1,1), (-1,-1)$. When $y = -x$ then $x = \pm 1$ also. Therefore the next set of critical points is $(1,-1), (-1,1)$

In summary, the following are the critical points found. There are 4 of them

$$\begin{aligned} \left(x,y\right)^* &= (1,1) \\ &= (-1,-1) \\ &= (1,-1) \\ &= (-1,1) \end{aligned}$$

To characterize the linearized system at these points, the Jacobian matrix is evaluated at each of point and from the nature of eigenvalues, the type of critical point is determined. Since $f_1 = 16x^2 + 9y^2 - 25, f_2 = 16x^2 - 16y^2$ then the Jacobian matrix is

$$\begin{aligned} J &= \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} \\ &= \begin{pmatrix} 32x & 18y \\ 32x & -32y \end{pmatrix} \end{aligned}$$

At Point (1,1) the Jacobian is

$$J = \begin{pmatrix} 32 & 18 \\ 32 & -32 \end{pmatrix}$$

Hence $|J - \lambda I| = 0$ gives

$$\begin{vmatrix} 32-\lambda & 18 \\ 32 & -32-\lambda \end{vmatrix} = 0$$

$$(32-\lambda)(-32-\lambda) - (18)(32) = 0$$

$$\lambda^2 - 1600 = 0$$

$$\lambda = \pm\sqrt{1600}$$

$$= \pm 40$$

This is <u>saddle</u> point because one eigenvalue is positive (not stable) and one is negative (stable). Since the problem asks also to sketch the phase plot, then the eigenvectors are now found as well. For $\lambda = 40$

$$\begin{pmatrix} 32-40 & 18 \\ 32 & -32-40 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Hence $-8v_1 + 18v_2 = 0$. Let $v_1 = 1$ then $v_2 = \frac{4}{9}$ and $\vec{v}_1 = \begin{pmatrix} 1 \\ \frac{4}{9} \end{pmatrix} = \begin{pmatrix} 9 \\ 4 \end{pmatrix}$.

For $\lambda = -40$

$$\begin{pmatrix} 32+40 & 18 \\ 32 & -32+40 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Hence $72v_1 + 18v_2 = 0$ or $4v_1 + v_2 = 0$. Let $v_1 = 1$ then $v_2 = -4$ and $\vec{v}_2 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$.

Summary for point (1,1) (Saddle)

λ_i	$\vec{v}_i$	direction
40	$\begin{pmatrix} 9 \\ 4 \end{pmatrix}$	not stable (move away from (1,1))
-40	$\begin{pmatrix} 1 \\ -4 \end{pmatrix}$	stable (move towards from (1,1))

Now that we know the eigenvectors, we can sketch them at (1,1) as follows

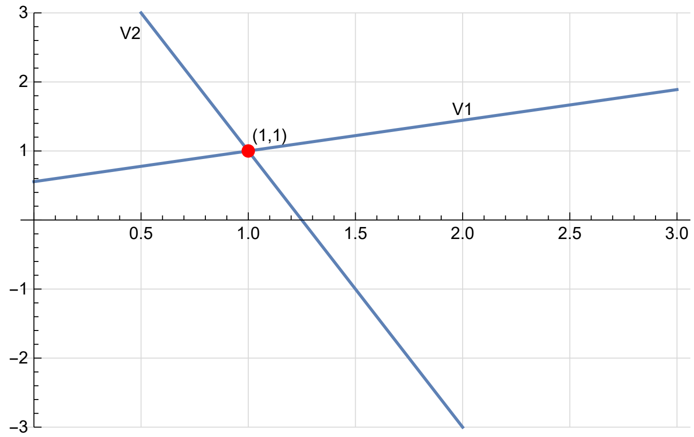

Figure 3: Eigenvectors around (1,1)

But we still do not know the directions along the eigenvectors. But we know that for negative eigenvalue, the solution is stable and for positive eigenvalue the solution is not stable. Hence along $\vec{v}_1$ the solution must move away from (1,1) since $\vec{v}_1$ is associated with an unstable λ.

For $\vec{v}_2$ the solution must be stable, therefore on $\vec{v}_2$ the solution must move towards (1,1). Now that we know the directions, we can update the above plot sketch by addition directions.

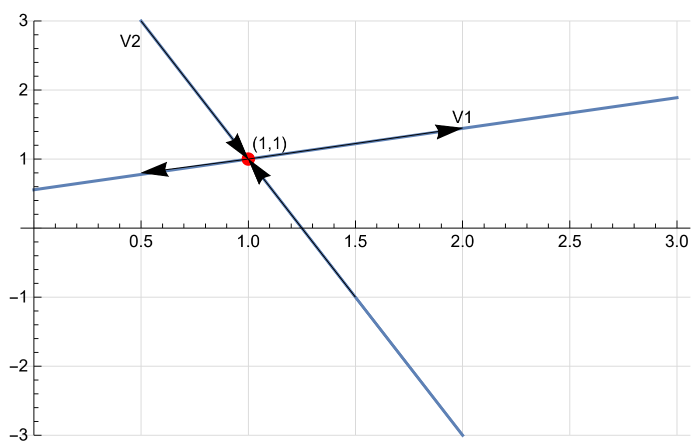

Figure 4: Eigenvectors around (1,1) with directions

Now the sketch is finished by adding stream lines that follow along the directions of the eigenvector due to continuity and because solution lines can not cross each others (due to uniqueness). This gives the phase plot around (1,1) found by linearization as follows

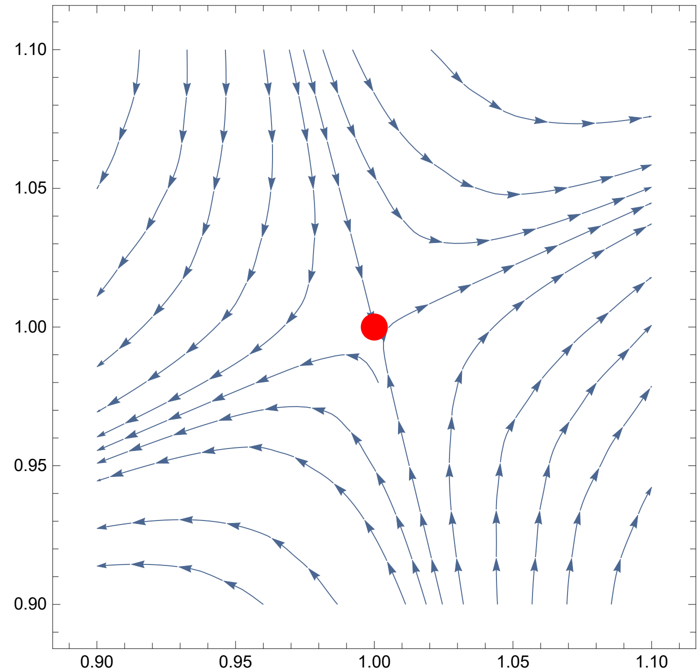

Figure 5: Adding more stream lines around (1,1)

The same steps above are now repeated for the next critical point $(-1,-1)$

At Point $(-1,-1)$ the Jacobian is

$$J = \begin{pmatrix} 32x & 18y \\ 32x & -32y \end{pmatrix}$$
$$= \begin{pmatrix} -32 & -18 \\ -32 & 32 \end{pmatrix}$$

Hence $|J - \lambda I| = 0$ gives

$$\begin{vmatrix} -32-\lambda & -18 \\ -32 & 32-\lambda \end{vmatrix} = 0$$
$$(-32-\lambda)(32-\lambda) - (18)(32) = 0$$
$$\lambda^2 - 1600 = 0$$
$$\lambda = \pm\sqrt{1600}$$
$$= \pm 40$$

This is the same result as the earlier point. This is a saddle point because one eigenvalue is positive (not stable) and one is negative (stable). Now the eigenvectors are found. For

$\lambda = 40$

$$\begin{pmatrix} -32-\lambda & -18 \\ -32 & 32-\lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -32-40 & -18 \\ -32 & 32-40 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -72 & -18 \\ -32 & -8 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Hence $-72v_1 - 18v_2 = 0$ or $-4v_1 - v_2 = 0$ Let $v_1 = 1$ then $v_2 = -4$. The eigenvector is $\vec{v}_1 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$.

For $\lambda = -40$

$$\begin{pmatrix} -32-\lambda & -18 \\ -32 & 32-\lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -32+40 & -18 \\ -32 & 32+40 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 8 & -18 \\ -32 & 72 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Hence $8v_1 - 18v_2 = 0$. Let $v_1 = 1$ then $v_2 = \frac{4}{9}$. The eigenvector is $\vec{v}_2 = \begin{pmatrix} 1 \\ \frac{4}{9} \end{pmatrix} = \begin{pmatrix} 9 \\ 4 \end{pmatrix}$.

Summary for point $(-1,-1)$ (Saddle, not stable).

λ_i	$\vec{v}_i$	direction
40	$\begin{pmatrix} 1 \\ -4 \end{pmatrix}$	Not stable. Move away from $(-1,-1)$
-40	$\begin{pmatrix} 9 \\ 4 \end{pmatrix}$	Stable. Move towards $(-1,-1)$

Now that we know the eigenvectors, we sketch them at $(-1,-1)$ as follows

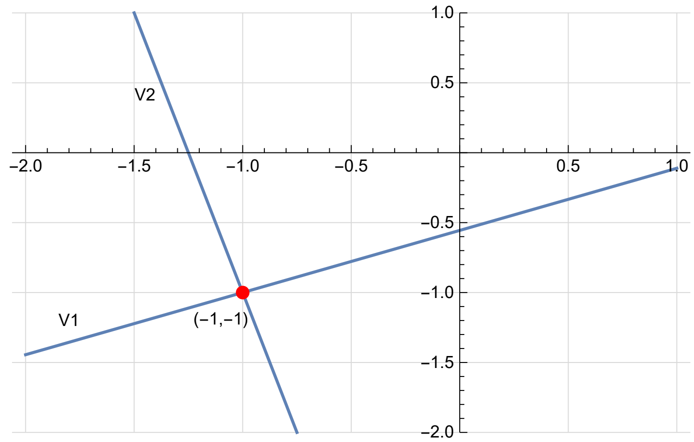

Figure 6: Eigenvectors around $(-1,-1)$

But we still do not know the directions along the eigenvectors. As was mentioned above, for negative eigenvalue, the solution is stable and for positive eigenvalue the solution is not stable. This means on $\vec{v}_1$ the solution moves away from $(-1,-1)$ since $\vec{v}_1$ is associated with unstable λ. For $\vec{v}_2$, since the solution is stable then on $\vec{v}_2$ the solution moves towards $(-1,-1)$. Now that the directions are known, the above sketch is updated giving

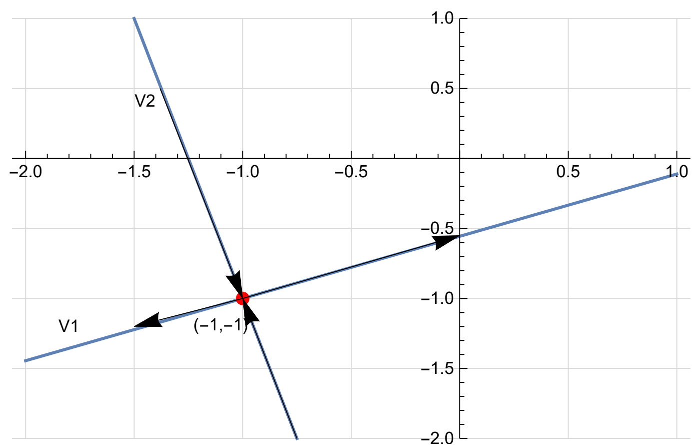

Figure 7: Eigenvectors around $(-1,-1)$ with directions

The sketch is finished by adding stream lines that follow along the directions of the eigenvector by continuity. This gives the phase plot around $(-1,-1)$ found by linearization as follows

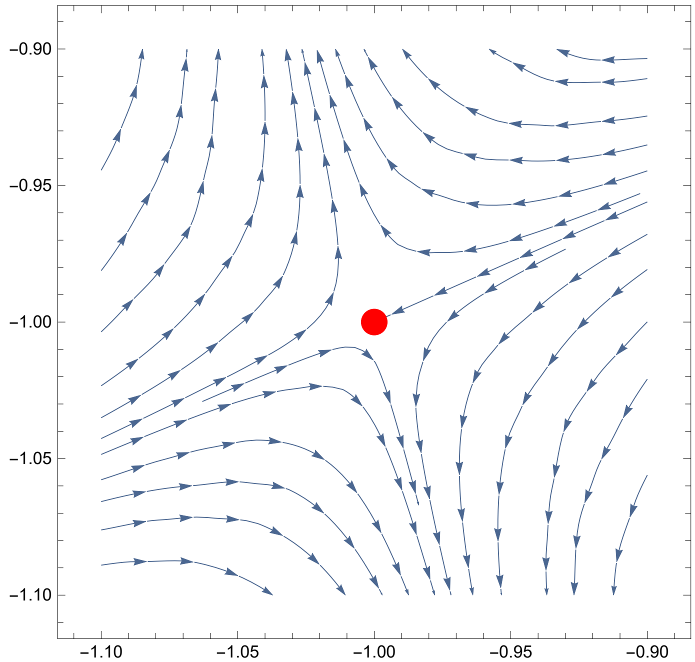

Figure 8: Adding more stream lines around $(-1,-1)$

The same steps are now repeated for the next critical point $(1,-1)$

At Point $(1,-1)$ the Jacobian is

$$J = \begin{pmatrix} 32x & 18y \\ 32x & -32y \end{pmatrix}$$
$$= \begin{pmatrix} 32 & -18 \\ 32 & 32 \end{pmatrix}$$

Hence $|J - \lambda I| = 0$ gives

$$\begin{vmatrix} 32-\lambda & -18 \\ 32 & 32-\lambda \end{vmatrix} = 0$$
$$(32-\lambda)^2 + (18)(32) = 0$$
$$\lambda^2 - 64\lambda + 1600 = 0$$
$$\lambda = 32 \pm 24i$$

This is unstable critical point, since since real part of the complex number is positive. This is spiral out point. Also called focus, with negative attraction. For $\lambda = 32 + 24i$ the eigenvector is

$$\begin{pmatrix} 32-\lambda & -18 \\ 32 & 32-\lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 32-(32+24i) & -18 \\ 32 & 32-(32+24i) \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -24i & -18 \\ 32 & -24i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Hence $-24iv_1 - 18v_2 = 0$. Let $v_1 = 1$ then $v_2 = -\frac{24}{18}i$ and $\vec{v}_1 = \begin{pmatrix} 1 \\ -\frac{4}{3}i \end{pmatrix} = \begin{pmatrix} 3 \\ -4i \end{pmatrix} = \begin{pmatrix} 3i \\ 4 \end{pmatrix}$

For $\lambda = 32 - 24i$

$$\begin{pmatrix} 32-\lambda & -18 \\ 32 & 32-\lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 32-(32-24i) & -18 \\ 32 & 32-(32-24i) \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 24i & -18 \\ 32 & 24i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Hence $24iv_1 - 18v_2 = 0$. Let $v_1 = 1$ then $v_2 = \frac{24}{18}i$ and $\vec{v}_1 = \begin{pmatrix} 1 \\ \frac{4}{3}i \end{pmatrix} = \begin{pmatrix} 3 \\ 4i \end{pmatrix} = \begin{pmatrix} 3i \\ -4 \end{pmatrix} = \begin{pmatrix} -3i \\ 4 \end{pmatrix}$. What is left to find out is to determine if the spiral is clockwise or anti-clockwise. One way to find this is to select a point (x, y) to the right of the critical point and then find if x is increasing or decreasing there and find out also if y is increasing or decreasing there. This gives the slop. Since the critical point is $(1, -1)$, let us pick point $(2, -1)$ to its right. From

$$\begin{aligned} \dot{x} &= 16x^2 + 9y^2 - 25 \\ \dot{y} &= 16x^2 - 16y^2 \end{aligned}$$

Then at $(2, -1)$ the above gives

$$\begin{aligned} \dot{x} &= 64 + 9 - 25 = 48 \\ \dot{y} &= 64 - 16 = 48 \end{aligned}$$

Hence $\dot{x} > 0$, then x is increasing and $\dot{y} > 0$, then y also increasing. This means the solution curve is moving in the NE direction ($\nearrow$). Hence the spiral is anti-clockwise direction around $(1, -1)$.

Summary for $(1, -1)$ (not stable, spiral out)

λ_i	$\vec{v}_i$	direction
$32+24i$	$\begin{pmatrix}3i\\4\end{pmatrix}$	Not stable. focus, negative attraction. Anti-clockwise direction
$32-24i$	$\begin{pmatrix}-3i\\4\end{pmatrix}$	Not stable. focus, negative attraction. Anti-clockwise direction

Now that we know the eigenvectors, we can sketch them at $(1,-1)$ as follows

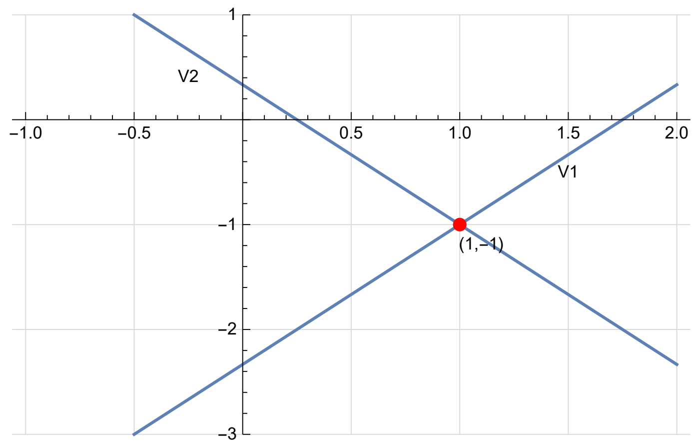

Figure 9: Eigenvectors around $(1,-1)$

Since both eigenvector are not stable, direction of solution near $(1,-1)$ is moving away from $(1,-1)$. Now that we know the directions, we can update the above plot sketch.

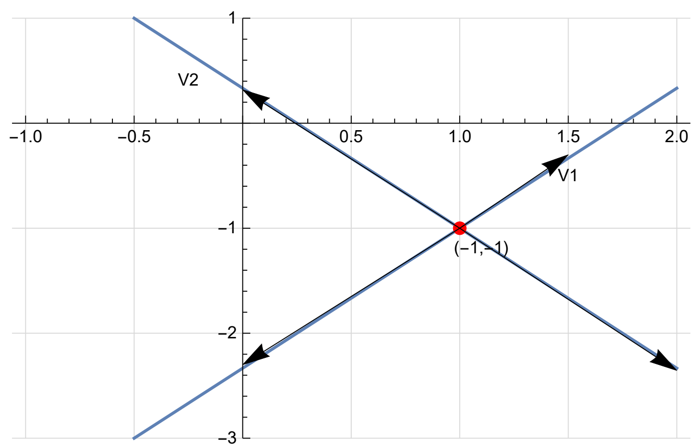

Figure 10: Eigenvectors around $(1,-1)$ with directions

Now the sketch is finished by adding the spiral out stream lines. This gives the phase plot around $(1,-1)$ found by linearization as follows

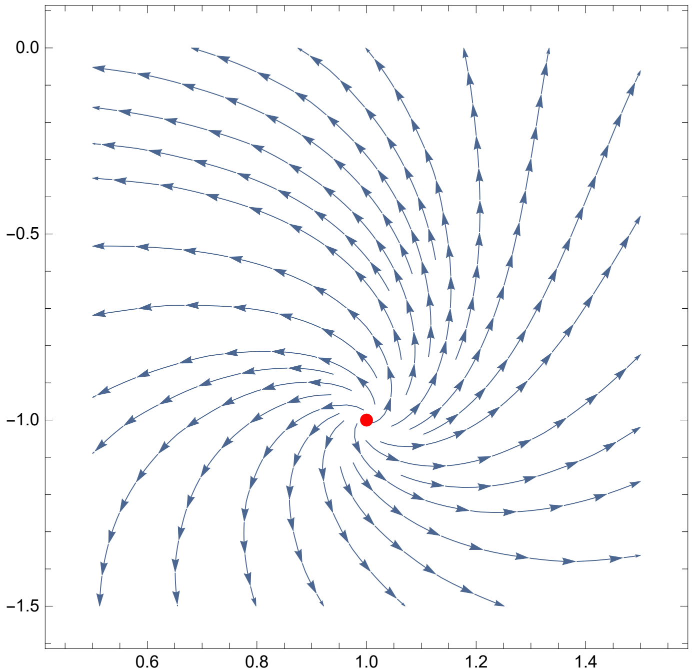

Figure 11: Adding more stream lines around $(1,-1)$

The same steps are now repeated for final critical point $(-1,1)$

At Point $(-1,1)$ the Jacobian is

$$J = \begin{pmatrix} 32x & 18y \\ 32x & -32y \end{pmatrix}$$
$$= \begin{pmatrix} -32 & 18 \\ -32 & -32 \end{pmatrix}$$

Hence $|J - \lambda I| = 0$ gives

$$\begin{vmatrix} -32-\lambda & 18 \\ -32 & -32-\lambda \end{vmatrix} = 0$$
$$(-32-\lambda)^2 + (18)(32) = 0$$
$$\lambda^2 + 64\lambda + 1600 = 0$$
$$\lambda = -32 \pm 24i$$

This is stable point, both eigenvalues has negative real part. The type is spiral in (focus, with positive attraction). For $\lambda_1 = -32 + 24i$ the eigenvector is

$$\begin{pmatrix} -32-\lambda & 18 \\ -32 & -32-\lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} -32-(-32+24i) & 18 \\ -32 & -32-(-32+24i) \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} -24i & 18 \\ -32 & -24i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Hence $-24iv_1 + 18v_2 = 0$. Let $v_1 = 1$ then $v_2 = \frac{24}{18}i$ and the eigenvector becomes $\vec{v}_1 = \begin{pmatrix} 1 \\ \frac{4}{3}i \end{pmatrix} = \begin{pmatrix} 3 \\ 4i \end{pmatrix} = \begin{pmatrix} 3i \\ -4 \end{pmatrix}$

For $\lambda_2 = -32 - 24i$ the eigenvector is

$$\begin{pmatrix} -32-\lambda & 18 \\ -32 & -32-\lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} -32-(-32-24i) & 18 \\ -32 & -32-(-32-24i) \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} 24i & 18 \\ -32 & 24i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Hence $24iv_1 + 18v_2 = 0$. Let $v_1 = 1$ then $v_2 = -\frac{24}{18}i$ and the second eigenvector becomes $\vec{v}_2 = \begin{pmatrix} 1 \\ -\frac{4}{3}i \end{pmatrix} = \begin{pmatrix} 3 \\ -4i \end{pmatrix} = \begin{pmatrix} 3i \\ 4 \end{pmatrix}$

The only thing left is to determine if the spiral is clockwise or anti-clockwise. One way to find out is to pick a point (x, y) to the right of the critical point and find if x is increasing or decreasing there and also find out if y is increasing or decreasing there. This gives the slope. Since the critical point is $(-1, 1)$, let us pick point $(0, 1)$ to its right. From

$$\begin{aligned} \dot{x} &= 16x^2 + 9y^2 - 25 \\ \dot{y} &= 16x^2 - 16y^2 \end{aligned}$$

Then at $(0, 1)$ the above gives

$$\begin{aligned} \dot{x} &= 9 - 25 = -16 \\ \dot{y} &= -16 = -16 \end{aligned}$$

Hence $\dot{x} < 0$, then x is decreasing and $\dot{y} < 0$, then y also decreasing. This means the solution curve is moving in the SW direction ($\swarrow$). Hence the spiral is in the clockwise direction around $(-1, 1)$.

Summary for $(-1, 1)$ (Stable)

λ_i	$\vec{v}_i$	direction
$-32+24i$	$\begin{pmatrix} 3i \\ -4 \end{pmatrix}$	Stable. Focus, positive attraction. Clockwise direction
$-32-24i$	$\begin{pmatrix} 3i \\ 4 \end{pmatrix}$	Stable. Focus, positive attraction. Clockwise direction

Now that we know the eigenvectors, we can sketch them at $(-1,1)$ as follows

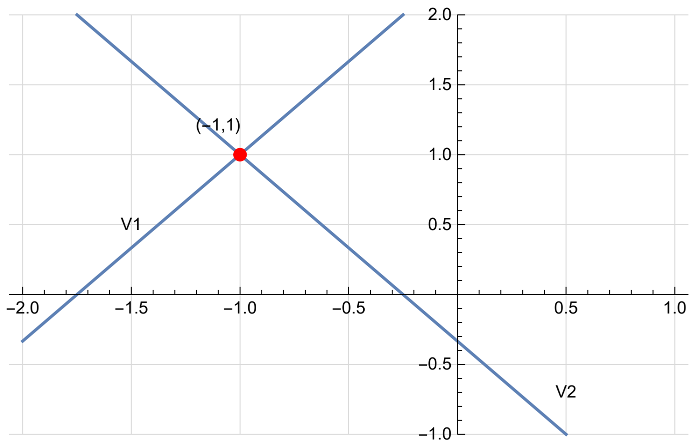

Figure 12: Eigenvectors around $(-1,1)$

Since both eigenvectors are stable, the direction along each is towards $(-1,1)$. Now that we know the directions, we can update the above plot sketch.

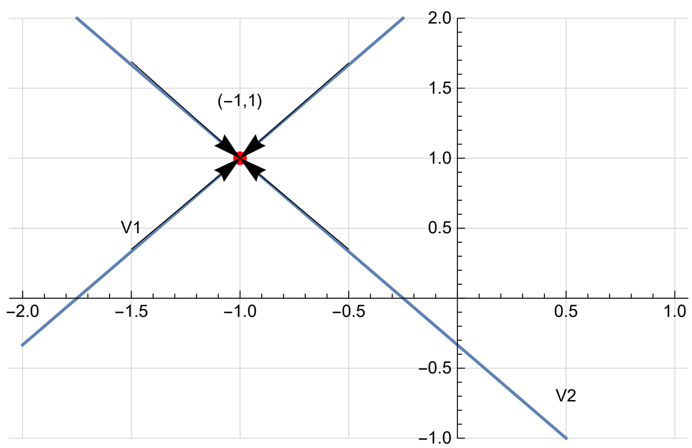

Figure 13: Eigenvectors around $(-1,1)$ with directions

The sketch is finished by adding the spiral stream lines. This gives the phase plot around $(-1,1)$ found by linearization as follows

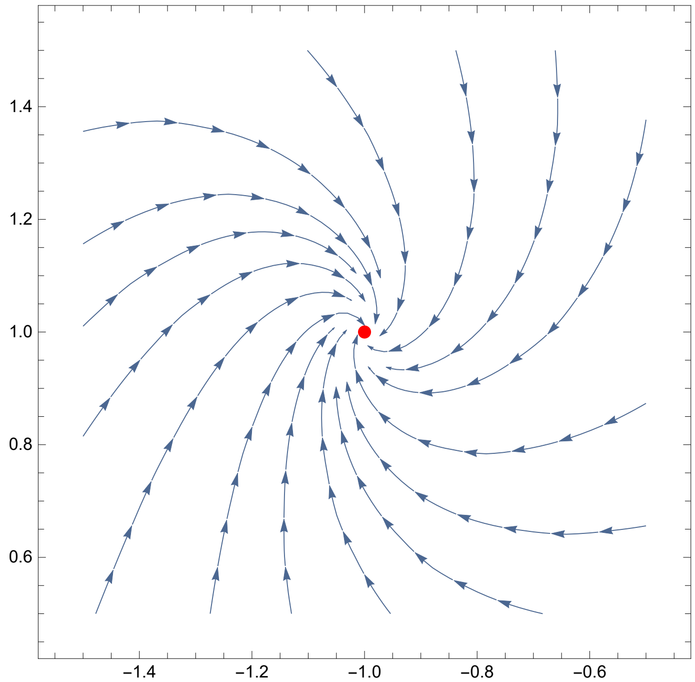

Figure 14: Adding more stream lines around (–1,1)

putting all the above result together gives the final sketch of phase plot as

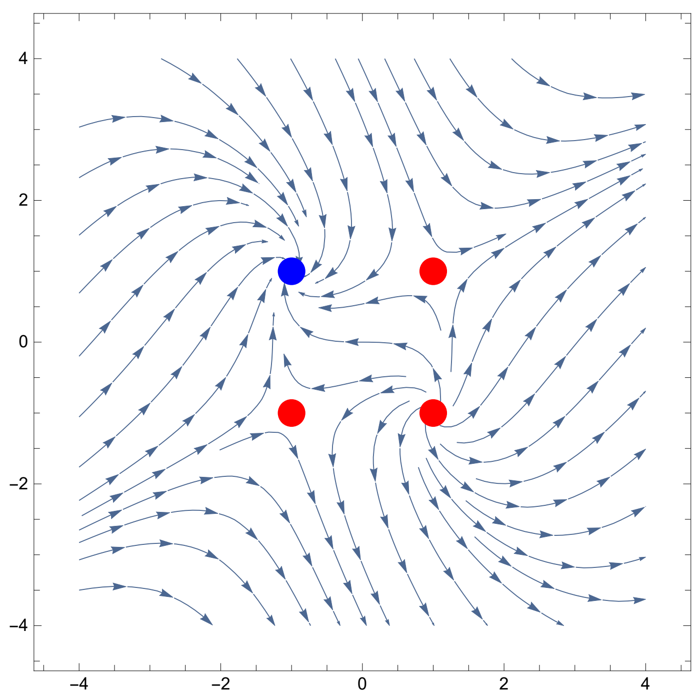

Figure 15: Final phase plot

The following is summary of result

critical point	stable/unstable	λ_1, λ_2	type
$(1,1)$	unstable	$40, -40$	Saddle node
$(-1,-1)$	unstable	$40, -40$	Saddle node
$(1,-1)$	unstable	$32 \pm 24i$	Spiral out (focus, negative attraction)
$(-1,1)$	Stable	$-32 \pm 24i$	Spiral in (focus, positive attraction)

3 Problem 3.5

In certain applications one studies the equation

$$\ddot{x} + c\dot{x} - x(1-x) = 0$$

with a special interest in solutions with the properties:

$$\lim_{t\to-\infty} x(t) = 0, \lim_{t\to\infty} x(t) = 1, \dot{x}(t) > 0 \text{ for } -\infty < t < \infty$$

Derive a necessary condition for the parameter c for such solutions to exist

solution

Using $x_1 = x, x_2 = \dot{x}$, the first step is to determine the critical points. Hence $\dot{x}_1 = x_2, \dot{x}_2 = \ddot{x} = -c\dot{x} + x(1-x) = -cx_2 + x_1(1-x_1)$. In state space the system becomes

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix} x_2 \\ -cx_2 + x_1(1-x_1)\end{pmatrix} = \begin{pmatrix} f_1 \\ f_2\end{pmatrix}$$

The first equation gives solution $x_2 = 0$. When $x_2 = 0$ the second equation gives $x_1(1-x_1) = 0$ or $x_1 = 0, x_1 = 1$. Hence the critical points are $(0,0), (1,0)$.

From the properties of the solutions, it shows that solutions that start with $x_1 = 0$ eventually go to $x_1 = 1$. Also, since $\dot{x}(t) > 0$ for $-\infty < t < \infty$ then this means $x_2 > 0$ for all time. Hence solution curves are in upper half of phase plane. Here is sketch of what phase plane should look like (I am taking $x_1 = 0$ as initial condition, at $t = -\infty$.)

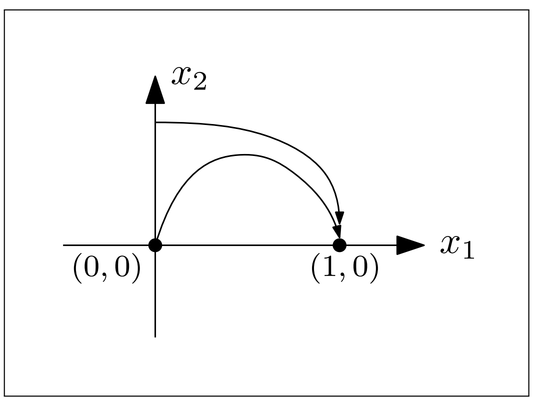

Figure 16: possible solution curves in phase plane

The Jacobian of the linearlized system is

$$\begin{aligned} J &= \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2}\end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ 1-2x_1 & -c\end{pmatrix}\end{aligned}$$

At $(0,0)$ the above becomes

$$J = \begin{pmatrix} 0 & 1 \\ 1 & -c \end{pmatrix}$$

Hence $|J - \lambda I| = 0$ gives

$$\begin{aligned} \begin{vmatrix} -\lambda & 1 \\ 1 & -c-\lambda \end{vmatrix} &= 0 \\ (-\lambda)(-c-\lambda) - 1 &= 0 \\ \lambda^2 + c\lambda - 1 &= 0 \\ \lambda &= -\frac{1}{2}c \pm \frac{1}{2}\sqrt{c^2+4} \end{aligned} \tag{1}$$

At $(1,0)$ the Jacobian becomes

$$\begin{aligned} J &= \begin{pmatrix} 0 & 1 \\ 1-2x_1 & -c \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ -1 & -c \end{pmatrix} \end{aligned}$$

Hence $|J - \lambda I| = 0$ gives

$$\begin{aligned} \begin{vmatrix} -\lambda & 1 \\ -1 & -c-\lambda \end{vmatrix} &= 0 \\ (-\lambda)(-c-\lambda) + 1 &= 0 \\ \lambda^2 + c\lambda + 1 &= 0 \\ \lambda &= -\frac{1}{2}c \pm \frac{1}{2}\sqrt{c^2-4} \end{aligned} \tag{2}$$

We know that (2) must give stable solution, because we want the solution to eventually move to that critical point $(1,0)$. Also, since we do not want to move into negative half plane because $x_2 > 0$ for all time, then this mean that we <u>can not have spiral solution</u> around $(1,0)$. Therefore $\sqrt{c^2-4}$ must be positive to avoid complex eigenvalue which gives spiral solutions. This means $c^2 \geq 4$ or

$$c \geq 2$$

Here is the phase plot for $c = 2.5$

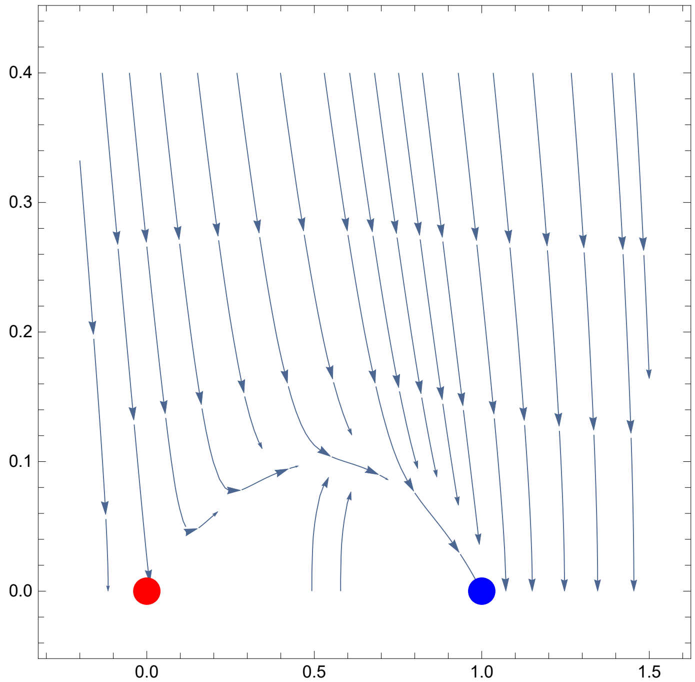

Figure 17: Phase plot for $c = 2.5$

We can now check that for such c value, periodic solutions do not exist. The gradient of the vector $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -cx_2 + x_1(1 - x_1) \end{pmatrix}$ is

$$\begin{aligned} \nabla \cdot \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} &= \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \\ &= 0 + -c \\ &= -c \end{aligned}$$

And since we determined that c must be positive, then $\nabla \cdot \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = -c$ do not change sign and remain negative. Hence by Bendixson's criterion (4.1 in book) no periodic solution is possible.

4 Problem 3.7

Determine the critical points of the system

$$\dot{x} = x\left(1 - x^2 - 6y^2\right)$$
$$\dot{y} = y\left(1 - 3x^2 - 3y^2\right)$$

And characterize them by linear analysis.

solution

Let

$$\dot{x} = x\left(1 - x^2 - 6y^2\right) = f_1\left(x, y\right) \tag{1}$$
$$\dot{y} = y\left(1 - 3x^2 - 3y^2\right) = f_2\left(x, y\right) \tag{2}$$

The critical points are found by solving $f_1 = 0, f_2 = 0$. Solving for x from $f_1 = 0$ gives

$$x = 0 \tag{3}$$
$$1 - 6y^2 = x^2 \tag{4}$$

From each solution above, we go to EQ (2) and solve for y. When $x = 0$ then (2) gives

$$y\left(1 - 3y^2\right) = 0$$

Hence $y = 0, y = \pm\sqrt{\frac{1}{3}}$. Therefore the first set of critical points is $(0,0), \left(0, \sqrt{\frac{1}{3}}\right), \left(0, -\sqrt{\frac{1}{3}}\right)$.

Now, when $1 - 6y^2 = x^2$ then (2) gives

$$y\left(1 - 3\left(1 - 6y^2\right) - 3y^2\right) = 0$$
$$y\left(15y^2 - 2\right) = 0$$

Hence $y = 0, y = \pm\sqrt{\frac{2}{15}}$. When $y = 0$ then $1 - 6y^2 = x^2$ gives $x = \pm 1$ and when $y = \sqrt{\frac{2}{15}}$ then $1 - 6y^2 = x^2$ gives $1 - 6\left(\frac{2}{15}\right) = x^2$, or $x = \pm\frac{1}{5}\sqrt{5} = \pm\frac{1}{\sqrt{5}}$ and when $y = -\sqrt{\frac{2}{15}}$ then $1 - 6y^2 = x^2$ gives same solution $x = \pm\frac{1}{\sqrt{5}}$. Therefore the second set of critical points is $(\pm 1, 0), \left(\pm\frac{1}{\sqrt{5}}, \sqrt{\frac{2}{15}}\right), \left(\pm\frac{1}{\sqrt{5}}, -\sqrt{\frac{2}{15}}\right)$. In summary, these are the critical points (9 in total)

$$(0,0), \left(0, \pm\sqrt{\frac{1}{3}}\right), (\pm 1, 0), \left(\pm\frac{1}{\sqrt{5}}, \sqrt{\frac{2}{15}}\right), \left(\pm\frac{1}{\sqrt{5}}, -\sqrt{\frac{2}{15}}\right)$$

Now that critical points are found, they are classified by linearizing the system and finding the eigenvalues of the Jacobian matrix which acts as the A matrix in $\dot{u} = Au$ of the linearized system. The Jacobian of the linearlized system is

$$\begin{aligned} J &= \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} \\ &= \begin{pmatrix} \left(1 - x^2 - 6y^2\right) - 2x^2 & -12xy \\ -6xy & \left(1 - 3x^2 - 3y^2\right) - y\left(6y\right) \end{pmatrix} \\ &= \begin{pmatrix} 1 - 3x^2 - 6y^2 & -12xy \\ -6xy & 1 - 3x^2 - 9y^2 \end{pmatrix} \end{aligned}$$

At point $(0,0)$

$$J = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Hence $|J - \lambda I| = 0$ gives

$$\begin{aligned} \begin{vmatrix} 1 - \lambda & 0 \\ 0 & 1 - \lambda \end{vmatrix} &= 0 \\ (1 - \lambda)^2 &= 0 \\ \lambda &= 1, 1 \end{aligned}$$

A repeated root. Since $\lambda > 1$ then this is unstable point. Negative attractor node. It is not spiral since pure real eigenvalues.

At point $\left(0, \sqrt{\frac{1}{3}}\right)$

$$\begin{aligned} J &= \begin{pmatrix} 1 - 3x^2 - 6y^2 & -12xy \\ -6xy & 1 - 3x^2 - 9y^2 \end{pmatrix} \\ &= \begin{pmatrix} 1 - 6\left(\frac{1}{3}\right) & 0 \\ 0 & 1 - 9\left(\frac{1}{3}\right) \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \end{aligned}$$

Hence $|J - \lambda I| = 0$ gives

$$\begin{aligned} \begin{vmatrix} -1 - \lambda & 0 \\ 0 & -2 - \lambda \end{vmatrix} &= 0 \\ (-1 - \lambda)(-2 - \lambda) &= 0 \\ (1 + \lambda)(2 + \lambda) &= 0 \\ \lambda &= -2, -1 \end{aligned}$$

Since both eigenvalues are negative then this is table point. Positive attractor node. Not a spiral node since pure real eigenvalues.

point $\left(0,-\sqrt{\frac{1}{3}}\right)$

This gives same result as above. Positive attractor node.

point $(1,0)$

$$\begin{aligned}
J &= \begin{pmatrix} 1-3x^2-6y^2 & -12xy \\ -6xy & 1-3x^2-9y^2 \end{pmatrix} \\
&= \begin{pmatrix} 1-3 & 0 \\ 0 & 1-3 \end{pmatrix} \\
&= \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}
\end{aligned}$$

Hence $|J-\lambda I|=0$ gives

$$\begin{aligned}
\begin{vmatrix} -2-\lambda & 0 \\ 0 & -2-\lambda \end{vmatrix} &= 0 \\
(-2-\lambda)^2 &= 0 \\
\lambda &= -2,-2
\end{aligned}$$

Repeated root. Since eigenvalue is negative then this is table point. Positive attractor node. Not a spiral node since pure real eigenvalues.

point $(-1,0)$

This gives same result as above. Positive attractor node.

point $\left(\frac{1}{\sqrt{5}},\sqrt{\frac{2}{15}}\right)$

$$\begin{aligned}
J &= \begin{pmatrix} 1-3x^2-6y^2 & -12xy \\ -6xy & 1-3x^2-9y^2 \end{pmatrix} \\
&= \begin{pmatrix} 1-3\left(\frac{1}{\sqrt{5}}\right)^2-6\left(\sqrt{\frac{2}{15}}\right)^2 & -12\left(\frac{1}{\sqrt{5}}\right)\left(\sqrt{\frac{2}{15}}\right) \\ -6\left(\frac{1}{\sqrt{5}}\right)\left(\sqrt{\frac{2}{15}}\right) & 1-3\left(\frac{1}{\sqrt{5}}\right)^2-9\left(\sqrt{\frac{2}{15}}\right)^2 \end{pmatrix} \\
&= \begin{pmatrix} -\frac{2}{5} & -\frac{4}{5}\sqrt{2}\sqrt{3} \\ -\frac{2}{5}\sqrt{2}\sqrt{3} & -\frac{4}{5} \end{pmatrix}
\end{aligned}$$

Hence $|J - \lambda I| = 0$ gives

$$\begin{vmatrix} -\frac{2}{5} - \lambda & -\frac{4}{5}\sqrt{2}\sqrt{3} \\ -\frac{2}{5}\sqrt{2}\sqrt{3} & -\frac{4}{5} - \lambda \end{vmatrix} = 0$$

$$\left(-\frac{2}{5} - \lambda\right)\left(-\frac{4}{5} - \lambda\right) - \left(-\frac{4}{5}\sqrt{2}\sqrt{3}\right)\left(-\frac{2}{5}\sqrt{2}\sqrt{3}\right) = 0$$

$$\lambda^2 + \frac{6}{5}\lambda - \frac{8}{5} = 0$$

$$\lambda = \frac{4}{5}, -2$$

Since one eigenvalue is negative (stable) but the other is positive (unstable), then this is saddle node. (considered unstable node).

point $\left(-\frac{1}{5}\sqrt{5}, \sqrt{\frac{2}{15}}\right)$

Same as above.

point $\left(\frac{1}{5}\sqrt{5}, \sqrt{\frac{2}{15}}\right)$

Same as above.

point $\left(-\frac{1}{5}\sqrt{5}, -\sqrt{\frac{2}{15}}\right)$

Same as above.

4.1 Summary of results

	critical point	stable/unstable	λ_1, λ_2	type
1	$(0,0)$	unstable	$1, 1$	node, negative attractor
2	$(1,0)$	Stable	$-2, -2$	node, positive attractor
3	$(-1,0)$	Stable	$-2, -2$	node, positive attractor
4	$\left(0, \frac{1}{\sqrt{3}}\right)$	Stable	$-2, -1$	node, positive attractor
5	$\left(0, \frac{-1}{\sqrt{3}}\right)$	Stable	$-2, -1$	node, positive attractor
6	$\left(\frac{1}{\sqrt{5}}, \sqrt{\frac{2}{15}}\right)$	Unstable	$-2, \frac{4}{5}$	Saddle
7	$\left(-\frac{1}{\sqrt{5}}, \sqrt{\frac{2}{15}}\right)$	Unstable	$-2, \frac{4}{5}$	Saddle
8	$\left(\frac{1}{\sqrt{5}}, \sqrt{\frac{2}{15}}\right)$	Unstable	$-2, \frac{4}{5}$	Saddle
9	$\left(-\frac{1}{\sqrt{5}}, -\sqrt{\frac{2}{15}}\right)$	Unstable	$-2, \frac{4}{5}$	Saddle

The following is phase plot, generated numerically directly from the non-linear system. A red dot indicates an unstable node and blue colored node is a stable node.

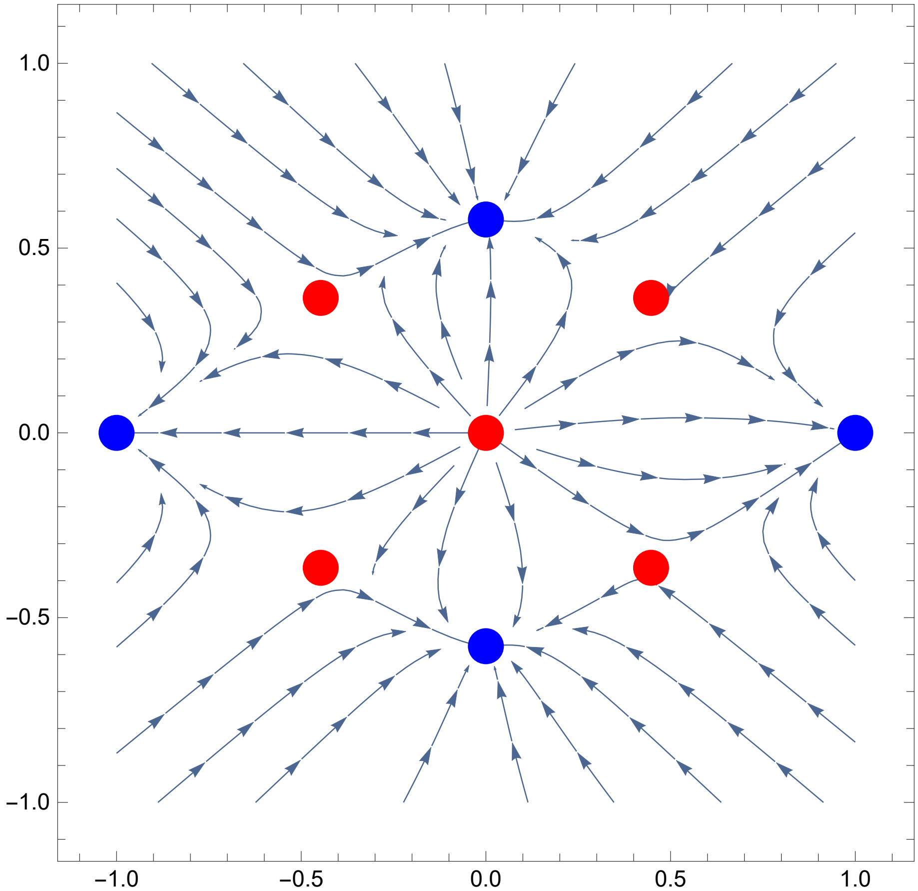

Figure 18: Phase plot