1) a) i) Solution 1 (Brute force)

Using the Discrete Fourier Transform analysis equation,

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

Since $x[n] = a^n(u[n] - u[n-5])$, the only non-zero points of $x[n]$ are $n \in \{0, 1, 2, 3, 4\}$, so

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=0}^{4} a^n e^{-j\omega n} \\ &= \sum_{n=0}^{4} (ae^{-j\omega})^n \end{aligned}$$

Since this is a partial sum of a geometric series, we can apply the formula

$$\sum_{k=0}^{N} b^k = \frac{1-b^{N+1}}{1-b}, \quad |b| < 1$$

to compute

$$X(e^{j\omega}) = \frac{1-(ae^{-j\omega})^5}{1-ae^{-j\omega}}$$

ii) Solution 2 (Transform pairs)

Observe that

$$\begin{aligned} x[n] &= a^n u[n] - a^n u[n-5] \\ &= a^n u[n] - a^5 a^{n-5} u[n-5] \\ &= (a^n u[n]) - a^5(a^n u[n]) * \delta[n-5] \\ &= x_1[n] - a^5 x_1[n] * \delta[n-5], \quad x_1[n] = a^n u[n] \end{aligned}$$

Using the Fourier transform pair,

$$\mathcal{F}\{x_1[n]\} = \mathcal{F}\{a^n u[n]\} = \frac{1}{1-ae^{-j\omega}}$$

and the time shift property,

$$\mathcal{F}\{x[n] * \delta[n-n_0]\} = e^{-j\omega n_0} X(e^{j\omega})$$

we can then write

$$\begin{aligned} X(e^{j\omega}) &= \mathcal{F}\{x[n]\} \\ &= \mathcal{F}\{x_1[n]\} - a^5 \mathcal{F}\{x_1[n] * \delta[n-5]\} \\ &= X_1(e^{j\omega}) - a^5 e^{-5j\omega} X_1(e^{j\omega}) \\ &= \frac{1}{1-ae^{-j\omega}} - \frac{a^5 e^{-5j\omega}}{a-ae^{-j\omega}} \end{aligned}$$

b) i) Solution 1 (Brute force)

I wrote all of this using $X(e^{j\omega})$ and $x[n]$ instead of $Z(e^{j\omega})$ and $z[n]$ and I don't want to go back and change all of them. Soorryyyyyyyyyyy.

We use the Continuous Fourier Transform synthesis equation,

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

We can express $X(e^{j\omega})$ in terms of it's phase and magnitude. By inspecting the slope of $\angle X(e^{j\omega})$, we see that $\angle X(e^{j\omega}) = -\omega$, so

$$\begin{aligned} X(e^{j\omega}) &= |X(e^{j\omega})| * e^{\angle X(e^{j\omega})} \\ &= e^{-j\omega} * \begin{cases} 2, & -\pi/2 < \omega < -\pi/4 \\ 1, & -\pi/4 < \omega < \pi/4 \\ 2, & \pi/4 < \omega < \pi/2 \\ 0, & \text{else} \end{cases} \end{aligned}$$

We take the integral in the Continuous Fourier Transform synthesis equation to be from $-\pi$ to π so that the bounds are symmetric. Using the piecewise form of $X(e^{j\omega})$ to break up this bound into intervals, we get

$$\begin{aligned} x[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega \\ &= \frac{1}{2\pi}\int_{-\pi/2}^{-\pi/4} 2 * e^{-j\omega} * e^{j\omega n}d\omega + \frac{1}{2\pi}\int_{-\pi/4}^{\pi/4} 1 * e^{-j\omega} * e^{j\omega n}d\omega + \frac{1}{2\pi}\int_{\pi/4}^{\pi/2} 2 * e^{-j\omega} * e^{j\omega n}d\omega \\ &= \frac{2}{2\pi}\int_{-\pi/2}^{-\pi/4} e^{j\omega(n-1)}d\omega + \frac{1}{2\pi}\int_{-\pi/4}^{\pi/4} e^{j\omega(n-1)}d\omega + \frac{2}{2\pi}\int_{\pi/4}^{\pi/2} *e^{j\omega(n-1)}d\omega \\ &= \frac{2}{2\pi j(n-1)}(e^{j\omega(n-1)})|_{-\pi/2}^{-\pi/4} + \frac{1}{j(n-1)}(e^{j\omega(n-1)})|_{-\pi/4}^{\pi/4} + \frac{2}{j(n-1)}(e^{j\omega(n-1)})|_{\pi/4}^{\pi/2} \\ &= \frac{1}{2\pi j(n-1)}(2e^{-j\pi(n-1)/4} - 2e^{-j\pi(n-1)/2} + e^{j\pi(n-1)/4} - e^{-j\pi(n-1)/4} + 2e^{j\pi(n-1)/2} - 2e^{j\pi(n-1)/4}) \end{aligned}$$

Rearranging terms,

$$\begin{aligned} x[n] &= \frac{1}{2\pi j(n-1)}((2e^{j\pi(n-1)/2} - 2e^{-j\pi(n-1)/2}) + (2e^{-j\pi(n-1)/4} - e^{-j\pi(n-1)/4}) + (e^{j\pi(n-1)/4} - 2e^{j\pi(n-1} \\ &= \frac{1}{2\pi j(n-1)}((2e^{j\pi(n-1)/2} - 2e^{-j\pi(n-1)/2}) + e^{-j\pi(n-1)/4} - e^{j\pi(n-1)/4}) \\ &= \frac{1}{\pi(n-1)}(2(\frac{1}{2j}e^{j\pi(n-1)/2} - \frac{1}{2j}e^{-j\pi(n-1)/2}) - (\frac{1}{2j}e^{j\pi(n-1)/4} - \frac{1}{2j}e^{-j\pi(n-1)/4}) \end{aligned}$$

Using the definition of the sin function,

$$\sin(x) = \frac{1}{2j}e^{jx} - \frac{1}{2j}e^{-jx}$$

, We can rewrite this as

$$\frac{1}{\pi(n-1)}(2\sin(\pi(n-1)/2) - \sin(\pi(n-1)/4))$$

ii) Solution 2 (Transform pairs) (Extra credit)

Consider a version $X_1(e^{j\omega}$ of $X(e^{j\omega})$ with zero phase, that is

$$|X_1(e^{j\omega})| = |X(e^{j\omega})|, \quad \angle X_1(e^{j\omega}) = 0$$

Such that

$$\begin{aligned} X_1(e^{j\omega}) &= |X_1(e^{j\omega})|e^{j\angle X_1(e^{j\omega})} \\ &= |X(e^{j\omega})|e^{j*0} \\ &= |X(e^{j\omega})| \end{aligned}$$

Then we can write $X(e^{j\omega})$ as

$$\begin{aligned} X(e^{j\omega}) &= |X(e^{j\omega})|e^{j\angle X(e^{j\omega})} \\ &= X_1(e^{j\omega})e^{-j\omega} \end{aligned}$$

When comparing this expression with the time delay property, we can see that the phase $e^{-j\omega}$ is really just shifting the signal $x_1[n]$ by 1 unit, so that

$$x[n] = x_1[n-1]$$

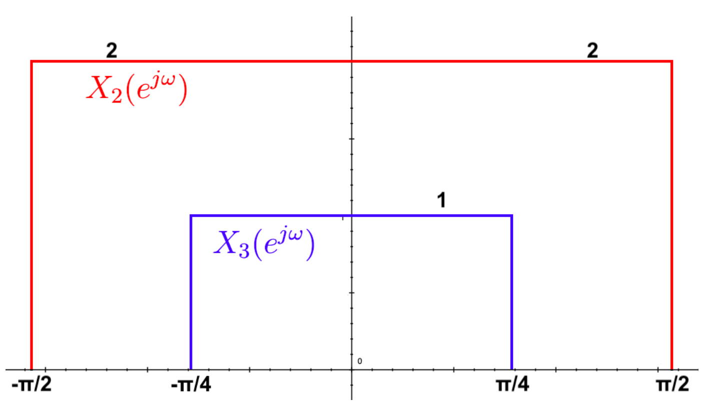
Effectively, this allows us to just find the inverse Fourier transform of the magnitude on it's own, and then apply a time shift of 1 unit at the end to account for the phase of the system. This is a nice property that we can use in general on systems that have linear phase

To determine $x_1[n]$, we consider the rectangle wave $X_1(e^{j\omega})$ as a difference of two rectangle waves. Specifically, define

$$X_2(e^{j\omega}) = \begin{cases} 2, & -\pi/2 < \omega < \pi/2 \\ 0, & \text{else} \end{cases}$$

$$X_3(e^{j\omega}) = \begin{cases} 1, & -\pi/4 < \omega < \pi/4 \\ 0, & \text{else} \end{cases}$$

or graphically,



Then it is clear that

$$X_1(e^{j\omega}) = X_2(e^{j\omega}) - X_3(e^{j\omega})$$

and therefore

$$x_1[n] = x_2[n] - x_3[n]$$

In table 5.2, we see the Fourier transform pair

$$\mathcal{F}^{-1}\{\begin{cases} 1, & -W < \omega < W \\ 0, & \text{else} \end{cases}\} = \frac{\sin(Wn)}{\pi n}$$

Since $X_2(e^{j\omega})$ and $X_3(e^{j\omega})$ are already in this form, we can easily find $x_2[n]$ and $x_3[n]$ as

$$x_2[n] = \frac{2\sin(\pi n/2)}{\pi n}$$

$$x_3[n] = \frac{\sin(\pi n/4)}{\pi n}$$

Then

$$x_1[n] = x_2[n] - x_3[n] = \frac{2\sin(\pi n/2)}{\pi n} - \frac{\sin(\pi n/4)}{\pi n}$$

and

$$x[n] = x_1[n-1] = \frac{2\sin(\pi(n-1)/2)}{\pi(n-1)} - \frac{\sin(\pi(n-1)/4)}{\pi(n-1)}$$

2) The main focus of this problem is to break the expression up fractally, handle it in small portions, and then build it back up into the final answer. From our initial expression,

$$X(j\omega) = (\frac{4}{9+\omega^2}) * (e^{-2j\omega}\frac{\sin(2\omega)}{\omega})$$

Make the definitions

$$X_1(j\omega) = \frac{4}{9+\omega^2}$$

$$X_2(j\omega) = e^{-2j\omega}\frac{\sin(2\omega)}{\omega}$$

So that

$$X(j\omega) = X_1(j\omega) * X_2(j\omega)$$

and, using the multiplication property,

$$x(t) = 2\pi x_1(t)x_2(t)$$

i) To handle $X_1(j\omega) = \frac{4}{9+\omega^2}$, we see that this expression has similar form to the Fourier Transform pair found in the provided tables,

$$\mathcal{F}\{e^{-\alpha|t|}\} = \frac{2\alpha}{\alpha^2+\omega^2}$$

In order to apply this pair to $X_1(j\omega)$, we need to have it in this exact form. Therefore, we need to say that

$$X_1(j\omega) = \frac{4}{9+\omega^2} = \frac{4}{6} * \frac{2(3)}{3^2+\omega^2}$$

to find that

$$x_1(t) = \frac{4}{6}e^{-3|t|}$$

ii) To handle $X_2(j\omega) = e^{-2j\omega}\frac{\sin(2\omega)}{\omega}$, we again need to break the problem up into smaller parts. We make the additional definition that

$$X_3(j\omega) = \frac{\sin(2\omega)}{\omega}$$

so that

$$X_2(j\omega) = e^{-2j\omega} X_3(j\omega)$$

From this, we can clearly invoke the same time shifting property that we used in problem 1(b) to show that

$$x_2(t) = x_3(t-2)$$

Then, to find $x_3(t)$, we see the Fourier transform pair

$$\mathcal{F}\{\begin{cases} 1, & -T_1 < t < T_1 \\ 0, & \text{else} \end{cases}\} = \frac{2\sin(\omega T_1)}{\omega}$$

We again need to exactly match $X_3(j\omega)$ to this form to invoke the property. By rewriting

$$X_3(j\omega) = \frac{\sin(2\omega)}{\omega} = \frac{1}{2} * \frac{2\sin(2\omega)}{\omega}$$

We can then use this property to say that

$$x_3(t) = \begin{cases} \frac{1}{2}, & -2 < t < 2 \\ 0, & \text{else} \end{cases}$$

then

$$x_2(t) = x_3(t-2) = \begin{cases} \frac{1}{2}, & -2 < t-2 < 2 \\ 0, & \text{else} \end{cases} = \begin{cases} \frac{1}{2}, & 0 < t < 4 \\ 0, & \text{else} \end{cases}$$

In general, if something says $|t| < T$, this is the same as $-T < t < T$.

Now that we know $x_1(t)$ and $x_2(t)$, we can use our original relation

$$x(t) = 2\pi x_1(t) x_2(t)$$

to write

$$x(t) = 2\pi * \frac{4}{6}e^{-3|t|} * \begin{cases} \frac{1}{2}, & 0 < t < 4 \\ 0, & \text{else} \end{cases} = \begin{cases} \frac{2\pi}{3}e^{-3|t|}, & 0 < t < 4 \\ 0, & \text{else} \end{cases}$$

4) We start by computing the Nyquist frequency for each signal $x_1(t)$, $x_2(t)$, $x_3(t)$ and $x(t)$. This is twice the maximum frequency present in each signal. Then

$$\omega_{\text{Nyquist, } x_1} = 2 * \pi, \quad f_{\text{Nyquist, } x_1} = \frac{1}{2\pi}\omega_{\text{Nyquist, } x_1} = 1\text{Hz}$$

$$\omega_{\text{Nyquist, } x_2} = 2 * 3\pi, \quad f_{\text{Nyquist, } x_2} = \frac{1}{2\pi}\omega_{\text{Nyquist, } x_2} = 3\text{Hz}$$
$$\omega_{\text{Nyquist, } x_3} = 2 * 5\pi, \quad f_{\text{Nyquist, } x_3} = \frac{1}{2\pi}\omega_{\text{Nyquist, } x_3} = 5\text{Hz}$$
$$\omega_{\text{Nyquist, } x} = 2 * 5\pi, \quad f_{\text{Nyquist, } x} = \frac{1}{2\pi}\omega_{\text{Nyquist, } x} = 5\text{Hz}$$

a) The sampling rate $T = 0.4$ corresponds to a sampling frequency $f_s = \frac{1}{T} = 2.5\text{Hz}$. This is less than the Nyquist frequencies of $x_2(t)$, $x_3(t)$, and $x(t)$, so these signals cannot be recovered in this sort of sampling scheme. However, $x_1(t)$ can be.

b) As we showed earlier, $f_{\text{Nyquist, } x} = \frac{1}{2\pi}\omega_{\text{Nyquist, } x} = 5\text{Hz}$

c) By the multiplication property, for $y(t) = x(t)c(t)$,

$$Y(j\omega) = \frac{1}{2\pi}X(j\omega) * C(j\omega)$$

Using the properties of cosines and sines, we can quickly find that

$$C(j\omega) = \pi\delta(\omega - 20\pi) + \pi\delta(\omega + 20\pi)$$

Then

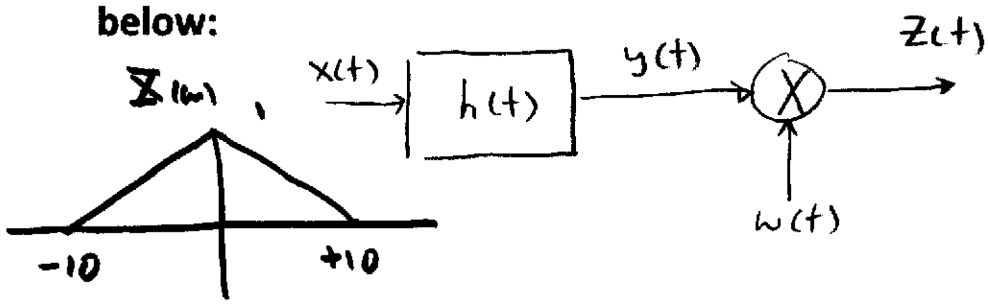
$$Y(j\omega) = \frac{1}{2\pi}(\pi\delta(\omega - 20\pi) + \pi\delta(\omega + 20\pi)) * X(j\omega)$$

Using the properties of convolution with delta functions,

$$Y(j\omega) = \frac{1}{2}X(j(\omega - 20\pi)) + \frac{1}{2}X(j(\omega + 20\pi))$$

This means that $Y(j\omega)$ will contain all the frequencies present in $X(j\omega)$, shifted either up or down by 20π radians. As the maximum frequency in $X(j\omega)$ is 5π, the maximum frequency in $Y(j\omega)$ will be that shifted up by 20π, so $20\pi + 5\pi = 25\pi$. This can be confirmed by finding $X(j\omega)$ and substituting this into the above expression for $Y(j\omega)$, which I will not do here for sake of legibility.

Problem 3. (25 pts.) Consider the filtering and modulation system below:



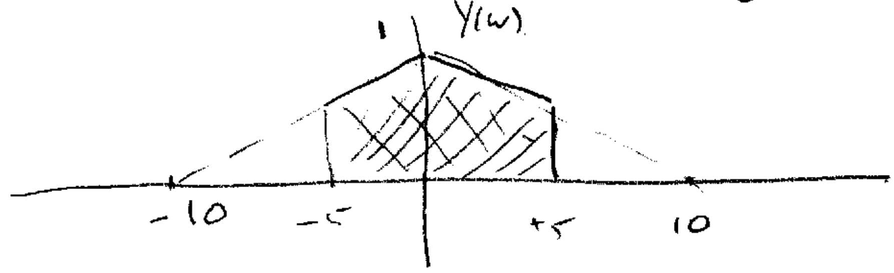
$h(t) = \frac{\sin(5t)}{\pi t}$

$w(t) = \cos(5t) + 2\cos(15t)$

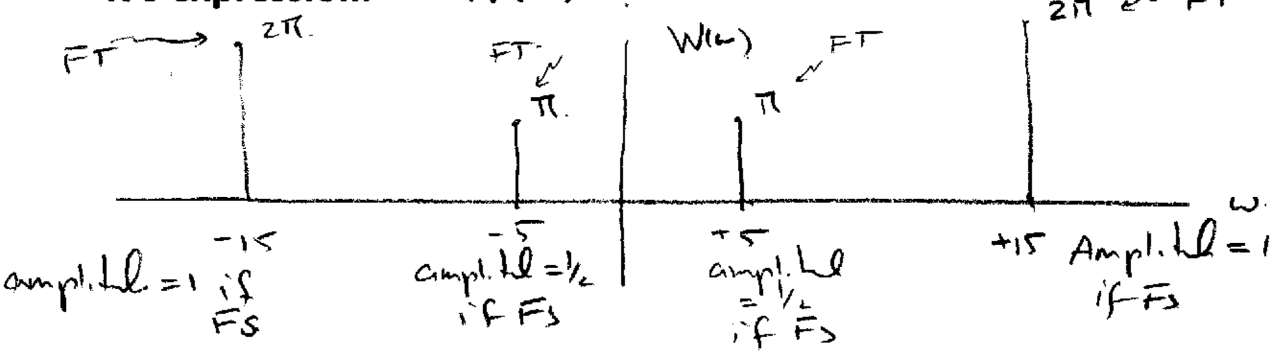
A. (10 pts.) Sketch the Fourier Transform of signal y(t) and write up it's expression. $Y(\omega) = ?$

$Y(\omega) = X(\omega) * H(\omega)$

$y(t) = x(t) \cdot h(t)$



B. (5 pts.) Sketch the Fourier transform of signal w(t) and write up it's expression. $W(\omega) = ?$



C. (10 pts.) Sketch the Fourier transform of signal z(t) and obtain it's expression. $Z(\omega) = ?$