HW 8

EE 3015
Signals and Systems

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Contents

1 Problem 9.3

Consider the signal $x(t) = e^{-5t}u(t) + e^{-\beta t}u(t)$ and denote its Laplace transform by $X(s)$ What are the constraints placed on the

real and imaginary parts of β if the region of convergence of $X(s)$ is $\mathrm{Re}(s) > -3$?

solution

The Laplace transform is

$$\begin{aligned} X(s) &= \int_{-\infty}^{\infty} x(t)\, e^{-st} dt \\ &= \int_0^{\infty} \left(e^{-5t} + e^{-\beta t}\right) e^{-st} dt \\ &= \int_0^{\infty} e^{-5t} e^{-st} dt + \int_0^{\infty} e^{-\beta t} e^{-st} dt \\ &= \int_0^{\infty} e^{-t(5+s)} dt + \int_0^{\infty} e^{-t\left(\beta+s\right)} dt \\ &= \frac{1}{-(s+5)} \left[e^{-t(5+s)}\right]_0^{\infty} - \frac{1}{\beta+s} \left[e^{-t\left(\beta+s\right)}\right]_0^{\infty} \end{aligned}$$

For the first term $\frac{1}{-(s+5)} \left[e^{-t(5+s)}\right]_0^{\infty} = \frac{1}{-(s+5)} \left[e^{-\infty(5+s)} - 1\right]$. For this term to converge we need $5 + \mathrm{Re}(s) > 0$ or

$$\mathrm{Re}(s) > -5$$

For the second term, let $\beta = a + ib$ and let $s = \sigma + j\omega$, hence the second term becomes

$$\begin{aligned} \frac{1}{\beta+s} \left[e^{-t\left(\beta+s\right)}\right]_0^{\infty} &= \frac{1}{\beta+s} \left[e^{-t\left((a+ib)+\left(\sigma+j\omega\right)\right)}\right]_0^{\infty} \\ &= \frac{1}{\beta+s} \left[e^{-t\left(a+\sigma+j(b+\omega)\right)}\right]_0^{\infty} \\ &= \frac{1}{\beta+s} \left[e^{-t(a+\sigma)} e^{-jt(b+\omega)}\right]_0^{\infty} \\ &= \frac{1}{\beta+s} \left[e^{-\infty(a+\sigma)} e^{-j\infty(b+\omega)} - 1\right] \end{aligned}$$

The complex exponential terms always converges since its norm is bounded by 1. For the real exponential term, we need $a + \sigma > 0$ or $a + \mathrm{Re}(s) > 0$ or $\mathrm{Re}(s) > -a$. Since we are told that $\mathrm{Re}(s) > -3$, then

$$a = 3$$

Is the requirement on real part of β. There is no restriction on the imaginary part of β.

2 Problem 9.9

Given that $e^{-at}u(t) \Longleftrightarrow \frac{1}{s+a}$ for $\operatorname{Re}(s) > \operatorname{Re}(-a)$, determine the inverse Laplace transform of

$$X(s) = \frac{2(s+2)}{s^2+7s+12} \qquad \operatorname{Re}(s) > -3$$

<u>solution</u>

Writing $X(s)$ as

$$\begin{aligned} X(s) &= \frac{2(s+2)}{(s+4)(s+3)} \\ &= \frac{A}{(s+4)} + \frac{B}{(s+3)} \end{aligned}$$

Hence $A = \left.\frac{2(s+2)}{(s+3)}\right|_{s=-4} = \frac{2(-4+2)}{(-4+3)} = 4$ and $B = \left.\frac{2(s+2)}{(s+4)}\right|_{s=-3} = \frac{2(-3+2)}{(-3+4)} = -2$, therefore the above becomes

$$X(s) = \frac{4}{(s+4)} - \frac{2}{(s+3)}$$

Using $e^{-at}u(t) \Longleftrightarrow \frac{1}{s+a}$ gives the inverse Laplace transform as

$$\begin{aligned} x(t) &= 4e^{-4t}u(t) - 2e^{-3t}u(t) \\ &= \left(4e^{-4t} - 2e^{-3t}\right)u(t) \end{aligned}$$

With $\operatorname{Re}(s) > -4$ and also $\operatorname{Re}(s) > -3$. Therefore the ROC for both is $\operatorname{Re}(s) > -3$.

3 Problem 9.15

Consider the two right-sides signals $x(t), y(t)$ related through the differential equations

$$\frac{dx(t)}{dt} = -2y(t) + \delta(t)$$
$$\frac{dy(t)}{dt} = 2x(t)$$

Determine $Y(s), X(s)$ along with their ROC.

solution

The Laplace transform of $\delta(t)$ is 1. Taking the Laplace transform of both the ODE's above, and assuming zero initial conditions gives

$$sX(s) = -2Y(s) + 1 \tag{1}$$
$$sY(s) = 2X(s) \tag{2}$$

Using the second equation in the first gives

$$\begin{aligned} sX(s) &= -2\left(\frac{2X(s)}{s}\right) + 1 \\ &= \frac{-4X(s) + s}{s} \\ s^2 X(s) &= -4X(s) + s \\ \left(s^2 + 4\right) X(s) &= s \\ X(s) &= \frac{s}{\left(s^2 + 4\right)} \end{aligned}$$

Using the above in (2) gives $Y(s)$

$$\begin{aligned} sY(s) &= 2\frac{s}{\left(s^2 + 4\right)} \\ Y(s) &= \frac{2}{\left(s^2 + 4\right)} \end{aligned}$$

Considering $X(s)$ to find its ROC, let us write it as

$$X(s) = \frac{s}{\left(s^2 + 4\right)} = \frac{s}{\left(s + 2j\right)\left(s - 2j\right)} = \frac{A}{\left(s + 2j\right)} + \frac{B}{\left(s - 2j\right)}$$

We see that the ROC for first term is $\operatorname{Re}(s) > -\operatorname{Re}\left(2j\right)$ which means $\operatorname{Re}(s) > 0$ since real part is zero. Same for the second term. Hence we see that for $X(s)$ the ROC is $\operatorname{Re}(s) > 0$. Similarly for $Y(s)$. Therefore the overall ROC is

$$\operatorname{Re}(s) > 0$$

4 Problem 9.32

A causal LTI system with impulse response $h(t)$ has the following properties: (1) When the input to the system is $x(t) = e^{2t}$ for all t, the output is $y(t) = \frac{1}{6}e^{2t}$ for all t. (2) The impulse response $h(t)$ satisfies the differential equation

$$\frac{dh(t)}{dt} + 2h(t) = e^{-4t}u(t) + bu(t)$$

Where b is unknown constant. Determine the system function $H(s)$ of the system, consistent with the information above. There should be no unknown constants in your answer; that is, the constant b should not appear in the answer

solution

First $H(s)$ is found from the differential equation. Taking Laplace transform gives (assuming zero initial conditions)

$$\begin{aligned} sH(s) + 2H(s) &= \frac{1}{s+4} + \frac{b}{s} \\ H(s)(2+s) &= \frac{1}{s+4} + \frac{b}{s} \\ H(s) &= \frac{1}{(s+4)(s+2)} + \frac{b}{s(s+2)} \\ &= \frac{s + b(s+4)}{s(s+4)(s+2)} \end{aligned} \tag{1}$$

Now we are told when the input is e^{2t} then the output is $\frac{1}{6}e^{2t}$. In Laplace domain this means $Y(s) = X(s)H(s)$. Therefore

$$\begin{aligned} Y(s) &= \frac{1}{6}\frac{1}{s-2} \qquad \operatorname{Re}(s) > 2 \\ X(s) &= \frac{1}{s-2} \qquad \operatorname{Re}(s) > 2 \end{aligned}$$

Hence

$$\begin{aligned} H(s) &= \frac{Y(s)}{X(s)} \\ H(s) &= \frac{\frac{1}{6}\frac{1}{s-2}}{\frac{1}{s-2}} \\ &= \frac{1}{6} \end{aligned} \tag{2}$$

Comparing (1,2) then

$$\frac{1}{6} = \frac{s + b(s+4)}{s(s+4)(s+2)}$$

Solving for b gives

$$\begin{aligned} \frac{s(s+4)(s+2)}{6} &= s + b(s+4) \\ \frac{s(s+4)(s+2)}{6(s+4)} - \frac{s}{(s+4)} &= b \\ b &= \frac{s(s+2)}{6} - \frac{s}{(s+4)} \\ &= \frac{s(s+2)(s+4) - 6s}{6(s+4)} \\ &= \frac{s\left(s^2 + 6s + 2\right)}{6(s+4)} \end{aligned}$$

This is true for $\operatorname{Re}(s) > 2$. Hence for $s = 2$ the above reduces to

$$\begin{aligned} b &= \frac{2(4+12+2)}{6(2+4)} \\ &= 1 \end{aligned}$$

Therefore (1) becomes

$$\begin{aligned}
H(s) &= \frac{s + (s+4)}{s(s+4)(s+2)} \\
&= \frac{2s+4}{s(s+4)(s+2)} \\
&= \frac{2(s+2)}{s(s+4)(s+2)} \\
&= \frac{2}{s(s+4)}
\end{aligned}$$

5 Problem 9.40

Consider the system S characterized by the differential equation

$$y'''(t) + 6y''(t) + 11y'(t) + 6y(t) = x(t)$$

(a) Determine the zero-state response of this system for the input $x(t) = e^{-4t}u(t)$ (b) Determine the zero-input response of the system for $t > 0^-$ given the initial conditions $y(0^-) = 1, \left.\frac{dy}{dt}\right|_{t=0^-} = -1, \left.\frac{d^2y}{dt^2}\right|_{t=0^-} = 1$. (c) Determine the output of S when the input is $x(t) = e^{-4t}u(t)$ and the initial conditions are the same as those specified in part (b).

Solution

5.1 Part a

Applying Laplace transform on the ODE and using zero initial conditions gives

$$\begin{aligned}
s^3Y(s) + 6s^2Y(s) + 11sY(s) + 6Y(s) &= \frac{1}{s+4} \\
Y(s)\left(s^3 + 6s^2 + 11s + 6\right) &= \frac{1}{s+4} \\
Y(s) &= \frac{1}{(s+4)\left(s^3 + 6s^2 + 11s + 6\right)} \\
&= \frac{1}{(s+4)(s+1)(s+2)(s+3)}
\end{aligned} \tag{1}$$

Using partial fractions

$$\frac{1}{(s+4)(s+1)(s+2)(s+3)} = \frac{A}{(s+4)} + \frac{B}{(s+1)} + \frac{C}{(s+2)} + \frac{D}{(s+3)}$$

Hence

$$\begin{aligned}
A &= \left.\frac{1}{(s+1)(s+2)(s+3)}\right|_{s=-4} = \frac{1}{(-4+1)(-4+2)(-4+3)} = -\frac{1}{6} \\
B &= \left.\frac{1}{(s+4)(s+2)(s+3)}\right|_{s=-1} = \frac{1}{(-1+4)(-1+2)(-1+3)} = \frac{1}{6} \\
C &= \left.\frac{1}{(s+4)(s+1)(s+3)}\right|_{s=-2} = \frac{1}{(-2+4)(-2+1)(-2+3)} = \frac{-1}{2} \\
D &= \left.\frac{1}{(s+4)(s+1)(s+2)}\right|_{s=-3} = \frac{1}{(-3+4)(-3+1)(-3+2)} = \frac{1}{2}
\end{aligned}$$

Hence (1) becomes

$$\begin{aligned}
Y(s) &= \frac{A}{(s+4)} + \frac{B}{(s+1)} + \frac{C}{(s+2)} + \frac{D}{(s+3)} \\
&= -\frac{1}{6}\frac{1}{(s+4)} + \frac{1}{6}\frac{1}{(s+1)} - \frac{1}{2}\frac{1}{(s+2)} + \frac{1}{2}\frac{1}{(s+3)} \qquad \operatorname{Re}(s) > -1
\end{aligned}$$

From tables, the inverse Laplace transform is (one sided) is

$$y(t) = -\frac{1}{6}e^{-4t}u(t) + \frac{1}{6}e^{-t}u(t) - \frac{1}{2}e^{-2t}u(t) + \frac{1}{2}e^{-3t}u(t)$$

5.2 Part b

Applying Laplace transform on the ODE $y'''(t) + 6y''(t) + 11y'(t) + 6y(t) = 0$ and using the non-zero initial conditions given above gives

$$\begin{aligned}
\left(s^3Y(s) - s^2y(0) - sy'(0) - y''(0)\right) + 6\left(s^2Y(s) - sy(0) - y'(0)\right) + 11\left(sY(s) - y(0)\right) + 6Y(s) &= 0 \\
\left(s^3Y(s) - s^2 + s - 1\right) + 6\left(s^2Y(s) - s + 1\right) + 11\left(sY(s) - 1\right) + 6Y(s) &= 0 \\
s^3Y(s) - s^2 + s - 1 + 6s^2Y(s) - 6s + 6 + 11sY(s) - 11 + 6Y(s) &= 0 \\
Y(s)\left(s^3 + 6s^2 + 11s + 6\right) - s^2 + s - 1 - 6s + 6 - 11 &= 0
\end{aligned} \tag{1}$$

Hence

$$\begin{aligned}
Y(s)\left(s^3+6s^2+11s+6\right) &= s^2-s+1+6s-6+11\\
Y(s) &= \frac{s^2+5s+6}{s^3+6s^2+11s+6}\\
&= \frac{(s+3)(s+2)}{(s+1)(s+2)(s+3)}\\
&= \frac{1}{s+1} \qquad \operatorname{Re}(s) > -1
\end{aligned}$$

Hence the inverse Laplace transform (one sided) gives

$$y(t) = e^{-t}u(t)$$

5.3 Part c

This is the sum of the response of part(a) and part(b) since the system is linear ODE. Hence

$$\begin{aligned}
y(t) &= -\frac{1}{6}e^{-4t}u(t)+\frac{1}{6}e^{-t}u(t)-\frac{1}{2}e^{-2t}u(t)+\frac{1}{2}e^{-3t}u(t)+e^{-t}u(t)\\
&= \left(-\frac{1}{6}e^{-4t}+\frac{1}{6}e^{-t}-\frac{1}{2}e^{-2t}+\frac{1}{2}e^{-3t}+e^{-t}\right)u(t)\\
&= \left(-\frac{1}{6}e^{-4t}+\frac{7}{6}e^{-t}-\frac{1}{2}e^{-2t}+\frac{1}{2}e^{-3t}\right)u(t)
\end{aligned}$$