HW 7

EE 3015
Signals and Systems

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Contents

1 Problem 7.2

A continuous-time signal $x(t)$ is obtained at the output of an ideal lowpass filter with cutoff frequency $\omega_c = 1000\pi$ rad/sec. If impulse-train sampling is performed on $x(t)$, which of the following sampling periods would guarantee that $x(t)$ can be recovered from its sampled version using an appropriate lowpass filter? (a) $T = 0.5 \times 10^{-3}$ sec. (b) $T = 2 \times 10^{-3}$ sec (c) $T = 10^{-4}$ sec

solution

Note: In all these problems, I will use Ω for the digital frequency and ω for the continuous frequency.

We want the Nyquist frequency to be larger than twice ω_c. Hence Nyquist frequency should be larger than 2000π rad/sec or larger than 1000 Hz.

Translating the given periods to hertz using $f = \frac{1}{T}$ relation, shows that (a) is $\frac{1}{0.5\times10^{-3}} = 2000$ Hz, (b) is $\frac{1}{2\times10^{-3}} = 500$ Hz, (c) is $\frac{1}{10^{-4}} = 10000$ Hz.

Therefore (a) and (c) would guarantee that $x(t)$ can be recovered.

2 Problem 7.6

filter that gives $x(t)$ as its output when $y(t)$ is the input.

7.6. In the system shown in Figure P7.6, two functions of time, $x_1(t)$ and $x_2(t)$, are multiplied together, and the product $w(t)$ is sampled by a periodic impulse train. $x_1(t)$ is band limited to ω_1, and $x_2(t)$ is band limited to ω_2; that is,

$$X_1(j\omega) = 0, |\omega| \geq \omega_1,$$

$$X_2(j\omega) = 0, |\omega| \geq \omega_2.$$

Determine the *maximum* sampling interval T such that $w(t)$ is recoverable from $w_p(t)$ through the use of an ideal lowpass filter.

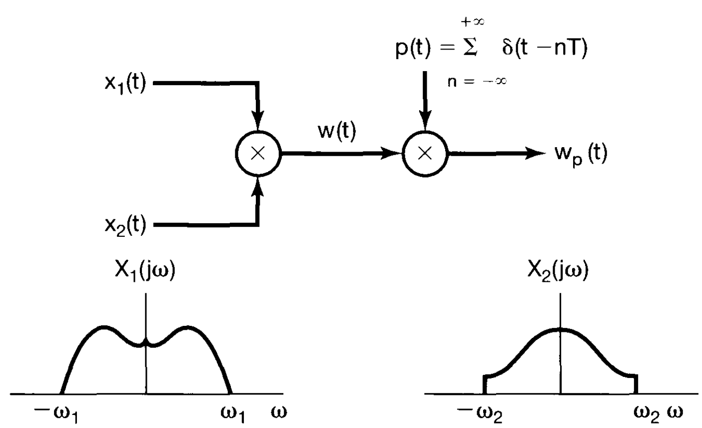

Figure P7.6

Figure 1: Problem description

<u>solution</u>

The multiplication of $x_1(t) \times x_2(t)$ becomes convolution in frequency domain $X_1(j\omega) \circledast X_2(j\omega)$. But we know when doing convolution the width of the result is the sum of each function width. This means the frequency spectrum of $w(t)$ will have width of $\omega_1 + \omega_2$.

Now by Nyquist theory, we know that the sampling frequency should be at least twice the largest frequency in the signal being sampled. This means

$$\omega_{sampling} > 2(\omega_1 + \omega_2)$$

Since $\omega_1 + \omega_2$ is now the largest frequency present in $w(t)$. But $\omega_{sampling} = \frac{2\pi}{T_{sampling}}$. Hence the above becomes

$$\frac{2\pi}{T_{sampling}} > 2\left(\omega_1 + \omega_2\right)$$
$$\frac{1}{T_{sampling}} > \frac{\omega_1 + \omega_2}{\pi}$$

Or

$$T_{sampling} < \frac{\pi}{\omega_1 + \omega_2}$$

This means the maximum possible sampling period is

$$T_{\max} = \frac{\pi}{\omega_1 + \omega_2}$$

In seconds.

3 Problem 7.11

Let $X_c(t)$ be a continuous-time signal whose Fourier transform has the property that $X_c(\omega) = 0$ for $|\omega| \geq 2000\pi$. A discrete-time signal

$$x_d[n] = x_n\left(n\left(0.5 \times 10^{-3}\right)\right)$$

Is obtained. For each of the following constraints on the Fourier transform $X_d(\Omega)$ of $x_d[n]$ determine the corresponding constraint on $X_c(\omega)$.

a $X_d(\Omega)$ is real

b The maximum value of $X_d(\Omega)$ over all Ω is 1.

c $X_d(\Omega) = 0$ for $\frac{3\pi}{4} \leq |\Omega| \leq \pi$

d $X_d(\Omega) = X_d(\Omega - \pi)$

solution

The main relation to translate between continuous time frequency ω (radians per second) and digital frequency Ω (radians per sample) which is used in all of these parts is the following

$$\Omega = \omega T$$

Where T is the sampling period (in seconds per sample). i.e. number of seconds to obtain one sample.

3.1 Part a

Since $X_d(\Omega)$ is the same as $X_c(\omega)$ (except for scaling factor) which contains replicated copies of $X_c(\omega)$ spaced at sampling frequencies intervals, then if $X_d(\Omega)$ is real, then this means $X_c(\omega)$ must also be real, since $X_d(\Omega)$ is just copies of $X_c(\omega)$.

3.2 Part b

If maximum value of $X_d(\Omega)$ is A then maximum value of $X_c(\omega)$ is given by AT where T is sampling period. Hence since $A = 1$ in this problem, then the maximum value of $X_c(\omega)$ will be 0.5×10^{-3}.

3.3 Part c

$X_d(\Omega) = 0$ for $\frac{3\pi}{4} \leq |\Omega| \leq \pi$ is translated to $X_c(\omega) = 0$ for $\frac{3\pi}{4} \leq |\omega T| \leq \pi$ since $\Omega = \omega T$. Therefore

$$\frac{3\pi}{4}\frac{1}{T} \leq |\omega| \leq \frac{\pi}{T}$$

But $T = 0.5 \times 10^{-3}$, hence the above becomes

$$\frac{3\pi}{4}(2000) \leq |\omega| \leq \pi(2000)$$
$$1500\pi \leq |\omega| \leq 2000\pi$$

Hence $X_c(\omega) = 0$ for $1500\pi \leq |\omega| \leq 2000\pi$. Actually, since $X_c(\omega) = 0$ for $|\omega| \geq 2000\pi$ from the problem statement, this can be simplified to

$$X_c(\omega) = 0 \qquad |\omega| \geq 500\pi$$

3.4 Part d

$X_d(\Omega) = X_d(\Omega - \pi)$ is translated to $X_c(\omega) = X_c\left(\omega - \frac{\pi}{T}\right) = X_c\left(\omega - \frac{\pi}{0.5 \times 10^{-3}}\right) = X_c(\omega - 2000\pi)$

Therefore

$$X_c(\omega) = X_c(\omega - 2000\pi)$$

4 Problem 7.15

Impulse-train sampling of $x[n]$ is used to obtain

$$g[n] = \sum_{k=-\infty}^{\infty} x[n]\,\delta[n-kN]$$

If $X(\Omega) = 0$ for $\frac{3\pi}{7} \leq |\Omega| \leq \pi$, determine the largest value for the sampling interval N which ensures that no aliasing takes place while sampling $x[n]$.

solution

This is similar to problem 7.6 above, but using digital frequency. By Nyquist theory, the sampling frequency must be larger than twice the largest frequency in the signal. We are given that $\frac{3\pi}{7} \leq |\Omega| \leq \pi$. Hence the largest frequency is $\frac{3\pi}{7}$. Hence,

$$\Omega_{sampling} > 2\left(\frac{3\pi}{7}\right) = \frac{6}{7}\pi$$

Therefore

$$\frac{2\pi}{N_{sampling}} > \frac{6}{7}\pi$$

Where $N_{sampling}$ is the discrete sampling period, which is number of samples. Therefore from above

$$\frac{1}{N_{sampling}} > \frac{3}{7}$$
$$N_{sampling} < \frac{7}{3}$$

But $N_{sampling}$ must be an integer (since it is number of samples, hence

$$N_{sampling} < 2$$

Therefore the maximum is

$$N = 2$$