HW 6

EE 3015
Signals and Systems

Spring 2020
University of Minnesota, Twin Cities

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May 27, 2020 Compiled on May 27, 2020 at 12:24am

Contents

1 Problem 6.2

Consider a discrete-time LTI system with frequency response $H(\Omega) = |H(\Omega)| e^{j\arg H(\Omega)}$ and real impulse response $h[n]$. Suppose that we apply the input $x[n] = \sin(\Omega_0 n + \phi_0)$ to this system. The resulting output can be shown to be of the form $y[n] = |H(\Omega_0)| x[n - n_0]$ provided that $\arg H(\Omega_0)$ and Ω_0 are related in a particular way. Determine this relationship.

solution

From standard LTI theory, the output is given by

$$y[n] = |H(\Omega_0)| \sin(\Omega_0 n + \phi_0 + \arg(H(\Omega_0))) \tag{1}$$

Comparing the above to

$$|H(\Omega_0)| x[n - n_0] = |H(\Omega_0)| \sin(\Omega_0 (n - n_0) + \phi_0) \tag{2}$$

Shows that

$$\begin{aligned} \sin(\Omega_0 (n - n_0) + \phi_0) &= \sin(\Omega_0 n + \phi_0 + \arg(H(\Omega_0))) \\ \sin(\Omega_0 n - \Omega_0 n_0 + \phi_0) &= \sin(\Omega_0 n + \phi_0 + \arg(H(\Omega_0))) \end{aligned}$$

Hence we need

$$-\Omega_0 n_0 = \arg(H(\Omega_0))$$

Since the input is periodic of period $2\pi k$ for k integer, then the above can also be written as

$$-\Omega_0 n_0 + 2\pi k = \arg(H(\Omega_0))$$

This is the relation needed.

2 Problem 6.5

Consider a continuous-time ideal bandpass filter whose frequency response is

$$H(\omega) = \begin{cases} 1 & \omega_c \leq |\omega| \leq 3\omega_c \\ 0 & \text{elsewhere} \end{cases}$$

(a) If $h(t)$ is the impulse response of this filter, determine a function $g(t)$ such that $h(t) = \left(\frac{\sin \omega_c t}{\pi t}\right) g(t)$. (b) As ω_c is increased, does the impulse response of the filter get more concentrated or less concentrated about the origin?

solution

2.1 Part a

Let $f(t) = \frac{\sin \omega_c t}{\pi t}$ which is a sinc function. The following is a sketch of $H(\omega)$ and of the CTFT of $f(t)$, i.e. $F(\omega)$ which we know will be rectangle since Fourier transform of rectangle is sinc.

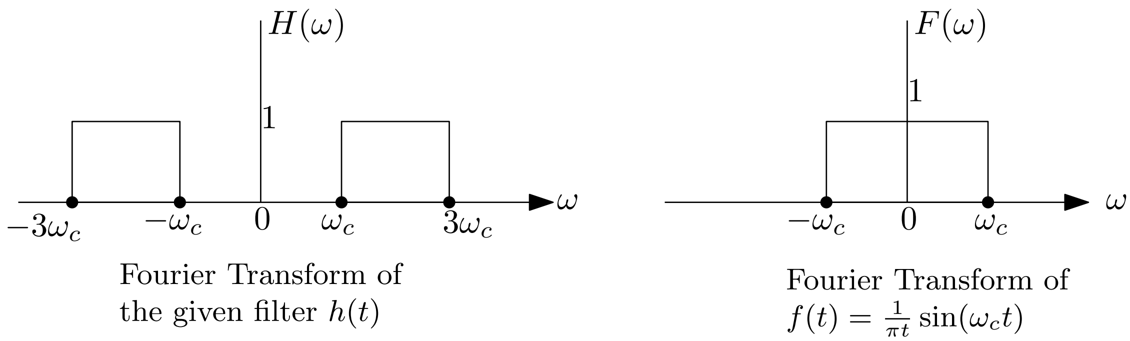
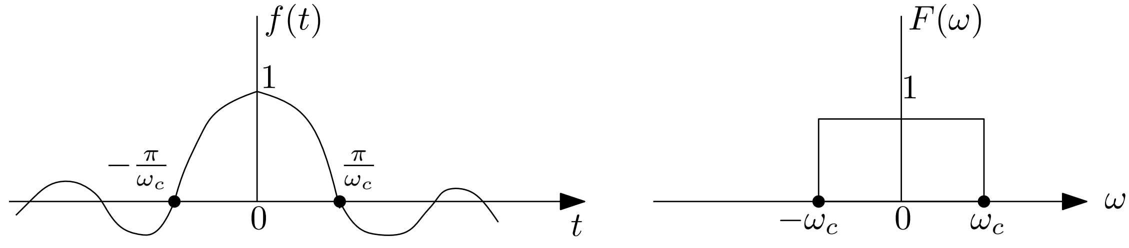


Figure 1: Sketch of CTFT of filter and sinc function

The relation between $\frac{\sin \omega_c t}{\pi t}$ and its CTFT is given in this sketch



$f(t) = \frac{1}{\pi t}\sin(\omega_c t)$

Fourier Transform of $f(t) = \frac{1}{\pi t}\sin(\omega_c t)$

Figure 2: Sketch of sinc function and its CTFT

Now, we know that, since $h(t) = f(t)\,g(t)$, then by modulation theory, this results in $2\pi H(\omega) = F(\omega) \circledast G(\omega)$. In this, we are given $H(\omega)$ and $F(\omega)$ but we do not know $G(\omega)$. But looking at $H(\omega)$ and $F(\omega)$, we see that if $G(\omega)$ happened to be two Dirac impulses, one at $-2\omega_c$ and one at $+2\omega_c$, then convolving $F(\omega)$ with it, will give $2\pi H(\omega)$. So we need $G(\omega)$ to be the following

$$G(\omega) = 2\pi\left(\delta(\omega + 2\omega_c) + \delta(\omega - 2\omega_c)\right)$$

And now $F(\omega) \circledast G(\omega)$ will result in $H(\omega)$. The factor 2π was added to cancel the 2π from the definition of modulation theory. But $\cos(2\omega_c t)$ has the CTFT of $\pi\left(\delta(\omega + 2\omega_c) + \delta(\omega - 2\omega_c)\right)$. This shows that $G(\omega)$ is the Fourier transform of $2\cos(2\omega_c t)$. Therefore

$$g(t) = 2\cos(2\omega_c t)$$

2.2 Part b

Since $f(t) = \frac{\sin \omega_c t}{\pi t}$ then we see as ω_c increases, the sinc function becomes more concentrated at origin, since the first loop cut off is given by $\frac{\pi}{\omega_c}$. So this gets closer to origin. And since $h(t) = f(t)\,(2\cos(2\omega_c t))$ then $h(t)$ becomes <u>more concentrated</u> around the origin as well.

3 Problem 6.7

A continuous-time lowpass filter has been designed with a passband frequency of 1000 Hz, a stopband frequency of 1200 Hz, passband ripple of 0.1, and stopband ripple of 0.05. Let the impulse response of this lowpass filter be denoted by $h(t)$. We wish to convert the filter into a bandpass filter with impulse response

$$g(t) = 2h(t)\cos(4000\pi t)$$

Assuming that $|H(\omega)|$ is negligible for for $|\omega| > 4000\pi$, answer the following questions. (a) If the passband ripple for the bandpass filter is constrained to be 0.1, what are

the two passband frequencies associated with the bandpass filter? (b) If the stopband ripple for the bandpass filter is constrained to be 0.05, what are the two stopband frequencies associated with the bandpass filter?

solution

3.1 Part a

Let $f(t) = 2\cos(4000\pi t)$. By modulation theory multiplication in time becomes convolution in frequency (with 2π factor)

$$g(t) = h(t)f(t) \rightarrow \frac{1}{2\pi}H(\omega) \circledast F(\omega) \tag{1}$$

Where $H(\omega)$ is the CTFT of $h(t)$ and $F(\omega)$ is the CTFT of $2\cos(4000\pi t)$ which is given by (since it is periodic)

$$F(\omega) = 2\sum_{n=-\infty}^{\infty} 2\pi a_k \delta(\omega - n\omega_0)$$

Where a_k are the Fourier series coefficients of $\cos(4000\pi t)$. In the above $\omega_0 = 4000\pi$. But we know that $a_{-1} = \frac{1}{2}$ and $a_1 = \frac{1}{2}$ from Euler relation. Hence the above becomes

$$\begin{aligned} F(\omega) &= 2\left(2\pi\frac{1}{2}\delta(\omega + 4000\pi) + 2\pi\frac{1}{2}\delta(\omega - 4000\pi)\right) \\ &= 2\pi(\delta(\omega + 4000\pi) + \delta(\omega - 4000\pi)) \end{aligned}$$

Now that we know $F(\omega)$ we go back to (1) and find the CTFT of $g(t)$ which is

$$\begin{aligned} G(\omega) &= \frac{1}{2\pi}H(\omega) \circledast (2\pi(\delta(\omega + 4000\pi) + \delta(\omega - 4000\pi))) \\ &= H(\omega) \circledast ((\delta(\omega + 4000\pi) + \delta(\omega - 4000\pi))) \end{aligned}$$

The above shows that bandpass filter is the lowpass filter spectrum but shifted to the right and to the left by 4000π. This is because convolution with impulse causes shifting. The following diagram shows the result

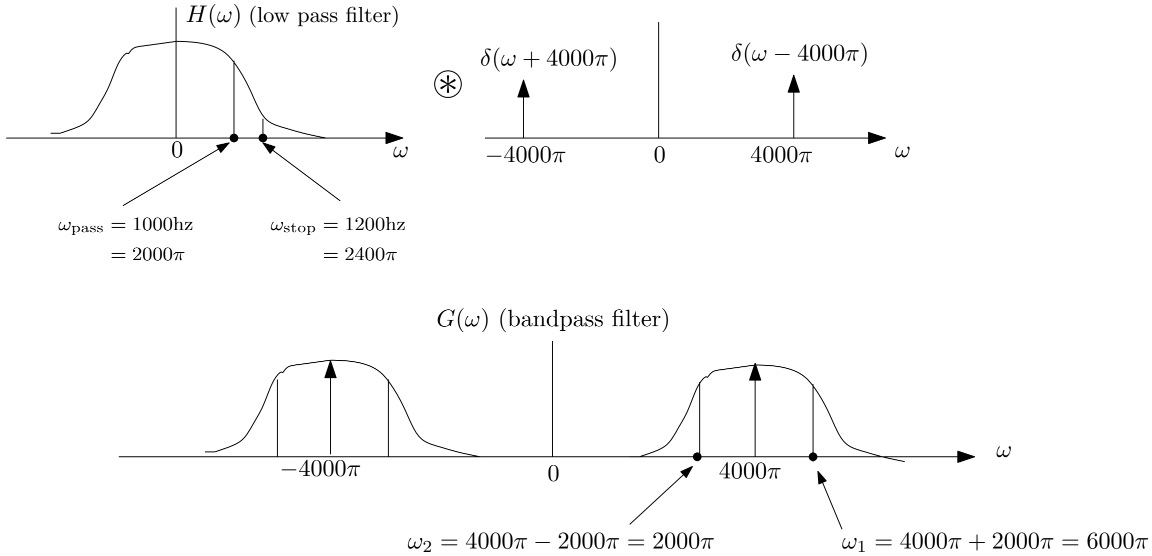


Figure 3: Sketch of bandpass filter

Therefore the two bandpass stop frequencies are

$$\omega_1 = 6000\pi = 3000 \text{ hz}$$
$$\omega_2 = 2000\pi = 1000 \text{ hz}$$

The same on the negative side.

3.2 Part b

Per instructor, we do not need to account for ripple effect in this problem. Therefore this is the same as part (a).

4 Problem 6.17

For each of the following second-order difference equations for causal and stable LTI systems, determine whether or not the step response of the system is oscillatory: (a) $y[n]+y[n-1]+\frac{1}{4}y[n-2]=x[n]$ (b) $y[n]-y[n-1]+\frac{1}{4}y[n-2]=x[n]$

solution

4.1 Part a

Taking the DFT of the difference equation $y[n]+y[n-1]+\frac{1}{4}y[n-2]=x[n]$ gives

$$Y(\Omega)+e^{-j\Omega}Y(\Omega)+\frac{1}{4}e^{-2j\Omega}Y(\Omega)=X(\Omega)$$

A step response means that the input is a step function. Hence $x[n]=u[n]$. Therefore $X(\Omega)=\sum_{n=-\infty}^{\infty}x[n]e^{-j\Omega n}=\sum_{n=0}^{\infty}e^{-j\Omega n}=\frac{1}{1-e^{-j\Omega}}$. The above becomes

$$\begin{aligned}Y(\Omega)\left(1+e^{-j\Omega}+\frac{1}{4}e^{-2j\Omega}\right)&=\frac{1}{1-e^{-j\Omega}}\\ Y(\Omega)&=\frac{1}{1-e^{-j\Omega}}\frac{1}{1+e^{-j\Omega}+\frac{1}{4}e^{-2j\Omega}}\end{aligned}$$

Let $e^{-j\Omega}=x$ for now to make it easier to factor the RHS. The above becomes

$$\begin{aligned}Y(\Omega)&=\frac{1}{1-x}\frac{1}{1+x+\frac{1}{4}x^2}\\ &=\frac{1}{1-x}\frac{4}{4+4x+x^2}\\ &=\frac{4}{(1-x)(x+2)^2}\end{aligned}$$

Using partial fractions on the RHS gives

$$\begin{aligned}\frac{4}{(1-x)(x+2)^2}&=\frac{A}{1-x}+\frac{B}{x+2}+\frac{C}{(x+2)^2}\\ &=\frac{A(x+2)^2+B(x+2)(1-x)+C(1-x)}{(1-x)(x+2)^2}\end{aligned}$$

Hence

$$\begin{aligned}4&=A(x+2)^2+B(x+2)(1-x)+C(1-x)\\ &=A\left(x^2+4+4x\right)+B\left(x-x^2+2-2x\right)+C-Cx\\ &=Ax^2+4A+4xA-Bx-Bx^2+2B+C-Cx\\ &=(4A+2B+C)+x(4A-B-C)+x^2(A-B)\end{aligned}$$

Comparing coefficients gives

$$\begin{aligned} 4A + 2B + C &= 4 \\ 4A - B - C &= 0 \\ A - B &= 0 \end{aligned}$$

Solving gives $A = \frac{4}{9}, B = \frac{4}{9}, C = \frac{4}{3}$. Hence

$$\begin{aligned} Y(\Omega) &= \frac{4}{(1-x)(x+2)^2} \\ &= \frac{A}{1-x} + \frac{B}{x+2} + \frac{C}{(x+2)^2} \\ &= \frac{4}{9}\frac{1}{1-x} + \frac{4}{9}\frac{1}{x+2} + \frac{4}{3}\frac{1}{(x+2)^2} \end{aligned}$$

But $x = e^{-j\Omega}$. The above becomes

$$\begin{aligned} Y(\Omega) &= \frac{4}{9}\frac{1}{1-e^{-j\Omega}} + \frac{4}{9}\frac{1}{2+e^{-j\Omega}} + \frac{4}{3}\frac{1}{\left(2+e^{-j\Omega}\right)^2} \\ &= \frac{4}{9}\frac{1}{1-e^{-j\Omega}} + \frac{4}{9}\frac{1}{2\left(1+\frac{1}{2}e^{-j\Omega}\right)} + \frac{4}{3}\frac{1}{\left(2\left(1+\frac{1}{2}e^{-j\Omega}\right)\right)^2} \\ &= \frac{4}{9}\frac{1}{1-e^{-j\Omega}} + \frac{4}{18}\frac{1}{1+\frac{1}{2}e^{-j\Omega}} + \frac{4}{3}\frac{1}{4\left(1+\frac{1}{2}e^{-j\Omega}\right)^2} \\ &= \frac{4}{9}\frac{1}{1-e^{-j\Omega}} + \frac{4}{18}\frac{1}{1+\frac{1}{2}e^{-j\Omega}} + \frac{1}{3}\frac{1}{\left(1+\frac{1}{2}e^{-j\Omega}\right)^2} \end{aligned}$$

From tables, using $au[n] \Longleftrightarrow \frac{1}{1-ae^{-j\Omega}}$ and $(n+1)a^n u[n] \Longleftrightarrow \frac{1}{\left(1-ae^{-j\Omega}\right)^2}$. Applying these to the above gives

$$\begin{aligned} y(n) &= \frac{4}{9}u[n] - \frac{4}{18}\frac{1}{2}u[n] + \frac{1}{3}(n+1)\left(-\frac{1}{2}\right)^n u[n] \\ &= \left(\frac{4}{9} - \frac{4}{36} + \frac{1}{3}(n+1)\left(-\frac{1}{2}\right)^n\right)u[n] \\ &= \left(\frac{1}{3} + \frac{1}{3}(n+1)\left(-\frac{1}{2}\right)^n\right)u[n] \\ &= \frac{1}{3}\left(1 + (n+1)\left(-\frac{1}{2}\right)^n\right)u[n] \end{aligned}$$

The following is a plot of $y[n]$

```
In[∘]:= y[n_] := 1/3 (1 + (n + 1) * (-1/2)^n)
DiscretePlot[y[n], {n, 0, 13}, PlotRange → All, AxesLabel → {"n", "y[n]"}]
```

Out[∘]=

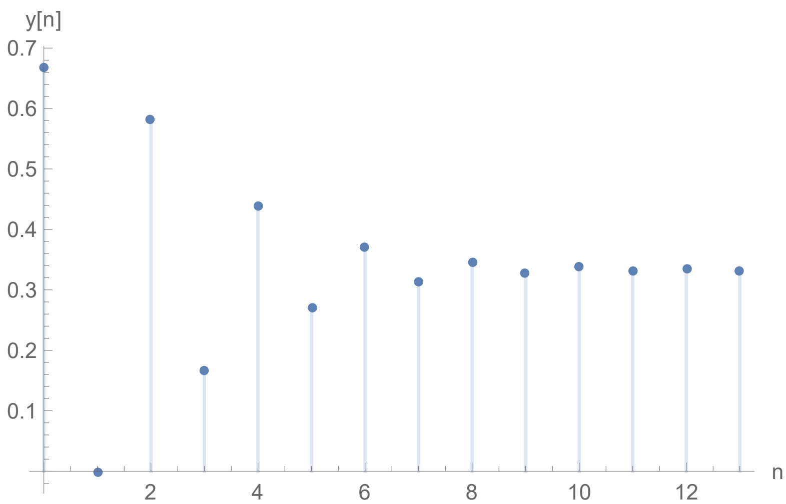


Figure 4: Plot of reponse $y[n]$ to step input

The above shows the response is oscillatory.

4.2 Part b

This is similar to part (a) except for sign difference. Taking the DFT of the difference equation $y[n] - y[n-1] + \frac{1}{4}y[n-2] = x[n]$ gives

$$Y(\Omega) - e^{-j\Omega}Y(\Omega) + \frac{1}{4}e^{-2j\Omega}Y(\Omega) = X(\Omega)$$

A step response means that the input is a step function. Hence $x[n] = u[n]$. Therefore $X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} = \sum_{n=0}^{\infty} e^{-j\Omega n} = \frac{1}{1-e^{-j\Omega}}$. The above becomes

$$\begin{aligned} Y(\Omega)\left(1 - e^{-j\Omega} + \frac{1}{4}e^{-2j\Omega}\right) &= \frac{1}{1-e^{-j\Omega}} \\ Y(\Omega) &= \frac{1}{1-e^{-j\Omega}}\frac{1}{1-e^{-j\Omega}+\frac{1}{4}e^{-2j\Omega}} \end{aligned}$$

Let $e^{-j\Omega} = x$ for now to make it easier to factor the RHS. The above becomes

$$\begin{aligned} Y(\Omega) &= \frac{1}{1-x}\frac{1}{1-x+\frac{1}{4}x^2} \\ &= \frac{1}{1-x}\frac{4}{4-4x+x^2} \\ &= \frac{4}{(1-x)(x-2)^2} \end{aligned}$$

Using partial fractions on the RHS gives

$$\begin{aligned}\frac{4}{(1-x)(x-2)^2} &= \frac{A}{1-x} + \frac{B}{x-2} + \frac{C}{(x-2)^2}\\ &= \frac{A(x-2)^2 + B(x-2)(1-x) + C(1-x)}{(1-x)(x-2)^2}\end{aligned}$$

Hence

$$\begin{aligned}4 &= A(x-2)^2 + B(x-2)(1-x) + C(1-x)\\ &= A\left(x^2+4-4x\right) + B\left(x - x^2 - 2 + 2x\right) + C - Cx\\ &= Ax^2 + 4A - 4xA + 3Bx - Bx^2 - 2B + C - Cx\\ &= (4A - 2B + C) + x(-4A + 3B - C) + x^2(A - B)\end{aligned}$$

Comparing coefficients gives

$$\begin{aligned}4A - 2B + C &= 4\\ -4A + 3B - C &= 0\\ A - B &= 0\end{aligned}$$

Solving gives $A = 4, B = 4, C = -4$. Hence

$$\begin{aligned}Y(\Omega) &= \frac{4}{(1-x)(x-2)^2}\\ &= \frac{A}{1-x} + \frac{B}{x-2} + \frac{C}{(x-2)^2}\\ &= 4\frac{1}{1-x} + 4\frac{1}{x-2} - 4\frac{1}{(x-2)^2}\end{aligned}$$

But $x = e^{-j\Omega}$. The above becomes

$$\begin{aligned}Y(\Omega) &= 4\frac{1}{1-e^{-j\Omega}} + 4\frac{1}{e^{-j\Omega}-2} - 4\frac{1}{\left(e^{-j\Omega}-2\right)^2}\\ &= 4\frac{1}{1-e^{-j\Omega}} + 4\frac{1}{-2\left(1-\frac{1}{2}e^{-j\Omega}\right)} - 4\frac{1}{\left(-2\left(1-\frac{1}{2}e^{-j\Omega}\right)\right)^2}\\ &= 4\frac{1}{1-e^{-j\Omega}} - 2\frac{1}{1-\frac{1}{2}e^{-j\Omega}} - 4\frac{1}{4\left(1-\frac{1}{2}e^{-j\Omega}\right)^2}\\ &= 4\frac{1}{1-e^{-j\Omega}} - 2\frac{1}{1-\frac{1}{2}e^{-j\Omega}} - \frac{1}{\left(1-\frac{1}{2}e^{-j\Omega}\right)^2}\end{aligned}$$

From tables, using $au[n] \iff \frac{1}{1-ae^{-j\Omega}}$ and $(n+1)a^n u[n] \iff \frac{1}{\left(1-ae^{-j\Omega}\right)^2}$. Applying these to the above gives

$$
\begin{aligned}
y(n) &= 4u[n] - 2\frac{1}{2}u[n] - \frac{1}{3}(n+1)\left(\frac{1}{2}\right)^n u[n] \\
&= \left(4 - 1 - \frac{1}{3}(n+1)\left(\frac{1}{2}\right)^n\right)u[n] \\
&= \left(3 - \frac{1}{3}(n+1)\left(\frac{1}{2}\right)^n\right)u[n]
\end{aligned}
$$

The following is a plot of $y[n]$

```
In[◦]:= y[n_] := (3 - 1 / 3 (n + 1) * (1 / 2) ^n)
        DiscretePlot[y[n], {n, 0, 40}, PlotRange → All, AxesLabel → {"n", "y[n]"}]
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Out[◦]=

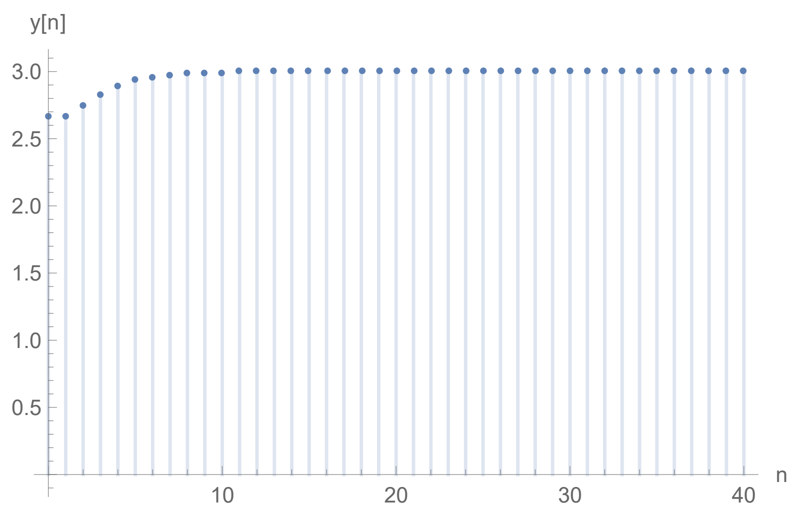


Figure 5: Plot of reponse $y[n]$ to step input

The above shows the response is not oscillatory. The reason is that sign difference in $y[n-1]$ term in the difference equation. $y[n] - y[n-1] + \frac{1}{4}y[n-2] = x[n]$.

5 Problem 6.22

Figure P6.21

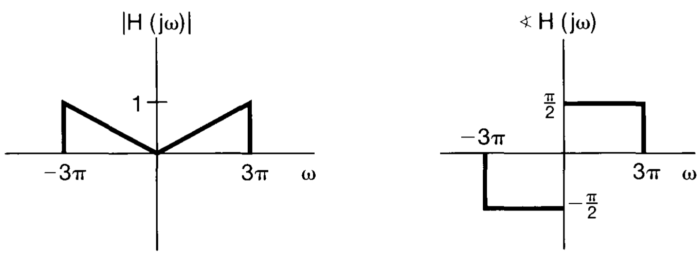
6.22. Shown in Figure P6.22(a) is the frequency response $H(j\omega)$ of a continuous-time filter referred to as a lowpass differentiator. For each of the input signals $x(t)$ below, determine the filtered output signal $y(t)$.

(a) $x(t) = \cos(2\pi t + \theta)$

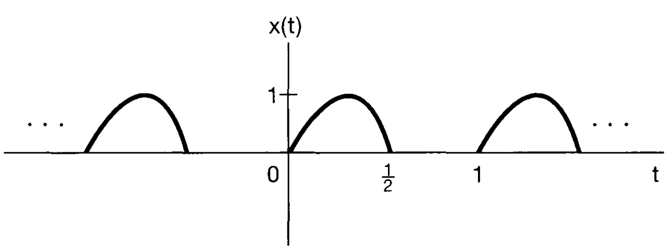
(b) $x(t) = \cos(4\pi t + \theta)$

(c) $x(t)$ is a half-wave rectified sine wave of period, as sketched in Figure P6.22(b).

$$x(t) = \begin{cases} \sin 2\pi t, & m \le t \le (m + \frac{1}{2}) \\ 0, & (m + \frac{1}{2}) \le t \le m \text{ for any integer } m \end{cases}$$



(a)



(b) **Figure P6.22**

6.23 Shown in Figure P6.23 is $|H(j\omega)|$ for a lowpass filter. Determine and sketch the

Figure 6: Problem description

solution

5.1 Part a

The relation between the input and output is given by

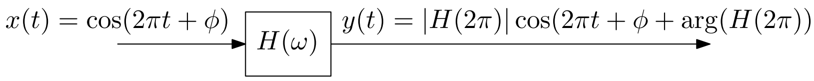


Figure 7: Output of LTI when input is sinuosidal

Since $|H(\omega)| = \frac{1}{3\pi}\omega$ then $|H(2\pi)| = \frac{2}{3}$ and from the phase diagram $\arg(H(2\pi)) = \frac{\pi}{2}$. Therefore

$$\begin{aligned} y(t) &= |H(2\pi)|\cos(2\pi t + \theta + \arg(H((2\pi)))) \\ &= \frac{2}{3}\cos\left(2\pi t + \theta + \frac{\pi}{2}\right) \end{aligned}$$

But $\cos\left(x + \frac{\pi}{2}\right) = -\sin(x)$, hence the above can be simplified to

$$y(t) = -\frac{2}{3}\sin(2\pi t + \theta)$$

5.2 Part b

Since $|H(\omega)| = 0$ for $\omega = 4\pi$, then $y(t) = 0$

5.3 Part c

$$X(\omega) = 2\pi \sum a_k \delta(\omega - k\omega_0)$$

Looking at $x(t)$ shows that its period is $T_0 = 1$. Hence $\omega_0 = 2\pi$. The above becomes

$$X(\omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k2\pi)$$

Hence

$$Y(\omega) = X(\omega) H(\omega)$$

But $|H(\omega)| = 0$ outside $|\omega| = 3\pi$. Then only $k = 0, k = -1, k = +1$ will go through the filter. Hence

$$\begin{aligned} Y(\omega) &= \left(2\pi \sum_{k=-1}^{1} a_k \delta(\omega - k2\pi)\right) H(\omega) \\ &= (2\pi(a_0\delta(\omega) + a_{-1}\delta(\omega + 2\pi) + a_1\delta(\omega - 2\pi)))H(\omega) \end{aligned} \tag{1}$$

To find a_0, a_{-1}, a_1. From

$$\begin{aligned} a_k &= \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jk\omega_0 t} dt \\ &= \int_0^1 x(t)\, e^{-jk2\pi t} dt \end{aligned}$$

Looking at $x(t)$ shows that its period is $T_0 = 1$. Hence $\omega_0 = 2\pi$. From above

$$\begin{aligned} a_0 &= \int_0^1 x(t)\, dt \\ &= \int_0^{\frac{1}{2}} \sin(2\pi t)\, dt \\ &= -\left[\frac{\cos(2\pi t)}{2\pi}\right]_0^{\frac{1}{2}} \\ &= -\frac{1}{2\pi}(\cos\pi - 1) \\ &= \frac{1}{\pi} \end{aligned}$$

And

$$\begin{aligned} a_{-1} &= \int_0^1 x(t)\, e^{j2\pi t} dt \\ &= \int_0^{\frac{1}{2}} \sin(2\pi t)\, e^{j2\pi t} dt \\ &= \frac{j}{4} \end{aligned}$$

And

$$\begin{aligned} a_1 &= \int_0^1 x(t)\, e^{-j2\pi t} dt \\ &= \int_0^{\frac{1}{2}} \sin(2\pi t)\, e^{-j2\pi t} dt \\ &= \frac{-j}{4} \end{aligned}$$

Hence (1) becomes

$$\begin{aligned} Y(\omega) &= \left(2\pi\left(\frac{1}{\pi}\delta(\omega) + \frac{j}{4}\delta(\omega+2\pi) - \frac{j}{4}\delta(\omega-2\pi)\right)\right) H(\omega) \\ &= \left(2\pi\left(\frac{1}{\pi}\delta(\omega) + \frac{j}{4}\delta(\omega+2\pi) - \frac{j}{4}\delta(\omega-2\pi)\right)\right) |H(\omega)|\, e^{j\arg H(\omega)} \end{aligned}$$

At $\omega = 0, Y(\omega) = 0$ since $|H(\omega)| = 0$ at $\omega = 0$. And at $\omega = 2\pi$,

$$
\begin{aligned}
Y(\omega) &= \left(2\pi\left(-\frac{j}{4}\right)\right)|H(2\pi)|\,e^{j\arg H(2\pi)} \\
&= \left(2\pi\left(-\frac{j}{4}\right)\right)\frac{2}{3}e^{j\frac{\pi}{2}} \\
&= -j\pi\frac{1}{3}e^{j\frac{\pi}{2}} \\
&= -j\pi\frac{1}{3}j \\
&= \frac{1}{3}\pi
\end{aligned}
$$

And at $\omega = -2\pi$,

$$
\begin{aligned}
Y(\omega) &= \left(2\pi\left(\frac{j}{4}\right)\right)|H(-2\pi)|\,e^{j\arg H(2\pi)} \\
&= \left(2\pi\left(\frac{j}{4}\right)\right)\frac{2}{3}e^{-j\frac{\pi}{2}} \\
&= j\pi\frac{1}{3}e^{-j\frac{\pi}{2}} \\
&= j\pi\frac{1}{3}(-j) \\
&= \frac{1}{3}\pi
\end{aligned}
$$

Hence the spectrum of $Y(\omega)$ is

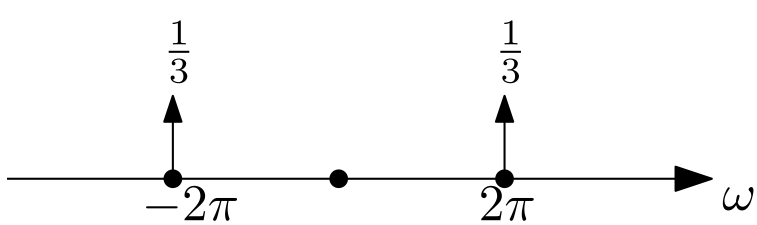


Figure 8: $Y(\omega)$

But the above is the Fourier transform of

$$y(t) = \frac{1}{3}\cos(2\pi t)$$

Which is therefore the output of the filter.

6 Problem 6.27 (a,b,c,d)

(c) Determine $s(0)$ and $s(\infty)$, where $s(t)$ is the step response of the filter.

6.27. The output $y(t)$ of a causal LTI system is related to the input $x(t)$ by the differential equation

$$\frac{dy(t)}{dt} + 2y(t) = x(t).$$

(a) Determine the frequency response

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}$$

of the system, and sketch its Bode plot.

(b) Specify, as a function of frequency, the group delay associated with this system.

(c) If $x(t) = e^{-t}u(t)$, determine $Y(j\omega)$, the Fourier transform of the output.

Figure 9: Problem description

solution

6.1 Part a

$$y'(t) + 2y(t) = x(t)$$

Taking the Fourier transform gives

$$\begin{aligned} j\omega Y(\omega) + 2Y(\omega) &= X(\omega) \\ Y(\omega)\left(2 + j\omega\right) &= X(\omega) \end{aligned}$$

Hence

$$\begin{aligned} H(\omega) &= \frac{Y(\omega)}{X(\omega)} \\ &= \frac{1}{2 + j\omega} \end{aligned}$$

Using Matlab, the following is the Bode plot

```
clear all;
s   = tf('s');
sys = 1 / (2+s);
bode(sys);
grid
```

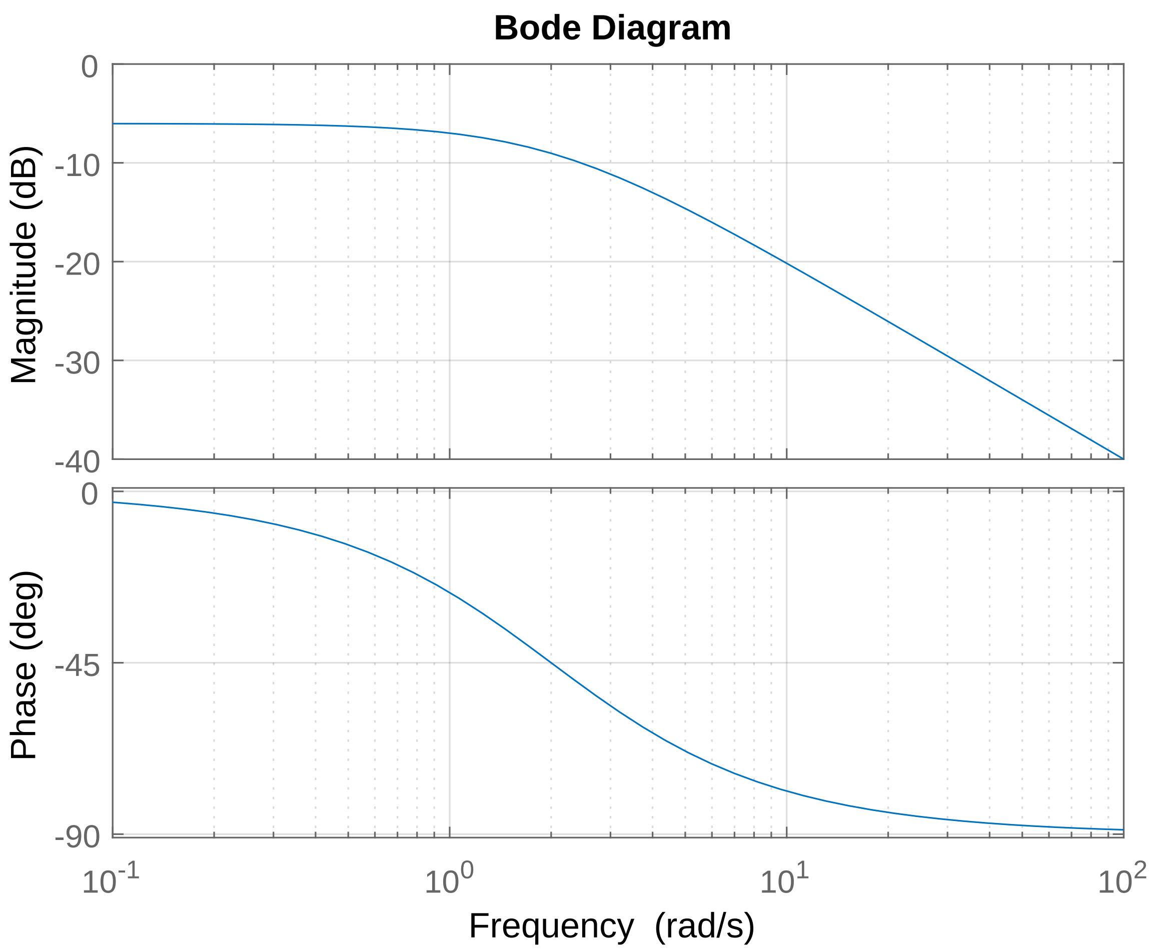


Figure 10: Bode plot

6.2 Part b

In class, it was mentioned that group delay is given by derivative of the Phase of the Fourier transform. Since $H(\omega) = \frac{1}{2+j\omega}$, then

$$\arg(H(\omega)) = -\arctan\left(\frac{\omega}{2}\right)$$

Hence, using the rule $\frac{d}{dx}\arctan(ax) = \frac{a}{1+a^2x^2}$, then using $a = \frac{1}{2}$ the derivative of the above becomes

$$\begin{aligned}\frac{d}{d\omega}\arg(H(\omega)) &= -\frac{\frac{1}{2}}{1+\left(\frac{1}{2}\right)^2\omega^2} \\ &= -\frac{2}{4+\omega^2}\end{aligned}$$

6.3 Part c

Since $x(t) = e^{-t}u(t)$ then now

$$Y(\omega) = X(\omega)H(\omega) \tag{1}$$

But

$$\begin{aligned} X(\omega) &= \int_0^\infty x(t)e^{-j\omega t}dt \\ &= \int_0^\infty e^{-t}e^{-j\omega t}dt \\ &= \int_0^\infty e^{-t(1+j\omega)}dt \\ &= \left[\frac{e^{-t(1+j\omega)}}{-(1+j\omega)}\right]_0^\infty \\ &= \frac{1}{-(1+j\omega)}\left[e^{-t(1+j\omega)}\right]_0^\infty \\ &= \frac{1}{-(1+j\omega)}(0-1) \\ &= \frac{1}{1+j\omega} \end{aligned}$$

Assuming $\operatorname{Re}(\omega) > 1$. Hence (1) becomes

$$Y(\omega) = \left(\frac{1}{1+j\omega}\right)\left(\frac{1}{2+j\omega}\right)$$

6.4 Part d

To find $y(t)$

$$\left(\frac{1}{1+j\omega}\right)\left(\frac{1}{2+j\omega}\right) = \frac{A}{1+j\omega} + \frac{B}{2+j\omega}$$

Hence $A = \left(\frac{1}{2+j\omega}\right)_{\omega=j} = 1$ and $B = \left(\frac{1}{1+j\omega}\right)_{\omega=2j} = -1$. Therefore

$$Y(\omega) = \frac{1}{1+j\omega} - \frac{1}{2+j\omega}$$

From tables

$$y(t) = \left(e^{-t} - e^{-2t}\right)u(t)$$