Homework 5 Solutions

5.3. We note from Section 5.2 that a periodic signal $x[n]$ with Fourier series representation

$$x[n] = \sum_{k=<N>} a_k e^{jk(2\pi/N)n}$$

has a Fourier transform

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta\left(\omega - \frac{2\pi k}{N}\right).$$

(a) Consider the signal $x_1[n] = \sin(\frac{\pi}{3}n + \frac{\pi}{4})$. We note that the fundamental period of the signal $x_1[n]$ is $N = 6$. The signal may be written as

$$x_1[n] = (1/2j)e^{j(\frac{\pi}{3}n+\frac{\pi}{4})} - (1/2j)e^{-j(\frac{\pi}{3}n+\frac{\pi}{4})} = (1/2j)e^{j\frac{\pi}{4}}e^{j\frac{2\pi}{6}n} - (1/2j)e^{-j\frac{\pi}{4}}e^{-j\frac{2\pi}{6}n}.$$

From this, we obtain the non-zero Fourier series coefficients a_k of $x_1[n]$ in the range $-2 \le k \le 3$ as

$$a_1 = (1/2j)e^{j\frac{\pi}{4}}, \quad a_{-1} = -(1/2j)e^{-j\frac{\pi}{4}}.$$

Therefore, in the range $-\pi \le \omega \le \pi$, we obtain

$$\begin{aligned} X(e^{j\omega}) &= 2\pi a_1 \delta(\omega - \frac{2\pi}{6}) + 2\pi a_{-1}\delta(\omega + \frac{2\pi}{6}) \\ &= (\pi/j)\{e^{j\pi/4}\delta(\omega - 2\pi/6) - e^{-j\pi/4}\delta(\omega + 2\pi/6)\} \end{aligned}$$

(b) Consider the signal $x_2[n] = 2 + \cos(\frac{\pi}{6}n + \frac{\pi}{8})$. We note that the fundamental period of the signal $x_1[n]$ is $N = 12$. The signal may be written as

$$x_1[n] = 2 + (1/2)e^{j(\frac{\pi}{6}n+\frac{\pi}{8})} + (1/2)e^{-j(\frac{\pi}{6}n+\frac{\pi}{8})} = 2 + (1/2)e^{j\frac{\pi}{8}}e^{j\frac{2\pi}{12}n} + (1/2)e^{-j\frac{\pi}{8}}e^{-j\frac{2\pi}{12}n}.$$

From this, we obtain the non-zero Fourier series coefficients a_k of $x_2[n]$ in the range $-5 \le k \le 6$ as

$$a_0 = 2, \quad a_1 = (1/2)e^{j\frac{\pi}{8}}, \quad a_{-1} = (1/2)e^{-j\frac{\pi}{8}}.$$

Therefore, in the range $-\pi \le \omega \le \pi$, we obtain

$$\begin{aligned} X(e^{j\omega}) &= 2\pi a_0\delta(\omega) + 2\pi a_1\delta(\omega - \frac{2\pi}{12}) + 2\pi a_{-1}\delta(\omega + \frac{2\pi}{12}) \\ &= 4\pi\delta(\omega) + \pi\{e^{j\pi/8}\delta(\omega - \pi/6) + e^{-j\pi/8}\delta(\omega + \pi/6)\} \end{aligned}$$

5.5. From the given information,

$$\begin{aligned} x[n] &= (1/2\pi)\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega \\ &= (1/2\pi)\int_{-\pi}^{\pi} |X(e^{j\omega})|e^{j\sphericalangle\{X(e^{j\omega})\}}e^{j\omega n}d\omega \\ &= (1/2\pi)\int_{-\pi/4}^{\pi/4} e^{-\frac{3}{2}\omega}e^{j\omega n}d\omega \\ &= \frac{\sin(\frac{\pi}{4}(n-3/2))}{\pi(n-3/2)} \end{aligned}$$

The signal $x[n]$ is zero when $\frac{\pi}{4}(n-3/2)$ is a nonzero integer multiple of π or when $|n| \to \infty$. The value of $\frac{\pi}{4}(n-3/2)$ can never be such that it is a nonzero integer multiple of π. Therefore, $x[n] = 0$ only for $n = \pm\infty$.

5.9. From Property 5.3.4 in Table 5.1, we know that for a real signal $x[n]$,

$$\mathcal{O}d\{x[n]\} \stackrel{FT}{\longleftrightarrow} j\mathcal{I}m\{X(e^{j\omega})\}$$

From the given information,

$$\begin{aligned} j\mathcal{I}m\{X(e^{j\omega})\} &= j\sin\omega - j\sin 2\omega \\ &= (1/2)(e^{j\omega} - e^{-j\omega} - e^{2j\omega} + e^{-2j\omega}) \end{aligned}$$

Therefore,

$$\mathcal{O}d\{x[n]\} = \mathcal{IFT}\{j\mathcal{I}m\{X(e^{j\omega})\}\} = (1/2)(\delta[n+1] - \delta[n-1] - \delta[n+2] + \delta[n-2])$$

We also know that

$$\mathcal{O}d\{x[n]\} = \frac{x[n] - x[-n]}{2}$$

and that $x[n] = 0$ for $n > 0$. Therefore,

$$x[n] = 2\mathcal{O}d\{x[n]\} = \delta[n+1] - \delta[n+2], \qquad \text{for } n < 0.$$

Now we only have to find $x[0]$. Using Parseval's relation, we have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |X(e^{j\omega})|^2 d\omega = \sum_{n=-\infty}^{\infty} |x[n]|^2.$$

From the given information, we can write

$$3 = (x[0])^2 + \sum_{n=-\infty}^{-1} |x[n]|^2 = (x[0])^2 + 2$$

This gives $x[0] = \pm 1$. But since we are given that $x[0] > 0$, we conclude that $x[0] = 1$. Therefore,

$$x[n] = \delta[n] + \delta[n+1] - \delta[n+2].$$

5.13. When two LTI systems are connected in parallel, the impulse response of the overall system is the sum of the impulse responses of the individual systems. Therefore,

$$h[n] = h_1[n] + h_2[n].$$

Using the linearity property (Table 5.1, Property 5.3.2),

$$H(e^{j\omega}) = H_1(e^{j\omega}) + H_2(e^{j\omega})$$

Given that $h_1[n] = (1/2)^n u[n]$, we obtain

$$H_1(e^{j\omega}) = \frac{1}{1 - \frac{1}{2}e^{-j\omega}}$$

Therefore,

$$H_2(e^{j\omega}) = \frac{-12 + 5^{-j\omega}}{12 - 7e^{-j\omega} + e^{-2j\omega}} - \frac{1}{1 - \frac{1}{2}e^{-j\omega}} = \frac{-2}{1 - \frac{1}{4}e^{-j\omega}}.$$

Taking the inverse Fourier transform,

$$h_2[n] = -2\left(\frac{1}{4}\right)^n u[n].$$

5.19. (a) Taking the Fourier transform of both sides of the difference equation, we have

$$Y(e^{j\omega})\left[1-\frac{1}{6}e^{-j\omega}-\frac{1}{6}e^{-2j\omega}\right]=X(e^{j\omega}).$$

Therefore,

$$H(ej\omega)=\frac{Y(e^{j\omega})}{X(e^{j\omega})}=\frac{1}{1-\frac{1}{6}e^{-j\omega}-\frac{1}{6}e^{-2j\omega}}=\frac{1}{(1-\frac{1}{2}e^{-j\omega})(1+\frac{1}{3}e^{-j\omega})}.$$

(b) Using Partial fraction expansion,

$$H(ej\omega)=\frac{3/5}{1-\frac{1}{2}e^{-j\omega}}+\frac{2/5}{1+\frac{1}{3}e^{-j\omega}}.$$

Using Table 5.2, and taking the inverse Fourier trasform, we obtain

$$h[n]=\frac{3}{5}\left(\frac{1}{2}\right)^n u[n]+\frac{2}{5}\left(-\frac{1}{3}\right)^n u[n].$$

(a) The frequency response of the system is as shown in Figure S5.30.

(b) The Fourier transform $X(e^{j\omega})$ of $x[n]$ is as shown in Figure S5.30.

(i) The frequency response $H(e^{j\omega})$ is as shown in Figure S5.30. Therefore, $y[n] = \sin(\pi n/8)$.

(ii) The frequency response $H(e^{j\omega})$ is as shown in Figure S5.30. Therefore, $y[n] = 2\sin(\pi n/8) - 2\cos(\pi n/4)$.