HW 5

EE 3015
Signals and Systems

Spring 2020
University of Minnesota, Twin Cities

Nasser M. Abbasi

May 27, 2020 Compiled on May 27, 2020 at 12:24am

Contents

1 Problem 5.3, Chapter 5

Determine the Fourier transform for $-\pi \leq \omega < \pi$ in the case of each of the following periodic signals (a) $\sin\left(\frac{\pi}{3}n + \frac{\pi}{4}\right)$ (b) $2 + \cos\left(\frac{\pi}{6}n + \frac{\pi}{8}\right)$

solution

1.1 Part a

Since the signal is periodic, then the Fourier transform is given by

$$X(\Omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\Omega - k\Omega_0) \tag{1}$$

Where a_k are the Fourier series coefficients of $x[n]$. To determine a_k we can expression $x[n]$ using Euler relation. To find the period, $\frac{\pi}{3}N = m2\pi$. Hence $\frac{m}{N} = \frac{1}{6}$. Hence

$$N = 6$$

Therefore $\Omega_0 = \frac{2\pi}{N} = \frac{\pi}{3}$. Now, using Euler relation

$$\begin{aligned}\sin\left(\frac{\pi}{3}n + \frac{\pi}{4}\right) &= \frac{e^{\left(\frac{\pi}{3}n+\frac{\pi}{4}\right)j} - e^{-\left(\frac{\pi}{3}n+\frac{\pi}{4}\right)j}}{2j} \\ &= \frac{1}{2j}e^{j\frac{\pi}{4}}\left(e^{j\frac{\pi}{3}n}\right) - \frac{1}{2j}e^{-j\frac{\pi}{4}}\left(e^{-j\frac{\pi}{3}n}\right) \end{aligned} \tag{2}$$

Comparing (2) to Fourier series expansion of periodic signal given by

$$\begin{aligned}x[n] &= \sum_{k=0}^{N-1} a_k e^{jk\Omega_0 n} \\ &= \sum_{k=0}^{5} a_k e^{jk\Omega_0 n} \\ &= \sum_{k=-2}^{3} a_k e^{jk\Omega_0 n}\end{aligned}$$

Since $\Omega_0 = \frac{\pi}{3}$ then the above becomes

$$x[n] = \sum_{k=-2}^{3} a_k e^{jk\frac{\pi}{3}n}$$

Comparing the above with (2) shows that $a_1 = \frac{1}{2j}e^{j\frac{\pi}{4}}$ and $a_{-1} = -\frac{1}{2j}e^{-j\frac{\pi}{4}}$ and all other $a_k = 0$ for $k = -2, 0, 2, 3$. Hence (1) becomes

$$\begin{aligned}X(\Omega) &= 2\pi\left(a_{-1}\delta(\Omega + \Omega_0) + a_1\delta(\Omega - \Omega_0)\right) \\ &= 2\pi\left(-\frac{1}{2j}e^{-j\frac{\pi}{4}}\delta\left(\Omega + \frac{\pi}{3}\right) + \frac{1}{2j}e^{j\frac{\pi}{4}}\delta\left(\Omega - \frac{\pi}{3}\right)\right) \\ &= \frac{\pi}{j}\left(-e^{-j\frac{\pi}{4}}\delta\left(\Omega + \frac{\pi}{3}\right) + e^{j\frac{\pi}{4}}\delta\left(\Omega - \frac{\pi}{3}\right)\right)\end{aligned}$$

1.2 Part b

Since the signal $2 + \cos\left(\frac{\pi}{6}n + \frac{\pi}{8}\right)$ is periodic, then the Fourier transform is given by

$$X(\Omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\Omega - k\Omega_0) \tag{1}$$

Where a_k are the Fourier series coefficients of $x[n]$. To determine a_k we can expression $x[n]$ using Euler relation. To find the period, $\frac{\pi}{6}N = m2\pi$. Hence $\frac{m}{N} = \frac{1}{12}$. Hence

$$N = 12$$

Therefore $\Omega_0 = \frac{2\pi}{N} = \frac{\pi}{6}$. Now, using Euler relation

$$\begin{aligned} 2 + \cos\left(\frac{\pi}{6}n + \frac{\pi}{8}\right) &= 2 + \frac{e^{\left(\frac{\pi}{6}n+\frac{\pi}{8}\right)j} + e^{-\left(\frac{\pi}{6}n+\frac{\pi}{8}\right)j}}{2} \\ &= 2 + \frac{1}{2}e^{j\frac{\pi}{8}}\left(e^{j\frac{\pi}{6}n}\right) + \frac{1}{2}e^{-j\frac{\pi}{8}}\left(e^{-j\frac{\pi}{6}n}\right) \end{aligned} \tag{2}$$

Comparing (2) to Fourier series expansion of periodic signal given by

$$\begin{aligned} x[n] &= \sum_{k=0}^{N-1} a_k e^{jk\Omega_0 n} \\ &= \sum_{k=0}^{11} a_k e^{jk\Omega_0 n} \\ &= \sum_{k=-5}^{6} a_k e^{jk\Omega_0 n} \end{aligned}$$

Since $\Omega_0 = \frac{\pi}{6}$ then the above becomes

$$x[n] = \sum_{k=-5}^{6} a_k e^{jk\frac{\pi}{6}n}$$

Comparing the above with (2) shows that $a_0 = 2, a_1 = \frac{1}{2}e^{j\frac{\pi}{8}}$ and $a_{-1} = \frac{1}{2}e^{-j\frac{\pi}{8}}$ and all other $a_k = 0$. Hence (1) becomes

$$\begin{aligned} X(\Omega) &= 2\pi\left(a_0\delta(\Omega) + a_{-1}\delta(\Omega + \Omega_0) + a_1\delta(\Omega - \Omega_0)\right) \\ &= 2\pi\left(2\delta(\Omega) + \frac{1}{2}e^{-j\frac{\pi}{8}}\delta\left(\Omega + \frac{\pi}{6}\right) + \frac{1}{2}e^{j\frac{\pi}{8}}\delta\left(\Omega - \frac{\pi}{6}\right)\right) \\ &= 4\pi\delta(\Omega) + \pi e^{-j\frac{\pi}{8}}\delta\left(\Omega + \frac{\pi}{6}\right) + \pi e^{j\frac{\pi}{8}}\delta\left(\Omega - \frac{\pi}{6}\right) \end{aligned}$$

2 Problem 5.5, Chapter 5

Use the Fourier transform synthesis equation (5.8)

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(\Omega)\, e^{j\Omega n} d\Omega \tag{5.8}$$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n} \tag{5.9}$$

To determine the inverse Fourier transform of $X(\Omega) = |X(\Omega)|\, e^{j\arg H(\Omega)}$, where $|X(\Omega)| = \begin{cases} 1 & 0 \le |\Omega| < \frac{\pi}{4} \\ 0 & \frac{\pi}{4} \le |\Omega| < \pi \end{cases}$ and $\arg H(\Omega) = \frac{-3\Omega}{2}$. Use your answer to determine the values of n for which $x[n] = 0$.

solution

$$\begin{aligned}
x[n] &= \frac{1}{2\pi}\int_{2\pi} X(\Omega)\, e^{j\Omega n} d\Omega \\
&= \frac{1}{2\pi}\int_{2\pi} |X(\Omega)|\, e^{j\arg H(\Omega)} e^{j\Omega n} d\Omega \\
&= \frac{1}{2\pi}\int_{0}^{\frac{\pi}{4}} e^{j\arg H(\Omega)} e^{j\Omega n} d\Omega \\
&= \frac{1}{2\pi}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} e^{j\frac{-3\Omega}{2}} e^{j\Omega n} d\Omega \\
&= \frac{1}{2\pi}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} e^{j\Omega\left(\frac{-3}{2}+n\right)} d\Omega \\
&= \frac{1}{2\pi}\frac{1}{j\left(\frac{-3}{2}+n\right)}\left[e^{j\Omega\left(\frac{-3}{2}+n\right)}\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \\
&= \frac{1}{2\pi}\frac{1}{j\left(\frac{-3}{2}+n\right)}\left[e^{j\frac{\pi}{4}\left(\frac{-3}{2}+n\right)} - e^{-j\frac{\pi}{4}\left(\frac{-3}{2}+n\right)}\right] \\
&= \frac{1}{\pi}\frac{1}{\left(\frac{-3}{2}+n\right)}\left[\frac{e^{j\frac{\pi}{4}\left(\frac{-3}{2}+n\right)} - e^{-j\frac{\pi}{4}\left(\frac{-3}{2}+n\right)}}{2j}\right] \\
&= \frac{1}{\pi}\frac{1}{\left(\frac{-3}{2}+n\right)}\sin\left(\frac{\pi}{4}\left(\frac{-3}{2}+n\right)\right) \\
&= \frac{1}{\pi}\frac{\sin\left(\frac{\pi}{4}\left(n-\frac{3}{2}\right)\right)}{n-\frac{3}{2}}
\end{aligned}$$

Now the above is zero when $\sin\left(\frac{\pi}{4}\left(n-\frac{3}{2}\right)\right) = 0$ or $\frac{\pi}{4}\left(n-\frac{3}{2}\right) = m\pi$ for integer m. Hence $n - \frac{3}{2} = 4m$. Or $n = 4m + \frac{3}{2}$. Since m is integer, and since n must be an integer as well, then there is no finite n where $\sin\left(\frac{\pi}{4}\left(n-\frac{3}{2}\right)\right) = 0$. The other option is to look at denominator of $\frac{\sin\left(\frac{\pi}{4}\left(n-\frac{3}{2}\right)\right)}{n-\frac{3}{2}}$ and ask where is that ∞. This happens when $n \to \pm\infty$ and only then $x[n] = 0$.

3 Problem 5.9, Chapter 5

The following four facts are given about a real signal $x[n]$ with Fourier transform $X(\Omega)$

1. $x[n] = 0$ for $n > 0$
2. $x[0] > 0$
3. $\operatorname{Im}(X(\Omega)) = \sin\Omega - \sin(2\Omega)$
4. $\frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\Omega)|^2 \, d\Omega = 3$

Determine $x[n]$

solution

From tables we know that the odd part of $x[n]$ has Fourier transform which is $j\operatorname{Im}(X(\Omega))$. Hence using (3) above, this means that odd part of $x[n]$ has Fourier transform of $j(\sin\Omega - \sin(2\Omega))$ or $j\left(\frac{e^{j\Omega}-e^{-j\Omega}}{2j} - \frac{e^{j2\Omega}-e^{-j2\Omega}}{2j}\right)$ or $\frac{1}{2}\left(e^{j\Omega} - e^{-j\Omega} - e^{j2\Omega} + e^{-j2\Omega}\right)$. From tables, we know find the inverse Fourier transform of this. Hence odd part of $x[n]$ is $\frac{1}{2}(\delta[n+1] - \delta[n-1] - \delta[n+2] + \delta[n-2])$. So now we know what the odd part of $x[n]$ is.

But since $x[n] = 0$ for $n > 0$ then the odd part of $x[n]$ reduces to $\frac{1}{2}(\delta[n+1] - \delta[n+2])$.

But we also know that any function can be expressed as the sum of its odd part and its even part. But since $x[n] = 0$ for $n > 0$ then this means $x[n] = 2\left(\frac{1}{2}(\delta[n+1] - \delta[n+2])\right)$ for $n < 0$. Hence

$$x[n] = \delta[n+1] - \delta[n+2] \qquad n < 0$$

Finally, using (4) above,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\Omega)|^2 \, d\Omega = 3 = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \sum_{n=-\infty}^{0} |x[n]|^2$$

Hence

$$\begin{aligned} 3 &= |\delta[-1]|^2 + |\delta[-2]|^2 + |x[0]|^2 \\ &= 1 + 1 + x[n]^2 \\ x[n]^2 &= 3 - 2 \\ &= 1 \end{aligned}$$

Therefore $x[n] = 1$ or $x[n] = -1$. But from (2) $x[0] > 0$. Hence $x[0]$. Therefore

$$x[n] = \delta[n+1] - \delta[n+2] + \delta[n] \qquad n \leq 0$$

4 Problem 5.13, Chapter 5

An LTI system with impulse response $h_1[n] = \left(\frac{1}{3}\right)^n u[n]$ is connected in parallel with another causal LTI system with impulse response $h_2[n]$. The resulting parallel interconnection has the frequency response

$$H(\Omega) = \frac{-12 + 5e^{-j\Omega}}{12 - 7e^{-j\Omega} + e^{-j2\Omega}}$$

Determine $h_2[n]$.

solution

Since the connection is parallel, then $h[n] = h_1[n] + h_2[n]$. Or $H(\Omega) = H_1(\Omega) + H_2(\Omega)$. Hence

$$H_2(\Omega) = H(\Omega) - H_1(\Omega) \tag{1}$$

But

$$\begin{aligned} H_1(\Omega) &= \sum_{n=-\infty}^{\infty} h_1[n] e^{-j\Omega n} \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n e^{-j\Omega n} \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{3} e^{-j\Omega}\right)^n \\ &= \sum_{n=0}^{\infty} a^n = \frac{1}{1-a} = \frac{1}{1 - \frac{1}{3}e^{-j\Omega}} \\ &= \frac{3}{3 - e^{-j\Omega}} \end{aligned}$$

Therefore from (1)

$$H_2(\Omega) = \frac{-12 + 5e^{-j\Omega}}{12 - 7e^{-j\Omega} + e^{-j2\Omega}} - \frac{3}{3 - e^{-j\Omega}}$$

Let $e^{-j\Omega} = x$ to simplify notation. The above becomes

$$\begin{aligned} H_2(\Omega) &= \frac{-12 + 5x}{12 - 7x + x^2} - \frac{3}{3 - x} \\ &= \frac{-12 + 5x}{(x-3)(x-4)} + \frac{3}{(x-3)} \\ &= \frac{-12 + 5x + 3(x-4)}{(x-3)(x-4)} \\ &= \frac{-12 + 5x + 3x - 12}{(x-3)(x-4)} \\ &= \frac{8x - 24}{(x-3)(x-4)} \\ &= \frac{8(x-3)}{(x-3)(x-4)} \\ &= \frac{8}{(x-4)} \\ &= -2\left(\frac{1}{1 - \frac{1}{4}x}\right) \end{aligned}$$

Hence

$$H_2(\Omega) = -2\left(\frac{1}{1 - \frac{1}{4}e^{-j\Omega}}\right)$$

from tables, $a^n u[n] \Longleftrightarrow \frac{1}{1-ae^{-j\Omega}}$ for $|a| < 1$. Comparing this to the above gives

$$h_2[n] = -2\left(\frac{1}{4}\right)^n u[n]$$

5 Problem 5.19, Chapter 5

Consider a causal and stable LTI system S whose input $x[n]$ and output $y[n]$ are related through the second-order difference equation

$$y[n]-\frac{1}{6}y[n-1]-\frac{1}{6}y[n-2]=x[n]$$

(a) Determine the frequency response $H(\Omega)$ for the system S. (b) Determine the impulse response $h[n]$ for the system S.

solution

5.1 part a

Taking DFT of the difference equation gives

$$\begin{aligned}
Y(\Omega)-\frac{1}{6}e^{-j\Omega}Y(\Omega)-\frac{1}{6}e^{-j2\Omega}Y(\Omega) &= X(\Omega)\\
Y(\Omega)\left(1-\frac{1}{6}e^{-j\Omega}-\frac{1}{6}e^{-j2\Omega}\right) &= X(\Omega)\\
\frac{Y(\Omega)}{X(\Omega)} &= \frac{1}{1-\frac{1}{6}e^{-j\Omega}-\frac{1}{6}e^{-j2\Omega}}
\end{aligned}$$

Let $e^{-j\Omega}=x$ to simplify the notation, then

$$\frac{Y(\Omega)}{X(\Omega)}=\frac{1}{1-\frac{1}{6}x-\frac{1}{6}x^2}=\frac{6}{6-x-x^2}=\frac{-6}{x^2+x-6}=\frac{-6}{(x-2)(x+3)}$$

Hence

$$\begin{aligned}
H(\Omega) &= \frac{Y(\Omega)}{X(\Omega)}\\
&= \frac{-6}{\left(e^{-j\Omega}-2\right)\left(e^{-j\Omega}+3\right)}
\end{aligned}$$

5.2 part b

Applying partial fractions

$$H(\Omega)=\frac{-6}{\left(e^{-j\Omega}-2\right)\left(e^{-j\Omega}+3\right)}=\frac{A}{(x-2)}+\frac{B}{(x+3)}$$

Hence $A=-\frac{6}{5}, B=\frac{6}{5}$. Therefore

$$\begin{aligned}
H(\Omega) &= -\frac{6}{5}\frac{1}{e^{-j\Omega}-2}+\frac{6}{5}\frac{1}{e^{-j\Omega}+3}\\
&= -\frac{3}{5}\frac{1}{\frac{1}{2}e^{-j\Omega}-1}+\frac{2}{5}\frac{1}{\frac{1}{3}e^{-j\Omega}+1}\\
&= \frac{3}{5}\frac{1}{1-\frac{1}{2}e^{-j\Omega}}+\frac{2}{5}\frac{1}{1+\frac{1}{3}e^{-j\Omega}}
\end{aligned}$$

Taking the inverse DFT using tables gives

$$\begin{aligned}
h[n] &= \frac{3}{5}\left(\frac{1}{2}\right)^n u[n]+\frac{2}{5}\left(-\frac{1}{3}\right)^n u[n]\\
&= \left(\frac{3}{5}\left(\frac{1}{2}\right)^n+\frac{2}{5}\left(-\frac{1}{3}\right)^n\right)u[n]
\end{aligned}$$

6 Problem 5.30, Chapter 5

In Chapter 4, we indicated that the continuous-time LTI system with impulse response

$$h(t) = \frac{W}{\pi} \sin c\left(\frac{Wt}{\pi}\right) = \frac{\sin(Wt)}{\pi t}$$

plays a very important role in LTI system analysis. The same is true of the discrete time LTI system with impulse response

$$h(n) = \frac{W}{\pi} \operatorname{sinc}\left(\frac{Wn}{\pi}\right) = \frac{\sin(Wn)}{\pi n}$$

(a) Determine and sketch the frequency response for the system with impulse response $h[n]$. (b) Consider the signal $x[n] = \sin\left(\frac{\pi n}{8}\right) - 2\cos\left(\frac{\pi n}{4}\right)$. Suppose that this signal is the input to LTI systems with the following impulse responses. Determine the output in each case (i) $h[n] = \frac{\sin\left(\frac{\pi n}{6}\right)}{\pi n}$. (ii) $h[n] = \frac{\sin\left(\frac{\pi n}{6}\right)}{\pi n} + \frac{\sin\left(\frac{\pi n}{2}\right)}{\pi n}$

solution

6.1 Part a

Given $h(n) = \frac{W}{\pi} \operatorname{sinc}\left(\frac{Wn}{\pi}\right) = \frac{\sin(Wn)}{\pi n}$. We will show that $H(\Omega)$ is the rectangle function by reverse. Assuming that $H(\Omega) = \begin{matrix} 1 & |\Omega| < 2W \\ 0 & \text{otherwise} \end{matrix}$ therefore

$$\begin{aligned} x[n] &= \frac{1}{2\pi} \int_{-W}^{W} X(\Omega) e^{j\Omega n} d\omega \\ &= \frac{1}{2\pi} \int_{-W}^{W} e^{j\Omega n} d\omega \\ &= \frac{1}{2\pi} \frac{e^{jWn} - e^{-jWn}}{jn} \\ &= \frac{1}{\pi n} \sin(Wn) \end{aligned}$$

Which is the $h[n]$ given. Therefore, the above shows that $\frac{\sin(Wn)}{\pi n}$ has DFT of $H(\Omega)$ as the rectangle function. Here is sketch

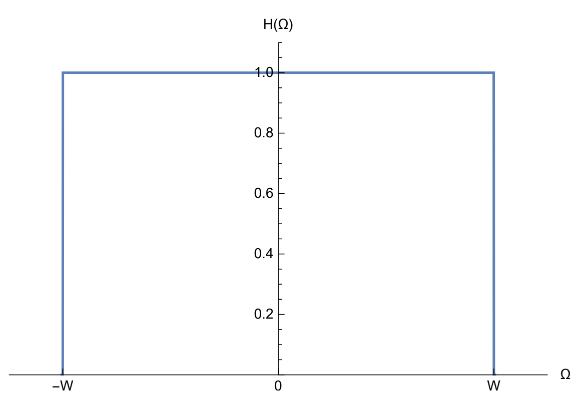

Figure 1: Plot of $H(\Omega)$

6.2 Part b

$$x[n] = \sin\left(\frac{\pi n}{8}\right) - 2\cos\left(\frac{\pi n}{4}\right)$$

(i) $h[n] = \frac{\sin\left(\frac{\pi n}{6}\right)}{\pi n}$. Hence $y[n] = x[n] \circledast h[n]$. Or $Y(\Omega) = X(\Omega) H(\Omega)$, and then we find $y[n]$ by taking the inverse discrete Fourier transform. Here is the result and the code used. The result is

$$y[n] = \sin\left(\frac{n\pi}{8}\right)$$

```
ClearAll[h, x, n];
x[n_] := Sin[π n/8] - 2 Cos[π n/4];
h[n_] := 1/(π n) Sin[π n/6];
X = FourierSequenceTransform[x[n], n, w, FourierParameters → {1, 1}];
H = FourierSequenceTransform[h[n], n, w, FourierParameters → {1, 1}];
y = InverseFourierSequenceTransform[X * H, w, n]

Out[∘]= Sin[n π/8]
```

Figure 2: Code used to generate $y[n]$

Here is plot of $y[n]$ for $n = -8 \cdots 8$

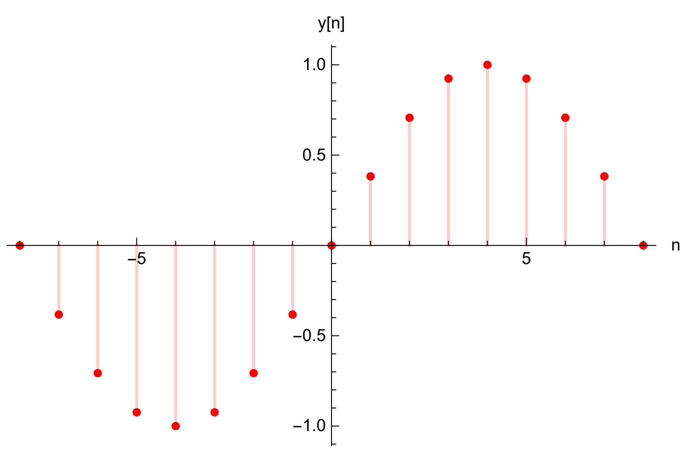

Figure 3: Plot of above $y[n]$

(ii) $h[n] = \frac{\sin\left(\frac{\pi n}{6}\right)}{\pi n} + \frac{\sin\left(\frac{\pi n}{2}\right)}{\pi n}$. Here is the result and the code used. The result is

$$y[n] = 2\sin\left(\frac{n\pi}{8}\right) - 2\cos\left(\frac{n\pi}{4}\right)$$

```
In[∘]:= ClearAll[h, x, n, w];
x[n_] := Sin[π n/8] - 2 Cos[π n/4];
h1[n_] := 1/(π n) Sin[π n/6]
h2[n_] := 1/(π n) Sin[π n/2];
X = FourierSequenceTransform[x[n], n, w, FourierParameters → {1, 1}];
H1 = FourierSequenceTransform[h1[n], n, w, FourierParameters → {1, 1}];
H2 = FourierSequenceTransform[h2[n], n, w, FourierParameters → {1, 1}];
y1 = InverseFourierSequenceTransform[X * H1, w, n];
y2 = InverseFourierSequenceTransform[X * H2, w, n];
y = y1 + y2

Out[∘]= -2 Cos[n π/4] + 2 Sin[n π/8]
```

Figure 4: Code used to generate $y[n]$

Here is plot of $y[n]$ for $n = -8 \cdots 8$

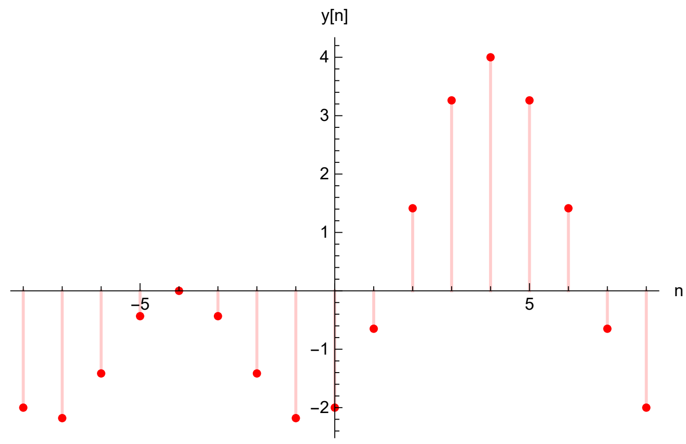

Figure 5: Plot of above $y[n]$