Homework 4 solutions.

If you catch any errors, please send me an email at leex8370@umn.edu.

4.3 Determine the Fourier Transforms of the following signals

a) $x(t) = \sin(2\pi t + \pi/4)$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} \sin(2\pi t + \pi/4) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} \left[\frac{e^{j(2\pi t + \pi/4)} - e^{-j(2\pi t + \pi/4)}}{2j}\right] e^{-j\omega t} dt$$

$$= \frac{e^{j\pi/4}}{2j} \int_{-\infty}^{\infty} e^{j2\pi t} e^{-j\omega t} dt + \frac{e^{-j\pi/4}}{2j} \int_{-\infty}^{\infty} e^{-j2\pi t} e^{-j\omega t} dt$$

Note that $\mathcal{F}^{-1}\{\delta(\omega - \omega_0)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega - \omega_0) e^{j\omega t} d\omega$
$= \frac{e^{j\omega_0 t}}{2\pi}$

Then $\int_{-\infty}^{\infty} e^{j\omega_0 t} e^{-j\omega t} = \mathcal{F}\{e^{j2\pi t}\} = 2\pi\,\delta(\omega - \omega_0)$

Then $X(j\omega) = \frac{\pi e^{j\pi/4}}{j} \delta(\omega - 2\pi) - \frac{\pi e^{-j\pi/4}}{j} \delta(\omega + 2\pi)$

b) $y(t) = 1 + \cos(6\pi t + \pi/8)$

$$Y(j\omega) = \int_{-\infty}^{\infty} [1 + \cos(6\pi t + \pi/8)] e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} e^{-j\omega t} dt + \int_{-\infty}^{\infty} \left[\frac{e^{j(6\pi t + \pi/8)} + e^{-j(6\pi t + \pi/8)}}{2}\right] e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} e^{-j\omega t} dt + \int_{-\infty}^{\infty} \frac{e^{j\pi/8}}{2} e^{j(6\pi - \omega)t} dt + \int_{-\infty}^{\infty} \frac{e^{-j\pi/8}}{2} e^{-j(6\pi + \omega)t} dt$$

$$= \int_{-\infty}^{\infty} e^{-j\omega t} dt + \frac{e^{j\pi/8}}{2}\int_{-\infty}^{\infty} e^{j(6\pi-\omega)t} dt + \frac{e^{-j\pi/8}}{2}\int_{-\infty}^{\infty} e^{-j(6\pi+\omega)} dt$$

We'll again use the fact that

$$\mathcal{F}^{-1}\{\delta(\omega-\omega_0)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega-\omega_0) e^{j\omega t} d\omega$$

$$= \frac{e^{j\omega_0 t}}{2\pi}$$

rewriting our expression for

$Y(j\omega)$ we see

$$Y(j\omega) = 2\pi \int_{-\infty}^{\infty} \frac{e^{j\cdot 0 t}}{2\pi} e^{-j\omega t} dt$$

$$+ \pi e^{j\pi/8} \int_{-\infty}^{\infty} e^{j6\pi t} e^{-j\omega t} dt$$

$$+ \pi e^{-j\pi/8} \int_{-\infty}^{\infty} e^{-6\pi t} e^{-j\omega t} dt$$

$$= 2\pi \cdot \mathcal{F}\left\{\frac{e^{j0t}}{2\pi}\right\} + \pi e^{j\pi/8} \mathcal{F}\left\{\frac{e^{j6\pi t}}{2\pi}\right\} + \pi e^{-j\pi/8} \mathcal{F}\left\{\frac{e^{-j6\pi t}}{2\pi}\right\}$$

$$= 2\pi\,\delta(\omega) + \pi e^{j\pi/8}\,\delta(\omega-6\pi) + \pi e^{-j\pi/8}\,\delta(\omega+6\pi)$$

3(b)

4.5 Use the synthesis equation to determine $X(j\omega) = |X(j\omega)| e^{j \measuredangle X(j\omega)}$

where $|X(j\omega)| = 2[u(\omega+3) - u(\omega-3)]$
$\measuredangle X(j\omega) = -\frac{3}{2}\omega + \pi$

$$X(j\omega) = 2[u(\omega+3) - u(\omega-3)] e^{-j\frac{3}{2}\omega + j\pi}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2[u(\omega+3) - u(\omega-3)] e^{-j\frac{3}{2}\omega + j\pi} e^{j\omega t} d\omega$$

* The integral is zero for $\omega < -3$ & $\omega > 3$.

$$= \frac{1}{2\pi}\int_{-3}^{3} 2 \cdot e^{j\pi} e^{j\omega(t-\frac{3}{2})} d\omega$$

$2e^{j\pi}$ is independent of ω, and can be brought outside of the integral. Also note $e^{j\pi} = -1$.

$$\Rightarrow x(t) = \frac{-1}{\pi}\int_{-3}^{3} e^{j\omega(t-\frac{3}{2})} d\omega$$

$$= \frac{-1}{\pi} \cdot \left[\frac{e^{j\omega(t-\frac{3}{2})}}{j(t-\frac{3}{2})}\right]\Bigg|_{\omega=-3}^{3}$$

$$= \frac{-1}{\pi} \frac{e^{j \cdot 3(t-\frac{3}{2})} - e^{-j3(t-\frac{3}{2})}}{j(t-\frac{3}{2})}$$

$$= \frac{-2}{\pi(t-\frac{3}{2})} \cdot \left[\frac{e^{j3(t-\frac{3}{2})} - e^{-j \cdot 3(t-\frac{3}{2})}}{2j}\right]$$

$$= \frac{-2\sin(3t - \frac{9}{2})}{\pi(t-\frac{3}{2})}$$

$$x(t) = 0 \Rightarrow \frac{\sin(3(t-\frac{3}{2}))}{(t-\frac{3}{2})} = 0$$

Clearly, we need $3(t-\frac{3}{2}) = \pi n$, where $n \in \mathbb{Z}$

Then $t = \frac{\pi n}{3} + \frac{3}{2}$

l'Hopital's rule.

but at $n=0$, we get $\lim_{t\to\frac{3}{2}} \frac{\sin(3(t-\frac{3}{2}))}{t-\frac{3}{2}} = \lim_{t\to\frac{3}{2}} \frac{3\cos(3(t-\frac{3}{2}))}{1} \neq 0$

Then we have $t = \frac{\pi n}{3} + \frac{3}{2}$ for $n \in \mathbb{Z} - \{0\}$.

1.11 $y(t) = x(t) * h(t)$
$g(t) = x(3t) * h(3t)$

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$
$$h(t) \longleftrightarrow H(j\omega)$$

use properties of the transform to show that $g(t) = Ay(Bt)$

The time scaling property tells us

$$x(3t) \longleftrightarrow \tfrac{1}{3} X(j\omega/3)$$
$$h(3t) \longleftrightarrow \tfrac{1}{3} H(j\omega/3)$$

$$g(t) = x(3t) * y(3t) \longleftrightarrow \tfrac{1}{3} X(j\omega/3) \cdot \tfrac{1}{3} H(j\omega/3)$$

$$y(t) = x(t) * g(t) \longleftrightarrow X(j\omega) H(j\omega)$$

Thus $G(j\omega) = \tfrac{1}{9} Y(j\omega/3)$

$$g(t) = Ay(B(t)) \longleftrightarrow \tfrac{1}{9} Y(j\omega/3)$$

$$\Rightarrow g(t) = \tfrac{1}{3} y(3t)$$

which was found using the frequency scaling property.

Thus $A = 1/3$, $B = 3$.

4.19 A causal LTI system has frequency response

$$H(j\omega) = \frac{1}{j\omega + 3}$$

The response to an input $x(t)$

is $y(t) = e^{-3t}u(t) - e^{-4t}u(t)$

$$y(t) = x(t) * h(t) \longleftrightarrow Y(j\omega) = H(j\omega)X(j\omega)$$

Thus $X(j\omega) = \dfrac{Y(j\omega)}{H(j\omega)}$

Using Transform tables:

$$Y(j\omega) = \frac{1}{j\omega+3} - \frac{1}{j\omega+4}$$

$$\Rightarrow X(j\omega) = \left(\tfrac{1}{j\omega+3} - \tfrac{1}{j\omega+4}\right) \Big/ \left(\tfrac{1}{j\omega+3}\right)$$

$$= \left(\frac{(j\omega+4)-(j\omega+3)}{(j\omega+3)(j\omega+4)}\right) \Big/ \left(\tfrac{1}{j\omega+3}\right)$$

$$= \frac{1}{j\omega+4}$$

$$x(t) = \mathcal{F}^{-1}\{X(j\omega)\} = e^{-4t}u(t)$$

4.23 $x_0(t) = \begin{cases} e^{-t} & 0 \le t \le 1 \\ 0 & \text{elsewhere} \end{cases}$

$$x_0(t) = [u(t) - u(t-1)]\, e^{-t}$$

$$X_0(j\omega) = \int_0^1 e^{-(j\omega+1)t}\, dt = \frac{-1}{j\omega+1} e^{-(j\omega+1)t}\Big|_{t=0}^{1}$$

$$= \frac{1 - e^{-(j\omega+1)}}{j\omega+1}$$

$$x_1(t) = x_0(t) + x_0(-t)$$

$$X_1(j\omega) = X_0(j\omega) + X_0(-j\omega) = \frac{1-e^{-(j\omega+1)}}{1+j\omega} + \frac{1-e^{-(1-j\omega)}}{1-j\omega}$$

$$= \frac{(1-j\omega)(1-e^{-1}e^{-j\omega}) + (1+j\omega)(1-e^{-1}e^{j\omega})}{1+\omega^2}$$

$$= \frac{2 - 2e^{-1}\left[\frac{e^{-j\omega}+e^{j\omega}}{2}\right] - 2\omega e^{-1}\left[\frac{e^{j\omega}-e^{-j\omega}}{2j}\right]}{1+\omega^2}$$

$$= \frac{2 - 2e^{-1}\cos(\omega) - 2\omega e^{-1}\sin(\omega)}{1+\omega^2}$$

$$x_2(t) = x_0(t) - x_0(-t)$$

$$X_2(j\omega) = X(j\omega) - X(-j\omega) = \frac{1-e^{-(j\omega+1)}}{1+j\omega} - \frac{1-e^{-(1-j\omega)}}{1-j\omega}$$

$$\frac{(1-j\omega)(1-e^{-(j\omega+1)}) - (1+j\omega)(1-e^{-(1-j\omega)})}{1+\omega^2}$$

$$= \frac{-2j\omega + 2je^{-1}\left[\frac{e^{j\omega}-e^{-j\omega}}{2j}\right] + 2j\omega e^{-1}\left[\frac{e^{j\omega}+e^{-j\omega}}{2}\right]}{1+\omega^2}$$

$$= \frac{-2j\omega + 2je^{-1}\sin(\omega) + 2j\omega\cos(\omega)}{1+\omega^2}$$

$$x_3(t) = x_0(t+1) + x_0(t)$$

$$X_3(j\omega) = X_0(j\omega) + e^{j\omega} X_0(-j\omega)$$

$$= \frac{1-e^{-(1+j\omega)}}{1+j\omega} + \frac{e^{j\omega}(1-e^{-(1+j\omega)})}{1+j\omega}$$

$$= \frac{1+e^{j\omega} - e^{-1}(1+e^{-j\omega})}{1+j\omega}$$

$$x_4(t) = t\, x_0(t)$$

$$X_4(j\omega) = j\frac{\partial}{\partial\omega} X_0(j\omega)$$

$$= j\frac{\partial}{\partial\omega}\left[\frac{1-e^{-1}e^{-j\omega}}{1+j\omega}\right]$$

$$= j\left[\frac{(1+j\omega)(je^{-1}e^{-j\omega}) - (1-e^{-1}e^{-j\omega})j}{(1+j\omega)^2}\right]$$

$$= \frac{[(j-\omega)(je^{-1}e^{-j\omega}) + (1-e^{-1}e^{-j\omega})]}{(1+j\omega)^2}$$

$$= \frac{1-2e^{-1}e^{-j\omega} - j\omega e^{-1}e^{-j\omega}}{(1+j\omega)^2}$$

4.26

a)

i) $x(t) = te^{-2t}u(t) \longleftrightarrow X(j\omega) = \frac{1}{(j\omega+2)^2}$

$h(t) = e^{-4t}u(t) \longleftrightarrow H(j\omega) = \frac{1}{j\omega+4}$

$y(t) = x(t) * h(t) \longleftrightarrow X(j\omega)H(j\omega)$

$$= \frac{1}{(j\omega+2)^2(j\omega+4)} = \frac{1/4}{j\omega+4} - \frac{1/4}{j\omega+2} + \frac{1/2}{(2+j\omega)^2}$$

$$y(t) = \left[\tfrac{1}{4}e^{-4t} - \tfrac{1}{2}e^{-2t} + \tfrac{1}{2}te^{-2t}\right]u(t)$$

ii)

$x(t) = te^{-2t}u(t) \longleftrightarrow X(j\omega) = \frac{1}{(j\omega+2)^2}$

$h(t) = te^{-4t}u(t) \longleftrightarrow H(j\omega) = \frac{1}{(j\omega+4)^2}$

$y(t) = x(t) * h(t) \longleftrightarrow Y(j\omega) = X(j\omega)H(j\omega)$

$$Y(j\omega) = \frac{1}{(j\omega+2)^2(j\omega+4)^2} = \frac{1/4}{2+j\omega} + \frac{1/4}{(2+j\omega)^2} - \frac{1/4}{4+j\omega} + \frac{1/4}{(4+j\omega)^2}$$

$$\Rightarrow y(t) = \left[\tfrac{1}{4}e^{-2t} + \tfrac{1}{4}te^{-2t} - \tfrac{1}{4}e^{-4t} + \tfrac{1}{4}te^{-4t}\right]u(t)$$

iii)

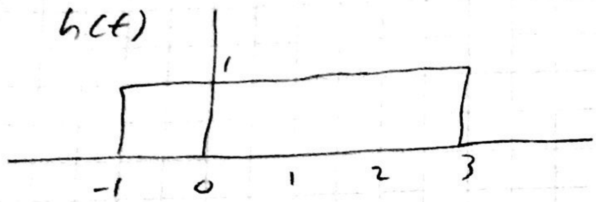
$x(t) = e^{-t}u(t) \longleftrightarrow X(j\omega) = \frac{1}{j\omega+1}$

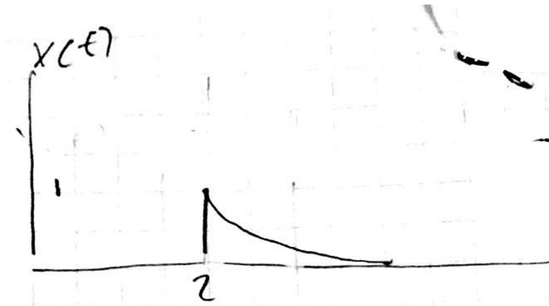
$h(t) = e^{t}u(-t) \longleftrightarrow H(j\omega) = \frac{1}{1-j\omega}$

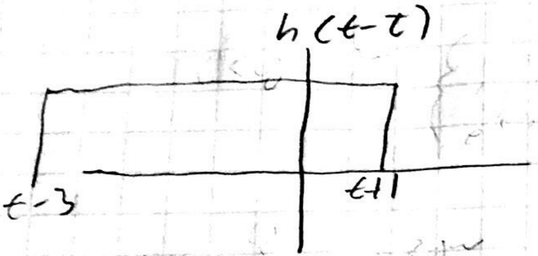
$$y(t) = x(t) * h(t) \longleftrightarrow \frac{1}{j\omega+1} \cdot \frac{1}{1-j\omega} = \frac{1/2}{1+j\omega} + \frac{1/2}{1-j\omega}$$

$$\Rightarrow y(t) = \tfrac{1}{2}e^{-|t|}$$

6. $x(t) = e^{-(t-2)}u(t-2)$







$t<1 \quad x(t)*h(t) = 0$

$1<t<5 \quad x(t)*h(t) = \int_2^{t+1} e^{-(\tau-2)} d\tau$

$= -e^{-(\tau-2)}\Big|_{\tau=2}^{t+1} = 1 - e^{-(t-1)}$

$t>5 \quad x(t)*h(t) = \int_{t-3}^{t+1} e^{-(\tau-2)} d\tau$

$= -e^{-(\tau-2)}\Big|_{\tau=t-3}^{t+1} = e^{-(t-5)} - e^{-(t-1)}$

$x(t) = e^{-(t-2)}u(t-2)$ is $e^{-t}u(t)$ shifted by 2

Then $X(j\omega) = e^{-2j\omega} \cdot \frac{1}{1+j\omega}$

$h(t) = e^{-t} \cdot b(t)$, where $b(t) = \begin{cases} 1 & |t|<2 \\ 0 & |t|>2 \end{cases}$

Then $H(j\omega) = \frac{e^{-j\omega} \cdot 2\sin(\omega \cdot 2)}{\omega} = e^{-j\omega} \cdot \frac{2}{\omega} \cdot \left[\frac{e^{j2\omega} - e^{-2j\omega}}{2j}\right]$

$= \frac{e^{j\omega} - e^{-3j\omega}}{j\omega}$

$X(j\omega)H(j\omega) = \frac{e^{-2j\omega}(e^{j\omega} - e^{-3j\omega})}{(1+j\omega)(j\omega)} = \frac{(e^{-j\omega} - e^{-5j\omega})}{(1+j\omega)j\omega}$

performing partial fraction decomposition on $\frac{1}{(1+j\omega)(j\omega)}$ gives

$$\frac{1}{(1+j\omega)(j\omega)} = \frac{A}{1+j\omega} + \frac{B}{j\omega}$$

$$A = \left.\frac{1}{j\omega}\right|_{\omega=j} = -1$$

$$B = \left.\frac{1}{(1+j\omega)}\right|_{\omega=0} = 1$$

Then we get

$$Y(j\omega) = X(j\omega)H(j\omega) = \left(e^{-j\omega} - e^{-5j\omega}\right)\cdot\left[\frac{-1}{1+j\omega} + \frac{1}{j\omega}\right]$$

$$= \left(e^{-j\omega} - e^{-5j\omega}\right)\left[\frac{-1}{1+j\omega} + \frac{1}{j\omega} + \pi\delta(\omega)\right]$$

$$\Rightarrow y(t) = \left(1 - e^{-(t-1)}\right)u(t-1) - \left(1 - e^{-(t-5)}\right)u(t-5)$$

which agrees with our convolution.

Note that the $\pi\delta(\omega)$ term was added to
⇒ match the form of the Fourier transform of $u(t)$.
It does not change the value of the expression because

$$\left(e^{-j\omega} - e^{-5j\omega}\right)\pi\delta(\omega) = \left(e^{0} - e^{0}\right)\cdot\pi\delta(\omega) = 0.$$