HW 4

EE 3015
Signals and Systems

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Contents

1 Problem 4.1(a), Chapter 4

Find Fourier transform of (a) $e^{-2(t-1)}u(t-1)$

Solution

$$
\begin{aligned}
X(\omega) &= \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt \\
&= \int_{1}^{\infty} e^{-2(t-1)} e^{-i\omega t} dt \\
&= \int_{1}^{\infty} e^{-2t} e^{2} e^{-i\omega t} dt \\
&= e^{2} \int_{1}^{\infty} e^{-t(2+i\omega)} dt \\
&= \frac{e^{2}}{-(2+i\omega)} \left[e^{-t(2+i\omega)} \right]_{1}^{\infty}
\end{aligned}
$$

Assuming $\operatorname{Im}(\omega) < 2$ then

$$
\begin{aligned}
X(\omega) &= \frac{e^{2}}{-(2+i\omega)} \left[0 - e^{-2} e^{-i\omega} \right] \\
&= \frac{e^{2}}{(2+i\omega)} \left[e^{-2} e^{-i\omega} \right] \\
&= \frac{e^{-i\omega}}{(2+i\omega)}
\end{aligned}
$$

2 Problem 4.3, Chapter 4

Determine the Fourier transform of each of the following periodic signals (a) $\sin\left(2\pi t + \frac{\pi}{4}\right)$ (b) $1 + \cos\left(6\pi t + \frac{\pi}{8}\right)$

Solution

2.1 Part a

Since this is periodic signal, then we can not use $X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt$ which is for aperiodic signal. Instead we need to use 4.22 in the textbook which is

$$
X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)
$$

From $\sin\left(2\pi t + \frac{\pi}{4}\right)$ we see that $\omega_0 = 2\pi$, hence

$$
X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - 2k\pi)
$$

Writing $\sin\left(\omega_0 t + \frac{\pi}{4}\right) = \frac{1}{2j} e^{j\left(\omega_0 t + \frac{\pi}{4}\right)} - \frac{1}{2j} e^{-j\left(\omega_0 t + \frac{\pi}{4}\right)} = \left(\frac{1}{2j} e^{j\frac{\pi}{4}}\right) e^{j\omega_0 t} - \left(\frac{1}{2j} e^{-j\frac{\pi}{4}}\right) e^{-j\omega_0 t}$ shows that $a_1 = \frac{1}{2j} e^{j\frac{\pi}{4}}$ and $a_{-1} = -\frac{1}{2j} e^{-j\frac{\pi}{4}}$ and $a_k = 0$ for all other k. Hence above simplifies to

$$
\begin{aligned}
X(\omega) &= 2\pi \left(\frac{1}{2j} e^{j\frac{\pi}{4}}\right) \delta(\omega - 2\pi) + 2\pi \left(-\frac{1}{2j} e^{-j\frac{\pi}{4}}\right) \delta(\omega + 2\pi) \\
&= \frac{\pi}{j} e^{j\frac{\pi}{4}} \delta(\omega - 2\pi) - \frac{\pi}{j} e^{-j\frac{\pi}{4}} \delta(\omega + 2\pi)
\end{aligned}
$$

2.2 Part b

Since this is periodic signal, then its Fourier transform is

$$
X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)
$$

From $1+\cos\left(6\pi t+\frac{\pi}{8}\right)$ we see that $\omega_0 = 6\pi$, hence

$$X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - 6k\pi)$$

Writing $1+\cos\left(6\pi t+\frac{\pi}{8}\right) = 1 + \left(\frac{1}{2}e^{j\left(\omega_0 t+\frac{\pi}{8}\right)} + \frac{1}{2}e^{-j\left(\omega_0 t+\frac{\pi}{8}\right)}\right) = 1 + \left(\frac{1}{2}e^{j\frac{\pi}{8}}e^{j\omega_0 t} + \frac{1}{2}e^{-j\frac{\pi}{8}}e^{-j\omega_0 t}\right)$ shows that $a_1 = \frac{1}{2}e^{j\frac{\pi}{8}}$ and $a_{-1} = \frac{1}{2}e^{-j\frac{\pi}{8}}$ and $a_0 = 1$. Therefore the above becomes

$$\begin{aligned}
X(\omega) &= 2\pi a_{-1}\delta(\omega+6\pi) + 2\pi a_0 \delta(\omega) + 2\pi a_1 \delta(\omega - 6\pi) \\
&= 2\pi\left(\frac{1}{2}e^{-j\frac{\pi}{8}}\right)\delta(\omega+6\pi) + 2\pi\delta(\omega) + 2\pi\left(\frac{1}{2}e^{j\frac{\pi}{8}}\right)\delta(\omega-6\pi) \\
&= \pi e^{-j\frac{\pi}{8}}\delta(\omega+6\pi) + 2\pi\delta(\omega) + \pi e^{j\frac{\pi}{8}}\delta(\omega-6\pi)
\end{aligned}$$

3 Problem 4.5, Chapter 4

Use the Fourier transform synthesis equation (4.8) to determine the inverse Fourier transform of $X\left(j\omega\right)=\left|X\left(j\omega\right)\right|e^{j\measuredangle X\left(j\omega\right)}$ where $\left|X\left(j\omega\right)\right|=2\left(u\left(\omega+3\right)-u\left(\omega-3\right)\right), \measuredangle X\left(j\omega\right)=-\frac{3}{2}\omega+\pi$

Solution

4.8 is

$$x\left(t\right)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X\left(\omega\right)e^{j\omega t}d\omega \tag{4.8}$$

Hence

$$\begin{aligned}x\left(t\right)&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|X\left(j\omega\right)\right|e^{j\measuredangle X\left(j\omega\right)}e^{j\omega t}d\omega\\&=\frac{1}{2\pi}\int_{-\infty}^{\infty}2\left(u\left(\omega+3\right)-u\left(\omega-3\right)\right)e^{j\left(-\frac{3}{2}\omega+\pi\right)}e^{j\omega t}d\omega\end{aligned}$$

But $u\left(\omega+3\right)-u\left(\omega-3\right)$ is one over $\omega=-3\cdots3$ and zero otherwise. The above simplifies to

$$\begin{aligned}x\left(t\right)&=\frac{1}{2\pi}\int_{-3}^{3}2e^{j\left(-\frac{3}{2}\omega+\pi\right)}e^{j\omega t}d\omega\\&=\frac{1}{\pi}\int_{-3}^{3}e^{j\pi}e^{j-\frac{3}{2}\omega}e^{j\omega t}d\omega\\&=\frac{e^{j\pi}}{\pi}\int_{-3}^{3}e^{j\left(-\frac{3}{2}+t\right)\omega}d\omega\end{aligned}$$

But $e^{j\pi}=-1$ and $\int e^{j\left(-\frac{3}{2}+t\right)\omega}d\omega=\frac{e^{j\left(-\frac{3}{2}+t\right)\omega}}{j\left(-\frac{3}{2}+t\right)}$, hence the above becomes

$$\begin{aligned}x\left(t\right)&=\frac{-1}{\pi}\frac{1}{j\left(-\frac{3}{2}+t\right)}\left[e^{j\left(-\frac{3}{2}+t\right)\omega}\right]_{-3}^{3}\\&=\frac{-1}{\pi\left(-\frac{3}{2}+t\right)}\left(\frac{e^{j3\left(-\frac{3}{2}+t\right)}-e^{-j3\left(-\frac{3}{2}+t\right)}}{j}\right)\\&=\frac{-1}{\pi\left(-\frac{3}{2}+t\right)}2\left(\sin\left(3\left(-\frac{3}{2}+t\right)\right)\right)\\&=\frac{-2}{\pi\left(t-\frac{3}{2}\right)}\sin\left(3\left(t-\frac{3}{2}\right)\right)\end{aligned}$$

4 Problem 4.11, Chapter 4

Given the relationships $y(t) = x(t) \circledast h(t)$ and $g(t) = x(3t) \circledast h(3t)$ and given that $x(t)$ has Fourier transform $X\left(j\omega\right)$ and $h(t)$ has Fourier transform $H\left(j\omega\right)$, use Fourier transform properties to show that $g(t)$ has the form $g(t) = Ay(Bt)$. Determine the values of A and B

Solution

The main relation to use is that if $y(t) \Longleftrightarrow Y(\omega)$ then $y(at) \Longleftrightarrow \frac{1}{a}Y\left(\frac{\omega}{a}\right)$. Therefore $x(3t) \Longleftrightarrow \frac{1}{3}X\left(\frac{\omega}{3}\right)$ and $h(3t) \Longleftrightarrow \frac{1}{3}H\left(\frac{\omega}{3}\right)$. Hence since $g(t) = x(3t) \circledast h(3t)$ then

$$\begin{aligned} G(\omega) &= \frac{1}{3}X\left(\frac{\omega}{3}\right)\frac{1}{3}H\left(\frac{\omega}{3}\right) \\ &= \frac{1}{3}\left(\frac{1}{3}X\left(\frac{\omega}{3}\right)H\left(\frac{\omega}{3}\right)\right) \end{aligned}$$

But $X\left(\frac{\omega}{3}\right)H\left(\frac{\omega}{3}\right) = Y\left(\frac{\omega}{3}\right)$. Therefore the above becomes

$$G(\omega) = \frac{1}{3}\left(\frac{1}{3}Y\left(\frac{\omega}{3}\right)\right)$$

Inverse Fourier transform gives

$$g(t) = \frac{1}{3}y(3t)$$

Where in the above, we used $\frac{1}{3}Y\left(\frac{\omega}{3}\right) \Longleftrightarrow y(3t)$. Hence $A = \frac{1}{3}, B = 3$

5 Problem 4.19, Chapter 4

Consider a causal LTI system with frequency response $H\left(j\omega\right) = \frac{1}{j\omega+3}$. For a particular input $x(t)$ this system is observed to produce the output $y(t) = e^{-3t}u(t) - e^{-4t}u(t)$. Determine $x(t)$.

<u>Solution</u>

$$y(t) = x(t) \circledast h(t)$$

Taking Fourier transform gives

$$Y(\omega) = X(\omega) H(\omega)$$

Hence

$$X(\omega) = \frac{Y(\omega)}{H(\omega)}$$

But $y(t) = e^{-3t}u(t) - e^{-4t}u(t)$, therefore, from table $Y(\omega) = \frac{1}{3+j\omega} - \frac{1}{4+j\omega} = \frac{\left(4+j\omega\right)-\left(3+j\omega\right)}{\left(3+j\omega\right)\left(4+j\omega\right)} = \frac{1}{\left(3+j\omega\right)\left(4+j\omega\right)}$ and the above becomes

$$X(\omega) = \frac{\frac{1}{\left(3+j\omega\right)\left(4+j\omega\right)}}{H(\omega)}$$

But we are given that $H\left(j\omega\right) = \frac{1}{j\omega+3}$. The above simplifies to

$$\begin{aligned} X(\omega) &= \frac{\frac{1}{\left(3+j\omega\right)\left(4+j\omega\right)}}{\frac{1}{j\omega+3}} \\ &= \frac{1}{4+j\omega} \end{aligned}$$

From tables

$$x(t) = e^{-4t}u(t)$$

6 Problem 4.23, Chapter 4

4.23. Consider the signal

$$x_0(t) = \begin{cases} e^{-t}, & 0 \le t \le 1 \\ 0, & \text{elsewhere} \end{cases}.$$

Determine the Fourier transform of each of the signals shown in Figure P4.23. You should be able to do this by explicitly evaluating *only* the transform of $x_0(t)$ and then using properties of the Fourier transform.

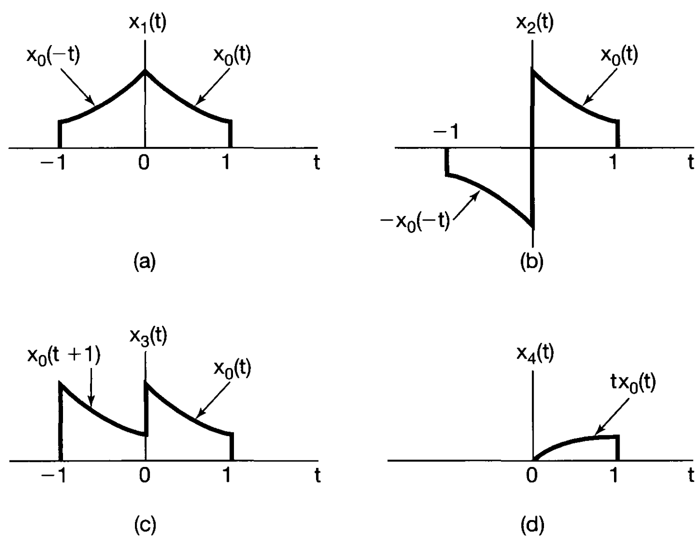

Figure P4.23

Figure 1: Problem description

Solution

6.1 Part a

First we find the Fourier transform of $x_0(t)$. Since this is a periodic, then $x_0(t) \Longleftrightarrow X_0(\omega)$ and

$$\begin{aligned}
X_0(\omega) &= \int_{-\infty}^{\infty} x_0(t)\, e^{-j\omega t} dt \\
&= \int_0^1 e^{-t} e^{-j\omega t} dt \\
&= \int_0^1 e^{-t(1+j\omega)} dt \\
&= \frac{-1}{1+j\omega}\left[e^{-t(1+j\omega)}\right]_0^1 \\
&= \frac{-1}{1+j\omega}\left(e^{-(1+j\omega)} - 1\right) \\
&= \frac{1-e^{-(1+j\omega)}}{1+j\omega}
\end{aligned}$$

From table 4.1, property 4.3.5, Fourier transform of $x(-t) = X(-\omega)$. Hence $x_0(-t) \Longleftrightarrow X_0(-\omega)$. Therefore, using the above result and taking its complex conjugate gives

$$X_0(-\omega) = \frac{1-e^{-(1-j\omega)}}{1-j\omega}$$

Therefore the Fourier transform of $x_0(t) + x_0(-t) \Longleftrightarrow X_1(\omega) = X_0(\omega) + X_0(-\omega)$ This is by linearity property. Hence

$$
\begin{aligned}
X_1(\omega) &= X_0(\omega) + X_0(-\omega) \\
&= \frac{1 - e^{-(1+j\omega)}}{1 + j\omega} + \frac{1 - e^{-(1-j\omega)}}{1 - j\omega} \\
&= \frac{(1 - j\omega)\left(1 - e^{-(1+j\omega)}\right) + (1 + j\omega)\left(1 - e^{-(1-j\omega)}\right)}{(1 + j\omega)(1 - j\omega)} \\
&= \frac{1 - e^{-(1+j\omega)} - j\omega\left(1 - e^{-(1+j\omega)}\right) + \left(1 - e^{-(1-j\omega)}\right) + j\omega\left(1 - e^{-(1-j\omega)}\right)}{1 + \omega^2} \\
&= \frac{1 - e^{-(1+j\omega)} - j\omega + j\omega e^{-(1+j\omega)} + \left(1 - e^{-(1-j\omega)}\right) + j\omega - j\omega e^{-(1-j\omega)}}{1 + \omega^2} \\
&= \frac{1 - e^{-(1+j\omega)} + j\omega e^{-(1+j\omega)} + 1 - e^{-(1-j\omega)} - j\omega e^{-(1-j\omega)}}{1 + \omega^2} \\
&= \frac{1 - e^{-1}e^{-j\omega} + j\omega e^{-1}e^{-j\omega} + 1 - e^{-1}e^{j\omega} - j\omega e^{-1}e^{j\omega}}{1 + \omega^2} \\
&= \frac{1 - e^{-1}\left(e^{j\omega} + e^{-j\omega}\right) - j\omega e^{-1}\left(e^{j\omega} - e^{-j\omega}\right)}{1 + \omega^2} \\
&= \frac{1}{1 + \omega^2} - \frac{e^{-1}}{1 + \omega^2}\left(e^{j\omega} + e^{-j\omega}\right) + \frac{\omega e^{-1}}{1 + \omega^2}\frac{\left(e^{j\omega} - e^{-j\omega}\right)}{j} \\
&= \frac{1}{1 + \omega^2} - \frac{2}{e\left(1 + \omega^2\right)}\cos\omega + \frac{2\omega}{e\left(1 + \omega^2\right)}\sin\omega
\end{aligned}
$$

The following is a plot of the above

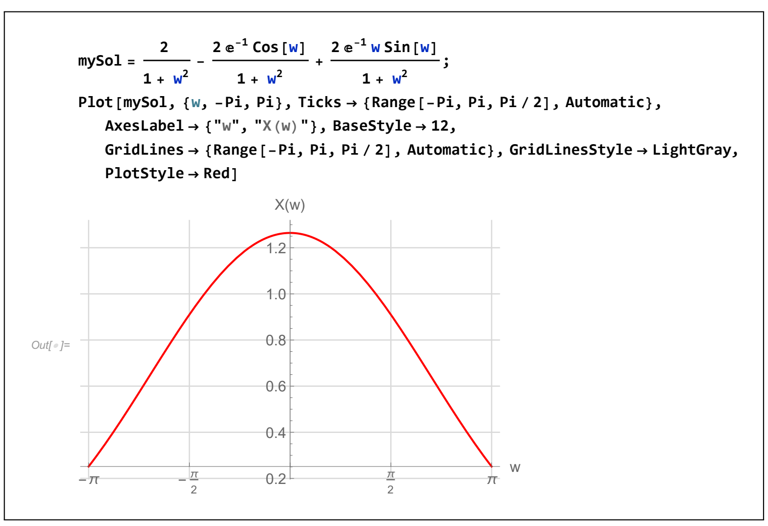

Figure 2: Plot of $X(\omega)$

We see that $X_1(\omega)$ is even and real. This agrees with table 4.1, property 4.3.3 which says that for real $x(t)$ which is even, then its Fourier transform is real and even.

6.2 Part b

We found $X_0(\omega) = \frac{1-e^{-(1+j\omega)}}{1+j\omega}$ above. Hence $-x_0(-t) \iff -X_0(-\omega) = -\frac{1-e^{-(1-j\omega)}}{1-j\omega} = \frac{1-e^{-(1-j\omega)}}{j\omega-1}$. Therefore

$$\begin{aligned}
X_2(\omega) &= X_0(\omega) - X_0(-\omega) \\
&= \frac{1-e^{-(1+j\omega)}}{1+j\omega} + \frac{1-e^{-(1-j\omega)}}{j\omega-1} \\
&= \frac{(j\omega-1)\left(1-e^{-(1+j\omega)}\right) + (1+j\omega)\left(1-e^{-(1-j\omega)}\right)}{(1+j\omega)(j\omega-1)} \\
&= \frac{j\omega\left(1-e^{-(1+j\omega)}\right) - \left(1-e^{-(1+j\omega)}\right) + \left(1-e^{-(1-j\omega)}\right) + j\omega\left(1-e^{-(1-j\omega)}\right)}{j\omega-1-\omega^2-j\omega} \\
&= \frac{j\omega - j\omega e^{-(1+j\omega)} - 1 + e^{-(1+j\omega)} + 1 - e^{-(1-j\omega)} + j\omega - j\omega e^{-(1-j\omega)}}{-\left(1+\omega^2\right)} \\
&= \frac{2j\omega - j\omega e^{-(1+j\omega)} + e^{-(1+j\omega)} - e^{-(1-j\omega)} - j\omega e^{-(1-j\omega)}}{-\left(1+\omega^2\right)} \\
&= \frac{2j\omega - j\omega e^{-1}e^{-j\omega} + e^{-1}e^{-j\omega} - e^{-1}e^{j\omega} - j\omega e^{-1}e^{j\omega}}{j\omega-\omega^2-2} \\
&= \frac{2j\omega - j\omega e^{-1}\left(e^{j\omega}+e^{-j\omega}\right) - e^{-1}\left(e^{j\omega}-e^{-j\omega}\right)}{-\left(1+\omega^2\right)} \\
&= \frac{2j\omega}{-\left(1+\omega^2\right)} - j\omega e^{-1}\frac{\left(e^{j\omega}+e^{-j\omega}\right)}{-\left(1+\omega^2\right)} - e^{-1}\frac{e^{j\omega}-e^{-j\omega}}{-\left(1+\omega^2\right)} \\
&= \frac{2j\omega}{-\left(1+\omega^2\right)} - 2j\omega e^{-1}\frac{\cos\omega}{-\left(1+\omega^2\right)} - 2je^{-1}\frac{\sin(\omega)}{-\left(1+\omega^2\right)}
\end{aligned}$$

Hence

$$\begin{aligned}
X_2(\omega) &= j\left(\frac{-2\omega}{\left(1+\omega^2\right)} + 2\omega\frac{\cos\omega}{e\left(1+\omega^2\right)} + 2\frac{\sin(\omega)}{e\left(1+\omega^2\right)}\right) \\
&= \frac{j}{e\left(1+\omega^2\right)}\left(-2e\omega + 2\omega\cos\omega + 2\sin(\omega)\right)
\end{aligned}$$

We see that $X_2(\omega)$ is pure imaginary. This agrees with table 4.1, property 4.3.3 which says that for real $x(t)$ which is odd, then its Fourier transform is pure imaginary and odd.

The following is a plot of the above which shows that the imaginary part of $X_2(\omega)$ is odd

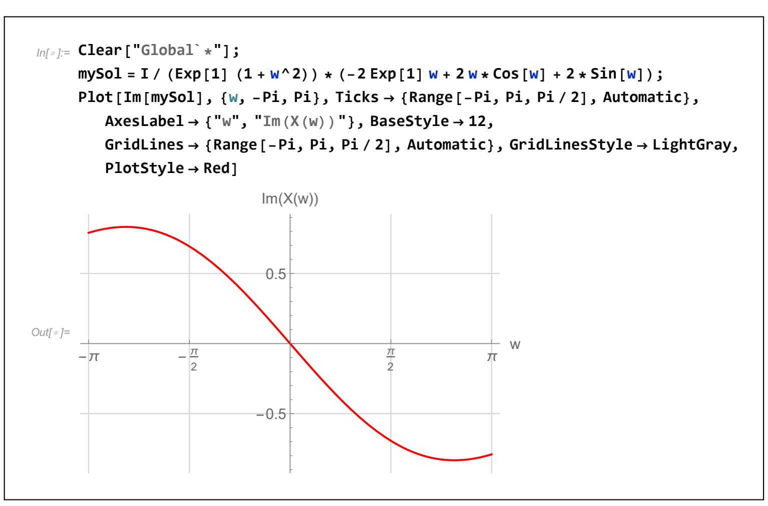

Figure 3: Plot of imaginary part of $X(\omega)$

7 Problem 4.26, Chapter 4

4.26. (a) Compute the convolution of each of the following pairs of signals $x(t)$ and $h(t)$ by calculating $X(j\omega)$ and $H(j\omega)$, using the convolution property, and inverse transforming.

(i) $x(t) = te^{-2t}u(t),\ h(t) = e^{-4t}u(t)$

(ii) $x(t) = te^{-2t}u(t),\ h(t) = te^{-4t}u(t)$

(iii) $x(t) = e^{-t}u(t),\ h(t) = e^{t}u(-t)$

(b) Suppose that $x(t) = e^{-(t-2)}u(t-2)$ and $h(t)$ is as depicted in Figure P4.26. Verify the convolution property for this pair of signals by showing that the Fourier transform of $y(t) = x(t) * h(t)$ equals $H(j\omega)X(j\omega)$.

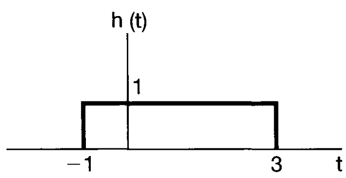

Figure P4.26

Figure 4: Problem description

Solution

7.1 Part a

Part i

$$y(t) = x(t) \circledast h(t)$$

Taking Fourier transform the above, convolution becomes multiplication

$$Y(\omega) = X(\omega) H(\omega)$$

Given that $x(t) = te^{-2t}u(t)$, then from tables $X(\omega) = \frac{1}{(2+j\omega)^2}$ and given that $h(t) = e^{-4t}u(t)$ then from table 4.2 $H(\omega) = \frac{1}{4+j\omega}$. Hence the above becomes

$$Y(\omega) = \frac{1}{\left(2 + j\omega\right)^2 \left(4 + j\omega\right)}$$

Doing partial fractions. Let $s = j\omega$ then

$$\begin{aligned}
\frac{1}{(2+s)^2(4+s)} &= \frac{A}{(2+s)^2} + \frac{B}{2+s} + \frac{C}{4+s} \\
&= \frac{A(4+s) + B((2+s)(4+s)) + C(2+s)^2}{(2+s)^2(4+s)}
\end{aligned}$$

Expanding numerator

$$\begin{aligned}
1 &= 4A + 8B + 4C + Bs^2 + Cs^2 + As + 6Bs + 4Cs \\
1 &= (4A + 8B + 4C) + s(A + 6B + 4C) + s^2(B + C)
\end{aligned}$$

Comparing coefficients

$$\begin{aligned}
1 &= 4A + 8B + 4C \\
0 &= A + 6B + 4C \\
0 &= B + C
\end{aligned}$$

Or

$$\begin{pmatrix} 4 & 8 & 4 \\ 1 & 6 & 4 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Gaussian elimination. Multiplying second row by 4 and subtracting result from first row gives

$$\begin{pmatrix} 4 & 8 & 4 \\ 0 & 16 & 12 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

Multiplying third row by 16 and subtracting result from second row gives

$$\begin{pmatrix} 4 & 8 & 4 \\ 0 & 16 & 12 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

Backsubstitution. Last row gives $C = \frac{1}{4}$. Second row gives $16B+12C = -1$ or $16B = -1-12\left(\frac{1}{4}\right)$, hence $16B = -4$, Hence $B = -\frac{1}{4}$. First row gives $4A + 8B + 4C = 1$ or $4A + 8\left(-\frac{1}{4}\right) + 4\left(\frac{1}{4}\right) = 1$ or $4A - 2 + 1 = 1$. Hence $A = \frac{1}{2}$. Therefore partial fractions gives

$$\frac{1}{(2+s)^2(4+s)} = \frac{\left(\frac{1}{2}\right)}{(2+s)^2} + \frac{\left(-\frac{1}{4}\right)}{2+s} + \frac{\left(\frac{1}{4}\right)}{4+s}$$

Replacing s back with $j\omega$

$$Y(\omega) = \frac{1}{2}\frac{1}{\left(2+j\omega\right)^2} - \frac{1}{4}\frac{1}{2+j\omega} + \frac{1}{4}\frac{1}{4+j\omega}$$

Applying inverse Fourier transform, using table gives

$$y(t) = \frac{1}{2}te^{-2t}u(t) - \frac{1}{4}e^{-2t}u(t) + \frac{1}{4}e^{-4t}u(t)$$

Part ii

$$y(t) = x(t) \circledast h(t)$$

Taking Fourier transform the above, convolution becomes multiplication

$$Y(\omega) = X(\omega)H(\omega)$$

Given that $x(t) = te^{-2t}u(t)$, then from tables $X(\omega) = \frac{1}{\left(2+j\omega\right)^2}$ and given that $h(t) = te^{-4t}u(t)$ then from table 4.2 $H(\omega) = \frac{1}{\left(4+j\omega\right)^2}$. Hence the above becomes

$$Y(\omega) = \frac{1}{\left(2+j\omega\right)^2\left(4+j\omega\right)^2}$$

Doing partial fractions. Let $s = j\omega$ then

$$\begin{aligned}\frac{1}{(2+s)^2(4+s)^2} &= \frac{A}{(2+s)^2} + \frac{B}{2+s} + \frac{C}{(4+s)^2} + \frac{D}{(4+s)} \\ &= \frac{A(4+s)^2 + B\left((2+s)(4+s)^2\right) + C(2+s)^2 + D\left((2+s)^2(4+s)\right)}{(2+s)^2(4+s)^2}\end{aligned}$$

Expanding numerator

$$\begin{aligned} 1 &= 16A + 32B + 4C + 16D + 20sD + As^2 + 10Bs^2 + Bs^3 + Cs^2 + 8s^2D + s^3D + 8As + 32Bs + 4Cs \\ 1 &= (16A + 32B + 4C + 16D) + s(20D + 8A + 32B + 4C) + s^2(A + 10B + C + 8D) + s^3(B + D) \end{aligned}$$

Comparing coefficients

$$
\begin{aligned}
1 &= 16A + 32B + 4C + 16D \\
0 &= 8A + 32B + 4C + 20D \\
0 &= A + 10B + C + 8D \\
0 &= B + D
\end{aligned}
$$

Or

$$
\begin{pmatrix} 16 & 32 & 4 & 16 \\ 8 & 32 & 4 & 20 \\ 1 & 10 & 1 & 8 \\ 0 & 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

Gaussian elimination. replacing row 2 by result of subtracting 1/2 times row 1 from row 2

$$
\begin{pmatrix} 16 & 32 & 4 & 16 \\ 0 & 16 & 2 & 12 \\ 1 & 10 & 1 & 8 \\ 0 & 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ 0 \\ 0 \end{pmatrix}
$$

Replacing row 3 by result of subtracting $\frac{1}{16}$ times row 1 from row 3

$$
\begin{pmatrix} 16 & 32 & 4 & 16 \\ 0 & 16 & 2 & 12 \\ 0 & 8 & \frac{3}{4} & 7 \\ 0 & 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ -\frac{1}{16} \\ 0 \end{pmatrix}
$$

Replacing row 3 by result of subtracting $\frac{1}{2}$ times row 2 from row 3

$$
\begin{pmatrix} 16 & 32 & 4 & 16 \\ 0 & 16 & 2 & 12 \\ 0 & 0 & \frac{-1}{4} & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ \frac{3}{16} \\ 0 \end{pmatrix}
$$

Replacing row 4 by result of subtracting $\frac{1}{16}$ times row 2 from row 4

$$
\begin{pmatrix} 16 & 32 & 4 & 16 \\ 0 & 16 & 2 & 12 \\ 0 & 0 & \frac{-1}{4} & 1 \\ 0 & 0 & \frac{-1}{8} & \frac{1}{4} \end{pmatrix}\begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ \frac{3}{16} \\ \frac{1}{32} \end{pmatrix}
$$

Replacing row 4 by result of subtracting $\frac{1}{2}$ times row 3 from row 4

$$
\begin{pmatrix} 16 & 32 & 4 & 16 \\ 0 & 16 & 2 & 12 \\ 0 & 0 & \frac{-1}{4} & 1 \\ 0 & 0 & 0 & \frac{-1}{4} \end{pmatrix}\begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ \frac{3}{16} \\ \frac{-1}{16} \end{pmatrix}
$$

Backsubstitution phase:

$$
\begin{aligned}
\frac{-1}{4}D &= \frac{-1}{16} \\
D &= \frac{1}{4}
\end{aligned}
$$

Third row gives $\frac{-1}{4}C + D = \frac{3}{16}$ or $\frac{-1}{4}C + \frac{1}{4} = \frac{3}{16} \rightarrow C = \frac{1}{4}$. Second row gives $16B + 2C + 12D = -\frac{1}{2}$ or $16B + 2\left(\frac{1}{4}\right) + 12\left(\frac{1}{4}\right) = -\frac{1}{2} \rightarrow B = -\frac{1}{4}$. First row gives $16A + 32B + 4C + 16D = 1$ or $16A + 32\left(-\frac{1}{4}\right) + 4\left(\frac{1}{4}\right) + 16\left(\frac{1}{4}\right) = 1, \rightarrow A = \frac{1}{4}$.

Therefore partial fractions gives

$$\frac{1}{(2+s)^2(4+s)^2} = \frac{A}{(2+s)^2} + \frac{B}{2+s} + \frac{C}{(4+s)^2} + \frac{D}{(4+s)}$$
$$= \frac{1}{4}\frac{1}{(2+s)^2} - \frac{1}{4}\frac{1}{2+s} + \frac{1}{4}\frac{1}{(4+s)^2} + \frac{1}{4}\frac{1}{(4+s)}$$

Replace s back with $j\omega$

$$Y(\omega) = \frac{1}{4}\frac{1}{\left(2+j\omega\right)^2} - \frac{1}{4}\frac{1}{2+j\omega} + \frac{1}{4}\frac{1}{\left(4+j\omega\right)^2} + \frac{1}{4}\frac{1}{\left(4+j\omega\right)}$$

Applying inverse Fourier transform, using table gives

$$y(t) = \frac{1}{4}te^{-2t}u(t) - \frac{1}{4}e^{-2t}u(t) + \frac{1}{4}te^{-4t}u(t) + \frac{1}{4}e^{-4t}u(t)$$

Part iii

$$y(t) = x(t) \circledast h(t)$$

Taking Fourier transform the above, convolution becomes multiplication

$$Y(\omega) = X(\omega)H(\omega)$$

Given that $x(t) = e^{-t}u(t)$, then from tables $X(\omega) = \frac{1}{\left(1+j\omega\right)}$ and given that $h(t) = e^{t}u(-t)$ then $H(\omega) = \frac{1}{1-j\omega}$. Hence the above becomes

$$Y(\omega) = \frac{1}{\left(1+j\omega\right)}\frac{1}{\left(1-j\omega\right)}$$

Doing partial fractions. Let $s = j\omega$ then

$$\frac{1}{(1+s)}\frac{1}{(1-s)} = \frac{A}{(1+s)} + \frac{B}{1-s}$$

Hence $A = \left(\frac{1}{(1-s)}\right)_{s=-1} = \frac{1}{2}$ and $B = \left(\frac{1}{(1+s)}\right)_{s=1} = \frac{1}{2}$. Hence

$$Y(\omega) = \frac{1}{2}\frac{1}{\left(1+j\omega\right)} + \frac{1}{2}\frac{1}{1-j\omega}$$

Therefore

$$y(t) = \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{t}u(-t)$$

The above means for $t < 0, y(t) = \frac{1}{2}e^{t}$ and for $t > 0, y(t) = \frac{1}{2}e^{-t}$.

7.2 Part b

$x(t) = e^{-(t-2)}u(t-2)$, $h(t) = u(t+1) - u(t-3)$. Let us first find $X(\omega)$ and $H(\omega)$

$$\begin{aligned}
X(\omega) &= \int_{-\infty}^{\infty} e^{-(t-2)}u(t-2)\,e^{-j\omega t}dt \\
&= \int_{2}^{\infty} e^{-(t-2)}e^{-j\omega t}dt \\
&= \int_{2}^{\infty} e^{-t}e^{2}e^{-j\omega t}dt \\
&= e^{2}\int_{2}^{\infty} e^{-t\left(1+j\omega\right)}dt \\
&= -\frac{e^{2}}{1+j\omega}\left[e^{-t\left(1+j\omega\right)}\right]_{2}^{\infty} \\
&= -\frac{e^{2}}{1+j\omega}\left(0-e^{-2\left(1+j\omega\right)}\right) \\
&= \frac{e^{2}e^{-2\left(1+j\omega\right)}}{1+j\omega} \\
&= \frac{e^{-2-2j\omega+2}}{1+j\omega} \\
&= \frac{e^{-2j\omega}}{1+j\omega}
\end{aligned}$$

And

$$\begin{aligned}
H(\omega) &= \int_{-\infty}^{\infty} u(t+1)-u(t-3)\,e^{-j\omega t}dt \\
&= \int_{-1}^{3} e^{-j\omega t}dt \\
&= \frac{1}{j\omega}\left(e^{-j\omega t}\right)_{-1}^{3} \\
&= \frac{1}{j\omega}\left(e^{-3j\omega}-e^{j\omega t}\right) \\
&= \frac{1}{j\omega}e^{-j\omega}\left(e^{-j2\omega}-e^{j2\omega t}\right) \\
&= -\frac{1}{j\omega}e^{-j\omega}\left(e^{j2\omega t}-e^{-j2\omega}\right) \\
&= -\frac{1}{\omega}e^{-j\omega}\left(\frac{e^{j2\omega t}-e^{-j2\omega}}{j}\right) \\
&= -\frac{2}{\omega}e^{-j\omega}\sin(2\omega)
\end{aligned}$$

Hence

$$\begin{aligned}
Y(\omega) &= H(\omega)\,X(\omega) \\
&= \frac{e^{-2j\omega}}{1+j\omega}\left(-\frac{2}{\omega}e^{-j\omega}\sin(2\omega)\right) \\
&= \frac{-2}{\omega\left(1+j\omega\right)}e^{-2j\omega}e^{-j\omega}\sin(2\omega) \\
&= \frac{-2}{\omega\left(1+j\omega\right)}e^{-3j\omega}\sin(2\omega) \qquad (1)
\end{aligned}$$

Now,

$$\begin{aligned}
y(t) &= x(t)\circledast h(t) \\
&= \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau
\end{aligned}$$

Folding $h(t)$. For $t<1$, then $y(t)=0$. For $2<1+t<6$ or $1<t<5$

$$
\begin{aligned}
y(t) &= \int_{2}^{1+t} x(\tau)\,d\tau \\
&= \int_{2}^{1+t} e^{-(\tau-2)} d\tau \\
&= -\left(e^{-(\tau-2)}\right)_{2}^{1+t} \\
&= -\left(e^{-(1+t-2)} - e^{-(2-2)}\right) \\
&= -\left(e^{-(-1+t)} - 1\right) \\
&= 1 - e^{1-t}
\end{aligned}
$$

For $6<1+t$ or $t>5$

$$
\begin{aligned}
y(t) &= \int_{t-3}^{t+1} x(\tau)\,d\tau \\
&= \int_{t-3}^{t+1} e^{-(\tau-2)} d\tau \\
&= -\left(e^{-(\tau-2)}\right)_{t-3}^{1+t} \\
&= -\left(e^{-(1+t-2)} - e^{-(t-3-2)}\right) \\
&= -\left(e^{-(-1+t)} - e^{-(t-5)}\right) \\
&= -\left(e^{1-t} - e^{5-t}\right) \\
&= e^{5-t} - e^{1-t}
\end{aligned}
$$

Hence

$$
y(t) = \begin{cases} 0 & t<1 \\ 1-e^{1-t} & 1<t<5 \\ e^{5-t}-e^{1-t} & t>5 \end{cases}
$$

The above can be written as $y(t)$

$$
\begin{aligned}
y(t) &= \left(1-e^{1-t}\right)\left(u(t-1)-u(t-5)\right) + \left(e^{5-t}-e^{1-t}\right)u(t-5) \\
&= u(t-1)-u(t-5)-e^{1-t}\left(u(t-1)-u(t-5)\right) + \left(e^{5-t}-e^{1-t}\right)u(t-5) \\
&= u(t-1)-u(t-5)-e^{1-t}u(t-1)+e^{1-t}u(t-5)+e^{5-t}u(t-5)-e^{1-t}u(t-5) \\
&= u(t-1)-u(t-5)-e^{1-t}u(t-1)+e^{5-t}u(t-5) \qquad (2)
\end{aligned}
$$

Taking the Fourier transform of the above from tables

$$
\begin{aligned}
u(t-1) &\Longleftrightarrow \frac{1}{j\omega}e^{-j\omega} + \pi\delta(\omega) \\
u(t-5) &\Longleftrightarrow \frac{1}{j\omega}e^{-5j\omega} + \pi\delta(\omega) \\
e^{1-t}u(t-1) &\Longleftrightarrow \frac{e^{-j\omega}}{1+j\omega} \\
e^{5-t}u(t-5) &\Longleftrightarrow \frac{e^{-5j\omega}}{1+j\omega}
\end{aligned}
$$

Hence $Y(\omega)$ is

$$\begin{aligned}
Y(\omega) &= \frac{1}{j\omega}e^{-j\omega} + \pi\delta(\omega) - \left(\frac{1}{j\omega}e^{-5j\omega} + \pi\delta(\omega)\right) - \frac{e^{-j\omega}}{1+j\omega} + \frac{e^{-5j\omega}}{1+j\omega} \\
&= \frac{1}{j\omega}e^{-j\omega} - \frac{1}{j\omega}e^{-5j\omega} - \frac{e^{-j\omega}}{1+j\omega} + \frac{e^{-5j\omega}}{1+j\omega} \\
&= \frac{e^{-3j\omega}}{\omega}\left(\frac{1}{j}e^{2j\omega} - \frac{1}{j}e^{-2j\omega}\right) - \frac{e^{-j\omega}}{1+j\omega} + \frac{e^{-5j\omega}}{1+j\omega} \\
&= \frac{e^{-3j\omega}}{\omega}2(\sin 2\omega) - \frac{e^{-3j\omega}}{1+j\omega}\left(e^{2j\omega} - e^{-2j\omega}\right) \\
&= \frac{e^{-3j\omega}}{\omega}2(\sin 2\omega) - 2\frac{e^{-3j\omega}}{j+\omega}\sin 2\omega \\
&= 2e^{-3j\omega}\sin 2\omega\left(\frac{1}{\omega} - \frac{1}{j+\omega}\right) \\
&= 2e^{-3j\omega}\sin 2\omega\left(\frac{j+\omega-\omega}{\omega\left(\omega+j\right)}\right) \\
&= 2e^{-3j\omega}\sin 2\omega\frac{j}{\omega\left(\omega+j\right)} \\
&= -2e^{-3j\omega}\sin 2\omega\frac{1}{\omega\left(1+j\omega\right)}
\end{aligned}$$

Comparing the above to (1) shows they are the same.

Hence this shows that Fourier transform of $x(t) \circledast h(t)$ gives same answer as $H(\omega)X(\omega)$