HW 4

EE 3015
Signals and Systems

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Contents

1 Problem 4.1(a), Chapter 4

Find Fourier transform of (a) $e^{-2(t-1)}u(t-1)$

Solution

$$
\begin{aligned}
X(\omega) &= \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \\
&= \int_{1}^{\infty} e^{-2(t-1)} e^{-i\omega t} dt \\
&= \int_{1}^{\infty} e^{-2t} e^{2} e^{-i\omega t} dt \\
&= e^{2} \int_{1}^{\infty} e^{-t(2+i\omega)} dt \\
&= \frac{e^{2}}{-(2+i\omega)} \left[e^{-t(2+i\omega)} \right]_{1}^{\infty}
\end{aligned}
$$

Assuming $\text{Im}(\omega) < 2$ then

$$
\begin{aligned}
X(\omega) &= \frac{e^{2}}{-(2+i\omega)} \left[0 - e^{-2} e^{-i\omega} \right] \\
&= \frac{e^{2}}{(2+i\omega)} \left[e^{-2} e^{-i\omega} \right] \\
&= \frac{e^{-i\omega}}{(2+i\omega)}
\end{aligned}
$$

2 Problem 4.3, Chapter 4

Determine the Fourier transform of each of the following periodic signals (a) $\sin\left(2\pi t + \frac{\pi}{4}\right)$
(b) $1 + \cos\left(6\pi t + \frac{\pi}{8}\right)$

Solution

2.1 Part a

Since this is periodic signal, then we can not use $X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$ which is for aperiodic signal. Instead we need to use 4.22 in the textbook which is

$$
X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)
$$

From $\sin\left(2\pi t + \frac{\pi}{4}\right)$ we see that $\omega_0 = 2\pi$, hence

$$
X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - 2k\pi)
$$

Writing $\sin\left(\omega_0 t + \frac{\pi}{4}\right) = \frac{1}{2j}e^{j\left(\omega_0 t + \frac{\pi}{4}\right)} - \frac{1}{2j}e^{-j\left(\omega_0 t + \frac{\pi}{4}\right)} = \left(\frac{1}{2j}e^{j\frac{\pi}{4}}\right)e^{j\omega_0 t} - \left(\frac{1}{2j}e^{-j\frac{\pi}{4}}\right)e^{-j\omega_0 t}$ shows that $a_1 = \frac{1}{2j}e^{j\frac{\pi}{4}}$

and $a_{-1} = -\frac{1}{2j}e^{-j\frac{\pi}{4}}$ and $a_k = 0$ for all other k. Hence above simplifies to

$$X(\omega) = 2\pi\left(\frac{1}{2j}e^{j\frac{\pi}{4}}\right)\delta(\omega - 2\pi) + 2\pi\left(-\frac{1}{2j}e^{-j\frac{\pi}{4}}\right)\delta(\omega + 2\pi)$$
$$= \frac{\pi}{j}e^{j\frac{\pi}{4}}\delta(\omega - 2\pi) - \frac{\pi}{j}e^{-j\frac{\pi}{4}}\delta(\omega + 2\pi)$$

2.2   Part b

Since this is periodic signal, then its Fourier transform is

$$X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

From $1 + \cos\left(6\pi t + \frac{\pi}{8}\right)$ we see that $\omega_0 = 6\pi$, hence

$$X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - 6k\pi)$$

Writing $1 + \cos\left(6\pi t + \frac{\pi}{8}\right) = 1 + \left(\frac{1}{2}e^{j\left(\omega_0 t + \frac{\pi}{8}\right)} + \frac{1}{2}e^{-j\left(\omega_0 t + \frac{\pi}{8}\right)}\right) = 1 + \left(\frac{1}{2}e^{j\frac{\pi}{8}}e^{j\omega_0 t} + \frac{1}{2}e^{-j\frac{\pi}{8}}e^{-j\omega_0 t}\right)$ shows that

$a_1 = \frac{1}{2}e^{j\frac{\pi}{8}}$ and $a_{-1} = \frac{1}{2}e^{-j\frac{\pi}{8}}$ and $a_0 = 1$. Therefore the above becomes

$$X(\omega) = 2\pi a_{-1}\delta(\omega + 6\pi) + 2\pi a_0\delta(\omega) + 2\pi a_1\delta(\omega - 6\pi)$$
$$= 2\pi\left(\frac{1}{2}e^{-j\frac{\pi}{8}}\right)\delta(\omega + 6\pi) + 2\pi\delta(\omega) + 2\pi\left(\frac{1}{2}e^{j\frac{\pi}{8}}\right)\delta(\omega - 6\pi)$$
$$= \pi e^{-j\frac{\pi}{8}}\delta(\omega + 6\pi) + 2\pi\delta(\omega) + \pi e^{j\frac{\pi}{8}}\delta(\omega - 6\pi)$$

3 Problem 4.5, Chapter 4

Use the Fourier transform synthesis equation (4.8) to determine the inverse Fourier transform of $X(j\omega) = |X(j\omega)| e^{j\angle X(j\omega)}$ where $|X(j\omega)| = 2(u(\omega + 3) - u(\omega - 3))$, $\angle X(j\omega) = -\frac{3}{2}\omega + \pi$

<u>Solution</u>

4.8 is

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \tag{4.8}$$

Hence

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)| e^{j\angle X(j\omega)} e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} 2(u(\omega + 3) - u(\omega - 3)) e^{j\left(-\frac{3}{2}\omega + \pi\right)} e^{j\omega t} d\omega$$

But $u(\omega + 3) - u(\omega - 3)$ is one over $\omega = -3\cdots 3$ and zero otherwise. The above simplifies to

$$x(t) = \frac{1}{2\pi} \int_{-3}^{3} 2e^{j\left(-\frac{3}{2}\omega + \pi\right)} e^{j\omega t} d\omega$$

$$= \frac{1}{\pi} \int_{-3}^{3} e^{j\pi} e^{j-\frac{3}{2}\omega} e^{j\omega t} d\omega$$

$$= \frac{e^{j\pi}}{\pi} \int_{-3}^{3} e^{j\left(-\frac{3}{2} + t\right)\omega} d\omega$$

But $e^{j\pi} = -1$ and $\int e^{j\left(-\frac{3}{2} + t\right)\omega} d\omega = \frac{e^{j\left(-\frac{3}{2} + t\right)\omega}}{j\left(-\frac{3}{2} + t\right)}$, hence the above becomes

$$x(t) = \frac{-1}{\pi} \frac{1}{j\left(-\frac{3}{2} + t\right)} \left[e^{j\left(-\frac{3}{2} + t\right)\omega} \right]_{-3}^{3}$$

$$= \frac{-1}{\pi\left(-\frac{3}{2} + t\right)} \left(\frac{e^{j3\left(-\frac{3}{2} + t\right)} - e^{-j3\left(-\frac{3}{2} + t\right)}}{j} \right)$$

$$= \frac{-1}{\pi\left(-\frac{3}{2} + t\right)} 2\left(\sin\left(3\left(-\frac{3}{2} + t\right)\right)\right)$$

$$= \frac{-2}{\pi\left(t - \frac{3}{2}\right)} \sin\left(3\left(t - \frac{3}{2}\right)\right)$$

4   Problem 4.11, Chapter 4

Given the relationships $y(t) = x(t) \circledast h(t)$ and $g(t) = x(3t) \circledast h(3t)$ and given that $x(t)$ has Fourier transform $X(j\omega)$ and $h(t)$ has Fourier transform $H(j\omega)$, use Fourier transform properties to show that $g(t)$ has the form $g(t) = Ay(Bt)$. Determine the values of A and B

<u>Solution</u>

The main relation to use is that if $y(t) \Longleftrightarrow Y(\omega)$ then $y(at) \Longleftrightarrow \frac{1}{a}Y\left(\frac{\omega}{a}\right)$. Therefore $x(3t) \Longleftrightarrow \frac{1}{3}X\left(\frac{\omega}{3}\right)$ and $h(3t) \Longleftrightarrow \frac{1}{3}H\left(\frac{\omega}{3}\right)$. Hence since $g(t) = x(3t) \circledast h(3t)$ then

$$G(\omega) = \frac{1}{3}X\left(\frac{\omega}{3}\right)\frac{1}{3}H\left(\frac{\omega}{3}\right)$$
$$= \frac{1}{3}\left(\frac{1}{3}X\left(\frac{\omega}{3}\right)H\left(\frac{\omega}{3}\right)\right)$$

But $X\left(\frac{\omega}{3}\right)H\left(\frac{\omega}{3}\right) = Y\left(\frac{\omega}{3}\right)$. Therefore the above becomes

$$G(\omega) = \frac{1}{3}\left(\frac{1}{3}Y\left(\frac{\omega}{3}\right)\right)$$

Inverse Fourier transform gives

$$g(t) = \frac{1}{3}y(3t)$$

Where in the above, we used $\frac{1}{3}Y\left(\frac{\omega}{3}\right) \Longleftrightarrow y(3t)$. Hence $A = \frac{1}{3}, B = 3$

5   Problem 4.19, Chapter 4

Consider a causal LTI system with frequency response $H(j\omega) = \frac{1}{j\omega+3}$. For a particular input $x(t)$ this system is observed to produce the output $y(t) = e^{-3t}u(t) - e^{-4t}u(t)$. Determine $x(t)$.

Solution

$$y(t) = x(t) \circledast h(t)$$

Taking Fourier transform gives

$$Y(\omega) = X(\omega)H(\omega)$$

Hence

$$X(\omega) = \frac{Y(\omega)}{H(\omega)}$$

But $y(t) = e^{-3t}u(t) - e^{-4t}u(t)$, therefore, from table $Y(\omega) = \frac{1}{3+j\omega} - \frac{1}{4+j\omega} = \frac{(4+j\omega)-(3+j\omega)}{(3+j\omega)(4+j\omega)} = \frac{1}{(3+j\omega)(4+j\omega)}$

and the above becomes

$$X(\omega) = \frac{\frac{1}{(3+j\omega)(4+j\omega)}}{H(\omega)}$$

But we are given that $H(j\omega) = \frac{1}{j\omega+3}$. The above simplifies to

$$X(\omega) = \frac{\frac{1}{(3+j\omega)(4+j\omega)}}{\frac{1}{j\omega+3}}$$

$$= \frac{1}{4+j\omega}$$

From tables

$$x(t) = e^{-4t}u(t)$$

6   Problem 4.23, Chapter 4

4.23. Consider the signal

$$x_0(t) = \begin{cases} e^{-t}, & 0 \le t \le 1 \\ 0, & \text{elsewhere} \end{cases}.$$

Determine the Fourier transform of each of the signals shown in Figure P4.23. You should be able to do this by explicitly evaluating *only* the transform of $x_0(t)$ and then using properties of the Fourier transform.
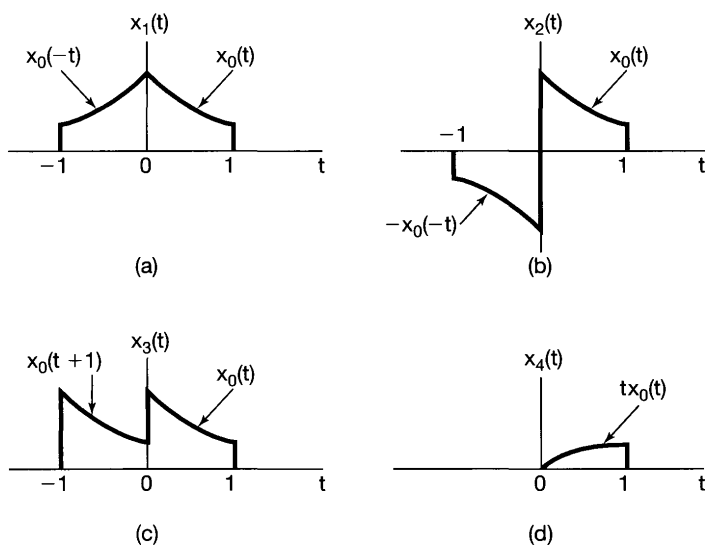

Figure 1: Problem description

Solution

6.1 Part a

First we find the Fourier transform of $x_0(t)$. Since this is a periodic, then $x_0(t) \Longleftrightarrow X_0(\omega)$ and

$$
\begin{aligned}
X_0(\omega) &= \int_{-\infty}^{\infty} x_0(t) e^{-j\omega t} dt \\
&= \int_0^1 e^{-t} e^{-j\omega t} dt \\
&= \int_0^1 e^{-t(1+j\omega)} dt \\
&= \frac{-1}{1+j\omega} \left[e^{-t(1+j\omega)} \right]_0^1 \\
&= \frac{-1}{1+j\omega} \left(e^{-(1+j\omega)} - 1 \right) \\
&= \frac{1 - e^{-(1+j\omega)}}{1+j\omega}
\end{aligned}
$$

From table 4.1, property 4.3.5, Fourier transform of $x(-t) = X(-\omega)$. Hence $x_0(-t) \Longleftrightarrow X_0(-\omega)$. Therefore, using the above result and taking its complex conjugate gives

$$
X_0(-\omega) = \frac{1 - e^{-(1-j\omega)}}{1-j\omega}
$$

Therefore the Fourier transform of $x_0(t) + x_0(-t) \iff X_1(\omega) = X_0(\omega) + X_0(-\omega)$ This is by linearity property. Hence

$$
\begin{aligned}
X_1(\omega) &= X_0(\omega) + X_0(-\omega) \\
&= \frac{1 - e^{-(1+j\omega)}}{1 + j\omega} + \frac{1 - e^{-(1-j\omega)}}{1 - j\omega} \\
&= \frac{(1 - j\omega)\left(1 - e^{-(1+j\omega)}\right) + (1 + j\omega)\left(1 - e^{-(1-j\omega)}\right)}{(1 + j\omega)(1 - j\omega)} \\
&= \frac{1 - e^{-(1+j\omega)} - j\omega\left(1 - e^{-(1+j\omega)}\right) + \left(1 - e^{-(1-j\omega)}\right) + j\omega\left(1 - e^{-(1-j\omega)}\right)}{1 + \omega^2} \\
&= \frac{1 - e^{-(1+j\omega)} - j\omega + j\omega e^{-(1+j\omega)} + \left(1 - e^{-(1-j\omega)}\right) + j\omega - j\omega e^{-(1-j\omega)}}{1 + \omega^2} \\
&= \frac{1 - e^{-(1+j\omega)} + j\omega e^{-(1+j\omega)} + 1 - e^{-(1-j\omega)} - j\omega e^{-(1-j\omega)}}{1 + \omega^2} \\
&= \frac{1 - e^{-1}e^{-j\omega} + j\omega e^{-1}e^{-j\omega} + 1 - e^{-1}e^{j\omega} - j\omega e^{-1}e^{j\omega}}{1 + \omega^2} \\
&= \frac{1 - e^{-1}\left(e^{j\omega} + e^{-j\omega}\right) - j\omega e^{-1}\left(e^{j\omega} - e^{-j\omega}\right)}{1 + \omega^2} \\
&= \frac{1}{1 + \omega^2} - \frac{e^{-1}}{1 + \omega^2}\left(e^{j\omega} + e^{-j\omega}\right) + \frac{\omega e^{-1}}{1 + \omega^2}\frac{\left(e^{j\omega} - e^{-j\omega}\right)}{j} \\
&= \frac{1}{1 + \omega^2} - \frac{2}{e\left(1 + \omega^2\right)}\cos\omega + \frac{2\omega}{e\left(1 + \omega^2\right)}\sin\omega
\end{aligned}
$$

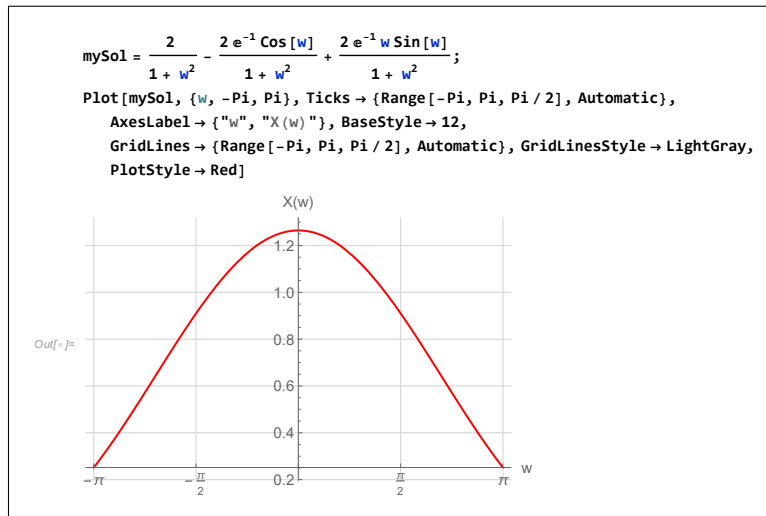The following is a plot of the above

Figure 2: Plot of $X(\omega)$

We see that $X_1(\omega)$ is even and real. This agrees with table 4.1, property 4.3.3 which says that for real $x(t)$ which is even, then its Fourier transform is real and even.

6.2 Part b

We found $X_0(\omega) = \frac{1-e^{-(1+j\omega)}}{1+j\omega}$ above. Hence $-x_0(-t) \Longleftrightarrow -X_0(-\omega) = -\frac{1-e^{-(1-j\omega)}}{1-j\omega} = \frac{1-e^{-(1-j\omega)}}{j\omega-1}$. Therefore

$$
\begin{aligned}
X_2(\omega) &= X_0(\omega) - X_0(-\omega) \\[6pt]
&= \frac{1-e^{-(1+j\omega)}}{1+j\omega} + \frac{1-e^{-(1-j\omega)}}{j\omega-1} \\[6pt]
&= \frac{\left(j\omega-1\right)\left(1-e^{-(1+j\omega)}\right)+\left(1+j\omega\right)\left(1-e^{-(1-j\omega)}\right)}{\left(1+j\omega\right)\left(j\omega-1\right)} \\[6pt]
&= \frac{j\omega\left(1-e^{-(1+j\omega)}\right)-\left(1-e^{-(1+j\omega)}\right)+\left(1-e^{-(1-j\omega)}\right)+j\omega\left(1-e^{-(1-j\omega)}\right)}{j\omega-1-\omega^2-j\omega} \\[6pt]
&= \frac{j\omega-j\omega e^{-(1+j\omega)}-1+e^{-(1+j\omega)}+1-e^{-(1-j\omega)}+j\omega-j\omega e^{-(1-j\omega)}}{-\left(1+\omega^2\right)} \\[6pt]
&= \frac{2j\omega-j\omega e^{-(1+j\omega)}+e^{-(1+j\omega)}-e^{-(1-j\omega)}-j\omega e^{-(1-j\omega)}}{-\left(1+\omega^2\right)} \\[6pt]
&= \frac{2j\omega-j\omega e^{-1}e^{-j\omega}+e^{-1}e^{-j\omega}-e^{-1}e^{j\omega}-j\omega e^{-1}e^{j\omega}}{j\omega-\omega^2-2} \\[6pt]
&= \frac{2j\omega-j\omega e^{-1}\left(e^{j\omega}+e^{-j\omega}\right)-e^{-1}\left(e^{j\omega}-e^{-j\omega}\right)}{-\left(1+\omega^2\right)} \\[6pt]
&= \frac{2j\omega}{-\left(1+\omega^2\right)}-j\omega e^{-1}\frac{\left(e^{j\omega}+e^{-j\omega}\right)}{-\left(1+\omega^2\right)}-e^{-1}\frac{e^{j\omega}-e^{-j\omega}}{-\left(1+\omega^2\right)} \\[6pt]
&= \frac{2j\omega}{-\left(1+\omega^2\right)}-2j\omega e^{-1}\frac{\cos\omega}{-\left(1+\omega^2\right)}-2je^{-1}\frac{\sin(\omega)}{-\left(1+\omega^2\right)}
\end{aligned}
$$

Hence

$$
\begin{aligned}
X_2(\omega) &= j\left(\frac{-2\omega}{\left(1+\omega^2\right)}+2\omega\frac{\cos\omega}{e\left(1+\omega^2\right)}+2\frac{\sin(\omega)}{e\left(1+\omega^2\right)}\right) \\[6pt]
&= \frac{j}{e\left(1+\omega^2\right)}\left(-2e\omega+2\omega\cos\omega+2\sin(\omega)\right)
\end{aligned}
$$

We see that $X_2(\omega)$ is pure imaginary. This agrees with table 4.1, property 4.3.3 which says that for real $x(t)$ which is odd, then its Fourier transform is pure imaginary and odd.

The following is a plot of the above which shows that the imaginary part of $X_2(\omega)$ is odd
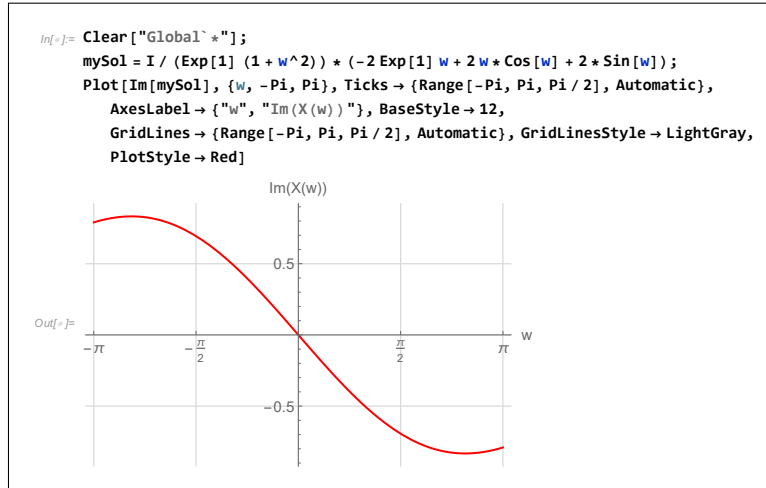
```
In[•]:= Clear["Global`*"];
       mySol = I / (Exp[1] (1 + w^2)) * (-2 Exp[1] w + 2 w * Cos[w] + 2 * Sin[w]);
       Plot[Im[mySol], {w, -Pi, Pi}, Ticks → {Range[-Pi, Pi, Pi / 2], Automatic},
         AxesLabel → {"w", "Im(X(w))"}, BaseStyle → 12,
         GridLines → {Range[-Pi, Pi, Pi / 2], Automatic}, GridLinesStyle → LightGray,
         PlotStyle → Red]
```


Figure 3: Plot of imaginary part of $X(\omega)$

7 Problem 4.26, Chapter 4

4.26. **(a)** Compute the convolution of each of the following pairs of signals $x(t)$ and $h(t)$ by calculating $X(j\omega)$ and $H(j\omega)$, using the convolution property, and inverse transforming.

(i) $x(t) = te^{-2t}u(t),\ h(t) = e^{-4t}u(t)$

(ii) $x(t) = te^{-2t}u(t),\ h(t) = te^{-4t}u(t)$

(iii) $x(t) = e^{-t}u(t),\ h(t) = e^{t}u(-t)$

(b) Suppose that $x(t) = e^{-(t-2)}u(t-2)$ and $h(t)$ is as depicted in Figure P4.26. Verify the convolution property for this pair of signals by showing that the Fourier transform of $y(t) = x(t) * h(t)$ equals $H(j\omega)X(j\omega)$.
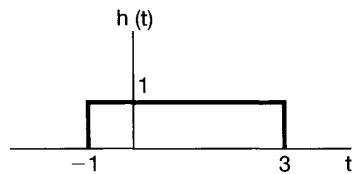
h (t)

Figure P4.26

Figure 4: Problem description

Solution

7.1 Part a

Part i

$$y(t) = x(t) \circledast h(t)$$

Taking Fourier transform the above, convolution becomes multiplication

$$Y(\omega) = X(\omega)H(\omega)$$

Given that $x(t) = te^{-2t}u(t)$, then from tables $X(\omega) = \dfrac{1}{(2+j\omega)^2}$ and given that $h(t) = e^{-4t}u(t)$ then from table 4.2 $H(\omega) = \dfrac{1}{4+j\omega}$. Hence the above becomes

$$Y(\omega) = \frac{1}{(2+j\omega)^2(4+j\omega)}$$

Doing partial fractions. Let $s = j\omega$ then

$$\frac{1}{(2+s)^2(4+s)} = \frac{A}{(2+s)^2} + \frac{B}{2+s} + \frac{C}{4+s}$$

$$= \frac{A(4+s) + B((2+s)(4+s)) + C(2+s)^2}{(2+s)^2(4+s)}$$

Expanding numerator

$$1 = 4A + 8B + 4C + Bs^2 + Cs^2 + As + 6Bs + 4Cs$$
$$1 = (4A + 8B + 4C) + s(A + 6B + 4C) + s^2(B + C)$$

Comparing coefficients

$$1 = 4A + 8B + 4C$$
$$0 = A + 6B + 4C$$
$$0 = B + C$$

Or

$$\begin{pmatrix} 4 & 8 & 4 \\ 1 & 6 & 4 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Gaussian elimination. Multiplying second row by 4 and subtracting result from first row gives

$$\begin{pmatrix} 4 & 8 & 4 \\ 0 & 16 & 12 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

Multiplying third row by 16 and subtracting result from second row gives

$$\begin{pmatrix} 4 & 8 & 4 \\ 0 & 16 & 12 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

Backsubstitution. Last row gives $C = \frac{1}{4}$. Second row gives $16B + 12C = -1$ or $16B = -1 - 12\left(\frac{1}{4}\right)$, hence $16B = -4$, Hence $B = -\frac{1}{4}$. First row gives $4A + 8B + 4C = 1$ or $4A + 8\left(-\frac{1}{4}\right) + 4\left(\frac{1}{4}\right) = 1$ or $4A - 2 + 1 = 1$. Hence $A = \frac{1}{2}$. Therefore partial fractions gives

$$\frac{1}{(2+s)^2(4+s)} = \frac{\left(\frac{1}{2}\right)}{(2+s)^2} + \frac{\left(-\frac{1}{4}\right)}{2+s} + \frac{\left(\frac{1}{4}\right)}{4+s}$$

Replacing s back with $j\omega$

$$Y(\omega) = \frac{1}{2}\frac{1}{(2+j\omega)^2} - \frac{1}{4}\frac{1}{2+j\omega} + \frac{1}{4}\frac{1}{4+j\omega}$$

Applying inverse Fourier transform, using table gives

$$y(t) = \frac{1}{2}te^{-2t}u(t) - \frac{1}{4}e^{-2t}u(t) + \frac{1}{4}e^{-4t}u(t)$$

Part ii

$$y(t) = x(t) \circledast h(t)$$

Taking Fourier transform the above, convolution becomes multiplication

$$Y(\omega) = X(\omega)H(\omega)$$

Given that $x(t) = te^{-2t}u(t)$, then from tables $X(\omega) = \frac{1}{(2+j\omega)^2}$ and given that $h(t) = te^{-4t}u(t)$ then from table 4.2 $H(\omega) = \frac{1}{(4+j\omega)^2}$. Hence the above becomes

$$Y(\omega) = \frac{1}{(2+j\omega)^2(4+j\omega)^2}$$

Doing partial fractions. Let $s = j\omega$ then

$$\frac{1}{(2+s)^2(4+s)^2} = \frac{A}{(2+s)^2} + \frac{B}{2+s} + \frac{C}{(4+s)^2} + \frac{D}{(4+s)}$$

$$= \frac{A(4+s)^2 + B\left((2+s)(4+s)^2\right) + C(2+s)^2 + D\left((2+s)^2(4+s)\right)}{(2+s)^2(4+s)^2}$$

Expanding numerator

$$1 = 16A + 32B + 4C + 16D + 20sD + As^2 + 10Bs^2 + Bs^3 + Cs^2 + 8s^2D + s^3D + 8As + 32Bs + 4Cs$$

$$1 = (16A + 32B + 4C + 16D) + s(20D + 8A + 32B + 4C) + s^2(A + 10B + C + 8D) + s^3(B + D)$$

Comparing coefficients

$$1 = 16A + 32B + 4C + 16D$$
$$0 = 8A + 32B + 4C + 20D$$
$$0 = A + 10B + C + 8D$$
$$0 = B + D$$

Or

$$\begin{pmatrix} 16 & 32 & 4 & 16 \\ 8 & 32 & 4 & 20 \\ 1 & 10 & 1 & 8 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Gaussian elimination. replacing row 2 by result of subtracting 1/2 times row 1 from row 2

$$\begin{pmatrix} 16 & 32 & 4 & 16 \\ 0 & 16 & 2 & 12 \\ 1 & 10 & 1 & 8 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ 0 \\ 0 \end{pmatrix}$$

Replacing row 3 by result of subtracting $\frac{1}{16}$ times row 1 from row 3

$$\begin{pmatrix} 16 & 32 & 4 & 16 \\ 0 & 16 & 2 & 12 \\ 0 & 8 & \frac{3}{4} & 7 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ -\frac{1}{16} \\ 0 \end{pmatrix}$$

Replacing row 3 by result of subtracting $\frac{1}{2}$ times row 2 from row 3

$$\begin{pmatrix} 16 & 32 & 4 & 16 \\ 0 & 16 & 2 & 12 \\ 0 & 0 & \frac{-1}{4} & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ \frac{3}{16} \\ 0 \end{pmatrix}$$

Replacing row 4 by result of subtracting $\frac{1}{16}$ times row 2 from row 4

$$\begin{pmatrix} 16 & 32 & 4 & 16 \\ 0 & 16 & 2 & 12 \\ 0 & 0 & \frac{-1}{4} & 1 \\ 0 & 0 & \frac{-1}{8} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ \frac{3}{16} \\ \frac{1}{32} \end{pmatrix}$$

Replacing row 4 by result of subtracting $\frac{1}{2}$ times row 3 from row 4

$$\begin{pmatrix} 16 & 32 & 4 & 16 \\ 0 & 16 & 2 & 12 \\ 0 & 0 & \frac{-1}{4} & 1 \\ 0 & 0 & 0 & \frac{-1}{4} \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ \frac{3}{16} \\ \frac{-1}{16} \end{pmatrix}$$

Backsubstitution phase:

$$\frac{-1}{4}D = \frac{-1}{16}$$
$$D = \frac{1}{4}$$

Third row gives $\frac{-1}{4}C + D = \frac{3}{16}$ or $\frac{-1}{4}C + \frac{1}{4} = \frac{3}{16} \rightarrow C = \frac{1}{4}$. Second row gives $16B + 2C + 12D =$ $-\frac{1}{2}$ or $16B + 2\left(\frac{1}{4}\right) + 12\left(\frac{1}{4}\right) = -\frac{1}{2} \rightarrow B = -\frac{1}{4}$. First row gives $16A + 32B + 4C + 16D = 1$ or $16A + 32\left(-\frac{1}{4}\right) + 4\left(\frac{1}{4}\right) + 16\left(\frac{1}{4}\right) = 1, \rightarrow A = \frac{1}{4}$.

Therefore partial fractions gives

$$\frac{1}{(2+s)^2(4+s)^2} = \frac{A}{(2+s)^2} + \frac{B}{2+s} + \frac{C}{(4+s)^2} + \frac{D}{(4+s)}$$

$$= \frac{1}{4}\frac{1}{(2+s)^2} - \frac{1}{4}\frac{1}{2+s} + \frac{1}{4}\frac{1}{(4+s)^2} + \frac{1}{4}\frac{1}{(4+s)}$$

Replace s back with $j\omega$

$$Y(\omega) = \frac{1}{4}\frac{1}{\left(2+j\omega\right)^2} - \frac{1}{4}\frac{1}{2+j\omega} + \frac{1}{4}\frac{1}{\left(4+j\omega\right)^2} + \frac{1}{4}\frac{1}{\left(4+j\omega\right)}$$

Applying inverse Fourier transform, using table gives

$$y(t) = \frac{1}{4}te^{-2t}u(t) - \frac{1}{4}e^{-2t}u(t) + \frac{1}{4}te^{-4t}u(t) + \frac{1}{4}e^{-4t}u(t)$$

Part iii

$$y(t) = x(t) \circledast h(t)$$

Taking Fourier transform the above, convolution becomes multiplication

$$Y(\omega) = X(\omega)H(\omega)$$

Given that $x(t) = e^{-t}u(t)$, then from tables $X(\omega) = \frac{1}{(1+j\omega)}$ and given that $h(t) = e^{t}u(-t)$ then $H(\omega) = \frac{1}{1-j\omega}$. Hence the above becomes

$$Y(\omega) = \frac{1}{\left(1+j\omega\right)}\frac{1}{\left(1-j\omega\right)}$$

Doing partial fractions. Let $s = j\omega$ then

$$\frac{1}{(1+s)}\frac{1}{(1-s)} = \frac{A}{(1+s)} + \frac{B}{1-s}$$

Hence $A = \left(\frac{1}{(1-s)}\right)_{s=-1} = \frac{1}{2}$ and $B = \left(\frac{1}{(1+s)}\right)_{s=1} = \frac{1}{2}$. Hence

$$Y(\omega) = \frac{1}{2}\frac{1}{\left(1+j\omega\right)} + \frac{1}{2}\frac{1}{1-j\omega}$$

Therefore

$$y(t) = \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{t}u(-t)$$

The above means for $t < 0, y(t) = \frac{1}{2}e^{t}$ and for $t > 0, y(t) = \frac{1}{2}e^{-t}$.

7.2 Part b

$x(t) = e^{-(t-2)}u(t-2)$, $h(t) = u(t+1) - u(t-3)$. Let us first find $X(\omega)$ and $H(\omega)$

$$
\begin{aligned}
X(\omega) &= \int_{-\infty}^{\infty} e^{-(t-2)}u(t-2)\, e^{-j\omega t} dt \\
&= \int_{2}^{\infty} e^{-(t-2)} e^{-j\omega t} dt \\
&= \int_{2}^{\infty} e^{-t} e^{2} e^{-j\omega t} dt \\
&= e^{2} \int_{2}^{\infty} e^{-t(1+j\omega)} dt \\
&= -\frac{e^{2}}{1+j\omega} \left[e^{-t(1+j\omega)} \right]_{2}^{\infty} \\
&= -\frac{e^{2}}{1+j\omega} \left(0 - e^{-2(1+j\omega)} \right) \\
&= \frac{e^{2} e^{-2(1+j\omega)}}{1+j\omega} \\
&= \frac{e^{-2-2j\omega+2}}{1+j\omega} \\
&= \frac{e^{-2j\omega}}{1+j\omega}
\end{aligned}
$$

And

$$
\begin{aligned}
H(\omega) &= \int_{-\infty}^{\infty} u(t+1) - u(t-3)\, e^{-j\omega t} dt \\
&= \int_{-1}^{3} e^{-j\omega t} dt \\
&= \frac{1}{j\omega} \left(e^{-j\omega t} \right)_{-1}^{3} \\
&= \frac{1}{j\omega} \left(e^{-3j\omega} - e^{j\omega t} \right) \\
&= \frac{1}{j\omega} e^{-j\omega} \left(e^{-j2\omega} - e^{j2\omega t} \right) \\
&= -\frac{1}{j\omega} e^{-j\omega} \left(e^{j2\omega t} - e^{-j2\omega} \right) \\
&= -\frac{1}{\omega} e^{-j\omega} \left(\frac{e^{j2\omega t} - e^{-j2\omega}}{j} \right) \\
&= -\frac{2}{\omega} e^{-j\omega} \sin(2\omega)
\end{aligned}
$$

Hence

$$Y(\omega) = H(\omega) X(\omega)$$

$$= \frac{e^{-2j\omega}}{1+j\omega} \left(-\frac{2}{\omega} e^{-j\omega} \sin(2\omega) \right)$$

$$= \frac{-2}{\omega(1+j\omega)} e^{-2j\omega} e^{-j\omega} \sin(2\omega)$$

$$= \frac{-2}{\omega(1+j\omega)} e^{-3j\omega} \sin(2\omega) \tag{1}$$

Now,

$$y(t) = x(t) \circledast h(t)$$

$$= \int_{-\infty}^{\infty} x(\tau) h(t-\tau) \, d\tau$$

Folding $h(t)$. For $t < 1$, then $y(t) = 0$. For $2 < 1 + t < 6$ or $1 < t < 5$

$$y(t) = \int_{2}^{1+t} x(\tau) \, d\tau$$

$$= \int_{2}^{1+t} e^{-(\tau-2)} d\tau$$

$$= -\left(e^{-(\tau-2)} \right)_{2}^{1+t}$$

$$= -\left(e^{-(1+t-2)} - e^{-(2-2)} \right)$$

$$= -\left(e^{-(-1+t)} - 1 \right)$$

$$= 1 - e^{1-t}$$

For $6 < 1 + t$ or $t > 5$

$$y(t) = \int_{t-3}^{t+1} x(\tau) \, d\tau$$

$$= \int_{t-3}^{t+1} e^{-(\tau-2)} d\tau$$

$$= -\left(e^{-(\tau-2)} \right)_{t-3}^{1+t}$$

$$= -\left(e^{-(1+t-2)} - e^{-(t-3-2)} \right)$$

$$= -\left(e^{-(-1+t)} - e^{-(t-5)} \right)$$

$$= -\left(e^{1-t} - e^{5-t} \right)$$

$$= e^{5-t} - e^{1-t}$$

Hence

$$y(t) = \begin{cases} 0 & t < 1 \\ 1 - e^{1-t} & 1 < t < 5 \\ e^{5-t} - e^{1-t} & t > 5 \end{cases}$$

The above can be written as $y(t)$

$$
\begin{aligned}
y(t) &= \left(1 - e^{1-t}\right)\left(u(t-1) - u(t-5)\right) + \left(e^{5-t} - e^{1-t}\right)u(t-5) \\
&= u(t-1) - u(t-5) - e^{1-t}\left(u(t-1) - u(t-5)\right) + \left(e^{5-t} - e^{1-t}\right)u(t-5) \\
&= u(t-1) - u(t-5) - e^{1-t}u(t-1) + e^{1-t}u(t-5) + e^{5-t}u(t-5) - e^{1-t}u(t-5) \\
&= u(t-1) - u(t-5) - e^{1-t}u(t-1) + e^{5-t}u(t-5)
\end{aligned}
\tag{2}
$$

Taking the Fourier transform of the above from tables

$$
u(t-1) \iff \frac{1}{j\omega}e^{-j\omega} + \pi\delta(\omega)
$$

$$
u(t-5) \iff \frac{1}{j\omega}e^{-5j\omega} + \pi\delta(\omega)
$$

$$
e^{1-t}u(t-1) \iff \frac{e^{-j\omega}}{1 + j\omega}
$$

$$
e^{5-t}u(t-5) \iff \frac{e^{-5j\omega}}{1 + j\omega}
$$

Hence $Y(\omega)$ is

$$
\begin{aligned}
Y(\omega) &= \frac{1}{j\omega}e^{-j\omega} + \pi\delta(\omega) - \left(\frac{1}{j\omega}e^{-5j\omega} + \pi\delta(\omega)\right) - \frac{e^{-j\omega}}{1 + j\omega} + \frac{e^{-5j\omega}}{1 + j\omega} \\
&= \frac{1}{j\omega}e^{-j\omega} - \frac{1}{j\omega}e^{-5j\omega} - \frac{e^{-j\omega}}{1 + j\omega} + \frac{e^{-5j\omega}}{1 + j\omega} \\
&= \frac{e^{-3j\omega}}{\omega}\left(\frac{1}{j}e^{2j\omega} - \frac{1}{j}e^{-2j\omega}\right) - \frac{e^{-j\omega}}{1 + j\omega} + \frac{e^{-5j\omega}}{1 + j\omega} \\
&= \frac{e^{-3j\omega}}{\omega}2(\sin 2\omega) - \frac{e^{-3j\omega}}{1 + j\omega}\left(e^{2j\omega} - e^{-2j\omega}\right) \\
&= \frac{e^{-3j\omega}}{\omega}2(\sin 2\omega) - 2\frac{e^{-3j\omega}}{j + \omega}\sin 2\omega \\
&= 2e^{-3j\omega}\sin 2\omega\left(\frac{1}{\omega} - \frac{1}{j + \omega}\right) \\
&= 2e^{-3j\omega}\sin 2\omega\left(\frac{j + \omega - \omega}{\omega(\omega + j)}\right) \\
&= 2e^{-3j\omega}\sin 2\omega\frac{j}{\omega(\omega + j)} \\
&= -2e^{-3j\omega}\sin 2\omega\frac{1}{\omega(1 + j\omega)}
\end{aligned}
$$

Comparing the above to (1) shows they are the same.

Hence this shows that Fourier transform of $x(t) \circledast h(t)$ gives same answer as $H(\omega)X(\omega)$