Homework 3 solutions

3. The given signal is

$$x(t) = 2 + \frac{1}{2}e^{j(2\pi/3)t} + \frac{1}{2}e^{-j(2\pi/3)t} - 2je^{j(5\pi/3)t} + 2je^{-j(5\pi/3)t}$$
$$= 2 + \frac{1}{2}e^{j2(2\pi/6)t} + \frac{1}{2}e^{-j2(2\pi/6)t} - 2je^{j5(2\pi/6)t} + 2je^{-j5(2\pi/6)t}$$

From this, we may conclude that the fundamental frequency of $x(t)$ is $2\pi/6 = \pi/3$. The non-zero Fourier series coefficients of $x(t)$ are:

$$a_0 = 2, \quad a_2 = a_{-2} = \frac{1}{2}, \quad a_5 = a_{-5}^* = -2j$$

3.10. Since the Fourier series coeffiecients repeat every N, we have

$$a_1 = a_{15}, \quad a_2 = a_{16} \quad \text{, and} \quad a_3 = a_{17}$$

Furthermore, since the signal is real and odd, the Fourier series coefficients a_k will be purely imaginary and odd. Therefore, $a_0 = 0$ and

$$a_1 = -a_{-1}, \quad a_2 = -a_{-2} \quad a_3 = -a_{-3}$$

Finally,

$$a_{-1} = -j, \quad a_{-2} = -2j, \quad a_{-3} = -3j$$

3.16. (a) The given signal $x_1[n]$ is

$$x_1[n] = (-1)^n = e^{j\pi n} = e^{j(2\pi/2)n}$$

Therefore, $x_1[n]$ is periodic with period $N = 2$ and it's Fourier series coefficients in the range $0 \le k \le 1$ are

$$a_0 = 0, \quad \text{and} \quad a_1 = 1$$

Using the results derived in Section 3.8, the output $y_1[n]$ is given by

$$y_1[n] = \sum_{k=0}^{1} a_k H(e^{j2\pi k/2}) e^{k(2\pi/2)}$$
$$= 0 + a_1 H(e^{j\pi}) e^{j\pi}$$
$$= 0$$

(b) The signal $x_2[n]$ is periodic with period $N = 16$. The signal $x_2[n]$ may be written as

$$x_2[n] = e^{j(2\pi/16)(0)n} - (j/2)e^{j(\pi/4)}e^{j(2\pi/16)(3)n} + (j/2)e^{-j(\pi/4)}e^{-j(2\pi/16)(3)n}$$
$$= e^{j(2\pi/16)(0)n} - (j/2)e^{j(\pi/4)}e^{j(2\pi/16)(3)n} + (j/2)e^{-j(\pi/4)}e^{j(2\pi/16)(13)n}$$

Therefore, the non-zero Fourier series coefficients of $x_2[n]$ in the range $0 \le k \le 15$ are

$$a_0 = 1, \quad a_3 = -(j/2)e^{j(\pi/4)}, \quad a_{13} = (j/2)e^{-j(\pi/4)}$$

Using the results derived in Section 3.8, the output $y_2[n]$ is given by

$$y_2[n] = \sum_{k=0}^{15} a_k H(e^{j2\pi k/16}) e^{k(2\pi/16)}$$
$$= 0 - (j/2)e^{j(\pi/4)}e^{j(2\pi/16)(3)n} + (j/2)e^{-j(\pi/4)}e^{j(2\pi/16)(13)n}$$
$$= \sin(\frac{3\pi}{8}n + \frac{\pi}{4})$$

3.20. (a) Current through the capacitor $= C\frac{dy(t)}{dt}$.
Voltage across resistor $= RC\frac{dy(t)}{dt}$.
Voltage across inductor $= LC\frac{d^2y(t)}{dt^2}$.
Input voltage = Voltage across resistor + Voltage across inductor + Voltage across capacitor.
Therefore,

$$x(t) = LC\frac{d^2y(t)}{dt^2} + RC\frac{dy(t)}{dt} + y(t)$$

Substituting for R, L and C, we have

$$\frac{d^2y(t)}{dt^2} + \frac{dy(t)}{dt} + y(t) = x(t)$$

(b) We will now use an approach similar to the one used in part (b) of the previous problem. If we assume that the input is of the form $e^{j\omega t}$, then the output will be of the form $H(j\omega)e^{j\omega t}$. Substituting in the above differential equation and simplifying, we obtain

$$H(j\omega) = \frac{1}{-\omega^2 + j\omega + 1}$$

(c) The signal $x(t)$ is periodic with period 2π. Since $x(t)$ can be expressed in the form

$$x(t) = \frac{1}{2j}e^{j(2\pi/2\pi)t} - \frac{1}{2j}e^{-j(2\pi/2\pi)t},$$

the non-zero Fourier series coefficients of $x(t)$ are

$$a_1 = a_{-1}^* = \frac{1}{2j}.$$

Using the results derived in Section 3.8 (see eq.(3.124)), we have

$$\begin{aligned} y(t) &= a_1 H(j)e^{jt} - a_{-1}H(-j)e^{-jt} \\ &= (1/2j)(\frac{1}{j}e^{jt} - \frac{1}{-j}e^{-jt}) \\ &= (-1/2)(e^{jt} + e^{-jt}) \\ &= -\cos(t) \end{aligned}$$

3.28. (a) $N = 7$,

$$a_k = \frac{1}{7}\frac{e^{-j4\pi k/7}\sin(5\pi k/7)}{\sin(\pi k/7)}.$$

(b) $N = 6$, a_k over one period $(0 \le k \le 5)$ may be specified as: $a_0 = 4/6$,

$$a_k = \frac{1}{6}e^{-j\pi k/2}\frac{\sin(\frac{2\pi k}{3})}{\sin(\frac{\pi k}{6})}, \qquad 1 \le k \le 5.$$

(c) $N = 6$,

$$a_k = 1 + 4\cos(\pi k/3) - 2\cos(2\pi k/3).$$

(d) $N = 12$, a_k over one period $(0 \le k \le 11)$ may be specified as: $a_1 = \frac{1}{4j} = a_{11}^*$, $a_5 = -\frac{1}{4j} = a_7^*$, $a_k = 0$ otherwise.

(e) $N = 4$.

$$a_k = 1 + 2(-1)^k(1 - \frac{1}{\sqrt{2}})\cos(\frac{\pi k}{2}).$$

(f) $N = 12$,

$$\begin{aligned} a_k &= 1 + (1 - \frac{1}{\sqrt{2}})2\cos(\frac{\pi k}{6}) + 2(1 - \frac{1}{\sqrt{2}})\cos(\frac{\pi k}{2}) \\ &+ 2(1 + \frac{1}{\sqrt{2}})\cos(\frac{5\pi k}{6}) + 2(-1)^k + 2\cos(\frac{2\pi k}{3}). \end{aligned}$$

3.47. Considering $x(t)$ to be periodic with period 1, the nonzero FS coefficients of $x(t)$ are $a_1 = a_{-1} = 1/2$. If we now consider $x(t)$ to be periodic with period 3, then the the nonzero FS coefficients of $x(t)$ are $b_3 = b_{-3} = 1/2$.