HW 3

EE 3015
Signals and Systems

Spring 2020
University of Minnesota, Twin Cities

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May 27, 2020 Compiled on May 27, 2020 at 12:22am

Contents

1 Problem 3 Chapter 3

For the continuous-time periodic signal $x(t) = 2 + \cos\left(\frac{2\pi}{3}t\right) + 4\sin\left(\frac{5\pi}{3}t\right)$ determine the fundamental frequency ω_0 and the Fourier series coefficients a_k such that $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

Solution

The signal $\cos\left(\frac{2\pi}{3}t\right)$ has period $\frac{2\pi}{T_1} = \frac{2\pi}{3}$. Hence $T_1 = 3$ and the signal $\sin\left(\frac{5\pi}{3}t\right)$ has period $\frac{2\pi}{T_2} = \frac{5\pi}{3}$ or $T_2 = \frac{6}{5}$. Therefore the LCM of $3, \frac{6}{5}$ is

$$\begin{aligned} 3m &= \frac{6}{5}n \\ \frac{m}{n} &= \frac{2}{5} \end{aligned}$$

Hence $m = 2$ and $n = 5$. Therefore $\underline{T_0 = 6}$. Therefore

$$\begin{aligned} \omega_0 &= \frac{2\pi}{T_0} \\ &= \frac{2\pi}{6} \\ &= \frac{\pi}{3} \end{aligned}$$

Hence

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \tag{1}$$

Where

$$a_k = \frac{1}{T_0}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\, e^{-jk\omega_0 t} \tag{2}$$

To find a_k for the given signal, instead of using the above integration formula, we could write the signal $x(t)$ in exponential form using Euler relation and just read the a_k coefficients directly from the result. The signal $x(t)$ can be written as

$$\begin{aligned} x(t) &= 2 + \frac{e^{j\frac{2\pi}{3}t} + e^{-j\frac{2\pi}{3}t}}{2} + 4\frac{e^{j\frac{5\pi}{3}t} - e^{-j\frac{5\pi}{3}t}}{2i} \\ &= 2 + \frac{e^{j2\omega_0 t} + e^{-j2\omega_0 t}}{2} + 4\frac{e^{j5\omega_0 t} - e^{-j5\omega_0 t}}{2i} \\ &= 2 + \frac{1}{2}e^{j2\omega_0 t} + \frac{1}{2}e^{-j2\omega_0 t} + 2ie^{j5\omega_0 t} - 2ie^{-j5\omega_0 t} \end{aligned} \tag{3}$$

Comparing (3) to (1) shows that the coefficients are

$$\begin{aligned} a_0 &= 2 \\ a_2 &= \frac{1}{2} \\ a_{-2} &= \frac{1}{2} \\ a_5 &= 2j \\ a_{-5} &= -2j \end{aligned}$$

2 Problem 10 Chapter 3

Let $x[n]$ be real and odd periodic signal with period $N = 7$ and Fourier coefficients a_k. Given that $a_{15} = j, a_{16} = 2j, a_{17} = 3j$, determine the values of $a_0, a_{-1}, a_{-2}, a_{-3}$.

Solution

For discrete signal

$$\begin{aligned} x[n] &= \sum_{k=0}^{N-1} a_k e^{jk\omega_0 n} \\ &= \sum_{k=0}^{N-1} a_k e^{jk\frac{2\pi}{N}n} \end{aligned}$$

Where

$$\begin{aligned} a_k &= \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-jk\omega_0 n} \\ &= \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-jk\frac{2\pi}{N}n} \end{aligned}$$

Since the signal $x[n]$ is real, then we know that $a_k = a_{-k}^*$. And since $x[n]$ is odd then we know that a_k is purely imaginary and odd. The Fourier coefficients repeat every N samples which is 7. Hence $a_{15} = a_9 = a_1$ and $a_{16} = a_9 = a_2$ and $a_{17} = a_{10} = a_3$. And since a_k is odd then

$$\begin{aligned} a_0 &= 0 \\ a_1 &= -a_{-1} \\ a_2 &= -a_{-2} \\ a_3 &= -a_{-3} \end{aligned}$$

But we know from above that $a_1 = a_{15} = j$ and $a_2 = a_{16} = 2j$ and $a_3 = a_{17} = 3j$ then the above gives

$$\begin{aligned} a_0 &= 0 \\ a_{-1} &= -j \\ a_{-2} &= -2j \\ a_{-3} &= -3j \end{aligned}$$

3 Problem 16 Chapter 3

For what values of k is it guaranteed that $a_k = 0$?

3.16. Determine the output of the filter shown in Figure P3.16 for the following periodic inputs:

(a) $x_1[n] = (-1)^n$

(b) $x_2[n] = 1 + \sin(\frac{3\pi}{8}n + \frac{\pi}{4})$

(c) $x_3[n] = \sum_{k=-\infty}^{\infty}(\frac{1}{2})^{n-4k}u[n-4k]$

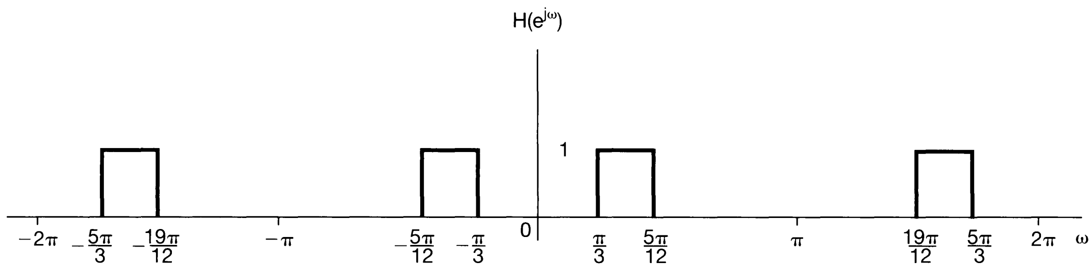

Figure P3.16

Figure 1: Problem description

Solution

The output of discrete LTI system when the input is $x[n] = a_n e^{jn\omega}$ is given by $y[n] = a_n H\left(e^{j\omega}\right) e^{jn\omega}$ where $H\left(e^{j\omega}\right)$ is given to us in the problem statement. Hence, to find $y[n]$ we need to express each input in its Fourier series representation in order to determine the a_n.

3.1 Part a

Here $x_1[n] = (-1)^n = \left(e^{j\pi}\right)^n = e^{jn\pi}$. To find the period$N$, let $x_1[n] = x_1[n+N]$ or

$$
\begin{aligned}
e^{jn\pi} &= e^{j(n+N)\pi} \\
&= e^{jn\pi} e^{jN\pi}
\end{aligned}
$$

Hence $\underline{N=2}$. Therefore $\omega_0 = \frac{2\pi}{N} = \frac{2\pi}{2} = \pi$ and $x_1[n] = \sum_{k=0}^{N-1} a_k e^{jk\omega_0 n} = a_0 + a_1 e^{j\pi n}$. Comparing this to $e^{jn\pi}$ shows that

$$
\begin{aligned}
a_0 &= 0 \\
a_1 &= 1
\end{aligned}
$$

Now that we found the Fourier coefficients for $x_1[n]$ then the output is

$$
\begin{aligned}
y_1[n] &= \sum_{k=0}^{N-1} a_n H\left(jk\omega_0\right) e^{jnk\omega_0} \\
&= a_0 H(0) e^0 + a_1 H\left(j\pi\right) e^{jn\pi}
\end{aligned}
$$

But $a_0 = 1, a_1 = 1$ and the above becomes

$$
y_1[n] = H\left(j\pi\right) e^{jn\pi}
$$

From the graph of $H\left(jk\omega_0\right)$ given, we see that at $\omega = \pi, H\left(j\pi\right) = 0$. Therefore

$$
y_1[n] = 0
$$

3.2 Part b

Here $x_2[n] = 1 + \sin\left(\frac{3\pi}{8}n + \frac{\pi}{4}\right)$. The first step is to find the period N

$$
\begin{aligned}
x_2[n] &= x_2[n+N] \\
1 + \sin\left(\frac{3\pi}{8}n + \frac{\pi}{4}\right) &= 1 + \sin\left(\frac{3\pi}{8}(n+N) + \frac{\pi}{4}\right) \\
&= 1 + \sin\left(\frac{3\pi}{8}n + \frac{3\pi}{8}N + \frac{\pi}{4}\right) \\
&= 1 + \sin\left(\left(\frac{3\pi}{8}n + \frac{\pi}{4}\right) + \frac{3\pi}{8}N\right)
\end{aligned}
$$

Hence $\frac{3\pi}{8}N = 2\pi m$ or $\frac{N}{m} = \frac{16}{3}$. Since these are relatively prime, then $N = 16$ is the fundamental period. Therefore

$$
x_2[n] = \sum_{k=0}^{N-1} a_k e^{jk\omega_0 n}
$$

where $\omega_0 = \frac{2\pi}{N} = \frac{2\pi}{16} = \frac{\pi}{8}$. The above becomes

$$
x_2[n] = \sum_{k=0}^{15} a_k e^{jk\frac{\pi}{8}n} \tag{1}
$$

But

$$
\begin{aligned}
1 + \sin\left(\frac{3\pi}{8}n + \frac{\pi}{4}\right) &= 1 + \frac{e^{j\left(\frac{3\pi}{8}n+\frac{\pi}{4}\right)} - e^{-j\left(\frac{3\pi}{8}n+\frac{\pi}{4}\right)}}{2j} \\
&= 1 + \frac{1}{2j}e^{j\frac{3\pi}{8}n}e^{j\frac{\pi}{4}} - \frac{1}{2j}e^{-j\frac{3\pi}{8}n}e^{-j\frac{\pi}{4}} \qquad (2)
\end{aligned}
$$

Comparing (1) and (2) shows that $a_0 = 1, a_3 = \frac{1}{2j}e^{j\frac{\pi}{4}}, a_{-3} = -\frac{1}{2j}e^{-j\frac{\pi}{4}}$. But $a_{-3} = a_{-3+16} = a_{13}$ due to periodicity (and since we want to keep the index from 0 to 15. Therefore

$$
\begin{aligned}
a_0 &= 1 \\
a_3 &= \frac{1}{2j}e^{j\frac{\pi}{4}} \\
a_{13} &= -\frac{1}{2j}e^{-j\frac{\pi}{4}}
\end{aligned}
$$

And all other $a_k = 0$. Now that we found the Fourier coefficient, then the response $y_2[n]$ is found from

$$
\begin{aligned}
y_2[n] &= \sum_{k=0}^{N-1} a_n H(jk\omega_0)\, e^{jkn\omega_0} \\
&= a_0 H(0) + a_3 H\left(j3\frac{\pi}{8}\right) e^{j3\frac{\pi}{8}n} + a_{13} H\left(j13\frac{\pi}{8}\right) e^{j13\frac{\pi}{8}n} \\
&= H(0) + \left(\frac{1}{2j}e^{j\frac{\pi}{4}}\right) H\left(j\frac{3\pi}{8}\right) e^{j\frac{3\pi}{8}n} + \left(-\frac{1}{2j}e^{-j\frac{\pi}{4}}\right) H\left(j\frac{13\pi}{8}\right) e^{j\frac{13\pi}{8}n}
\end{aligned}
$$

From the graph of $H(jk\omega_0)$ given, we see that at $\omega = 0, H(0) = 0$ and at $\omega = \frac{3\pi}{8}, H\left(j\frac{3\pi}{8}\right) = 1$ and that at $\omega = \frac{13\pi}{8}, H\left(j\frac{13\pi}{8}\right) = 1$. Hence the above becomes

$$
y_2[n] = \left(\frac{1}{2j}e^{j\frac{\pi}{4}}\right) e^{j\frac{3\pi}{8}n} + \left(-\frac{1}{2j}e^{-j\frac{\pi}{4}}\right) e^{j\frac{13\pi}{8}n}
$$

But $e^{j\frac{13\pi}{8}n} = e^{j\frac{-3\pi}{8}n}$ since period is $N = 16$. Therefore the above simplifies to

$$
\begin{aligned}
y_2[n] &= \left(\frac{1}{2j}e^{j\frac{\pi}{4}}\right) e^{j\frac{3\pi}{8}n} + \left(-\frac{1}{2j}e^{-j\frac{\pi}{4}}\right) e^{j\frac{-3\pi}{8}n} \\
&= \frac{e^{j\left(\frac{\pi}{4}+\frac{3\pi}{8}n\right)} - e^{-j\left(\frac{\pi}{4}+\frac{3\pi}{8}n\right)}}{2j} \\
&= \sin\left(\frac{3\pi}{8}n + \frac{\pi}{4}\right)
\end{aligned}
$$

4 Problem 20 Chapter 3

(c) Determine the output $y(t)$ if $x(t) = \cos(t)$.

3.20. Consider a causal LTI system implemented as the *RLC* circuit shown in Figure P3.20. In this circuit, $x(t)$ is the input voltage. The voltage $y(t)$ across the capacitor is considered the system output.

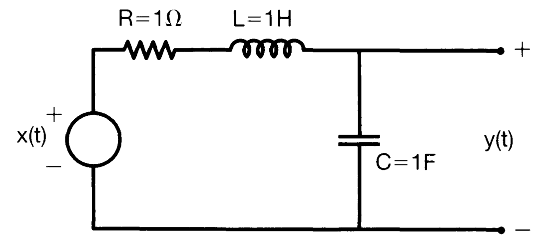

Figure P3.20

(a) Find the differential equation relating $x(t)$ and $y(t)$.

(b) Determine the frequency response of this system by considering the output of the system to inputs of the form $x(t) = e^{j\omega t}$.

(c) Determine the output $y(t)$ if $x(t) = \sin(t)$.

Figure 2: Problem description

Solution

4.1 Part a

Input voltage is $x(t)$. Hence drop in voltage around circuit is

$$x(t) = Ri(t) + L\frac{di}{dt} + y(t)$$

Now we need to relate the current $i(t)$ to $y(t)$. Since current across the capacitor is given by $i(t) = C\frac{dy}{dt}$ then replacing $i(t)$ in the above by $C\frac{dy}{dt}$ gives the diffeential equation

$$x(t) = RC\frac{dy}{dt} + LC\frac{d^2y}{dt^2} + y(t)$$

Or

$$LCy''(t) + RCy'(t) + y(t) = x(t)$$

But $L = 1, R = 1, C = 1$ therefore

$$y''(t) + y'(t) + y(t) = x(t)$$

4.2 Part b

Let the input $x(t) = e^{j\omega t}$. Therefore $y(t) = H(\omega)e^{j\omega t}$ where $H(\omega)$ is the frequency response (Book writes this as $H\left(e^{j\omega}\right)$ but $H(\omega)$ is simpler notation).

Hence

$$\begin{aligned} y'(t) &= H(\omega) j\omega e^{j\omega t} \\ y''(t) &= H(\omega)\left(j\omega\right)^2 e^{j\omega t} \\ &= -H(\omega)\omega^2 e^{j\omega t} \end{aligned}$$

Substituting the above into the ODE gives

$$-H(\omega)\,\omega^2 e^{j\omega t} + H(\omega)\,j\omega e^{j\omega t} + H(\omega)\,e^{j\omega t} = e^{j\omega t}$$

Dividing by $e^{j\omega t} \neq 0$ results in

$$-H(\omega)\,\omega^2 + H(\omega)\,j\omega + H(\omega) = 1$$

Solving for $H(\omega)$ gives

$$\begin{aligned} H(\omega)\left(-\omega^2 + j\omega + 1\right) &= 1 \qquad (1)\\ H(\omega) &= \frac{1}{-\omega^2 + j\omega + 1} \end{aligned}$$

4.3 Part c

Since we now know $H(\omega)$ then the output $y(t)$ when the input is $x(t) = \sin(t)$ is given by

$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(k\omega_0)\,e^{jk\omega_0 t} \qquad (2)$$

Where a_k are the Fourier coefficients of $\sin(t)$ and ω_0 is the fundamental frequency of $x(t)$. Since $\sin(t) = \sin\left(\frac{2\pi}{T}t\right)$ then $\frac{2\pi}{T} = 1$ and $T = 2\pi$. Hence $\omega_0 = 1$. And since $\sin(t) = \frac{1}{2j}\left(e^{jt} - e^{-jt}\right)$ then $a_1 = \frac{1}{2j}, a_{-1} = -\frac{1}{2j}$. Eq. (2) becomes

$$\begin{aligned} y(t) &= a_{-1}H(-\omega_0)\,e^{-j\omega_0 t} + a_1 H(\omega_0)\,e^{j\omega_0 t}\\ &= -\frac{1}{2j}H(-1)\,e^{-jt} + \frac{1}{2j}H(1)\,e^{jt} \qquad (3) \end{aligned}$$

Now we need to find $H(-1), H(1)$. From (1)

$$\begin{aligned} H(-1) &= \frac{1}{-(-1)^2 - j(-1) + 1}\\ &= \frac{1}{-1 + j + 1}\\ &= \frac{1}{j} \end{aligned}$$

And

$$\begin{aligned} H(+1) &= \frac{1}{-(+1)^2 - j(+1) + 1}\\ &= \frac{1}{-1 - j + 1}\\ &= \frac{1}{j} \end{aligned}$$

Therefore (3) becomes

$$\begin{aligned} y(t) &= -\frac{1}{2j}\frac{1}{j}e^{-jt} + \frac{1}{2j}\frac{1}{j}e^{jt}\\ &= -\frac{1}{2j^2}e^{-jt} + \frac{1}{2j^2}e^{jt}\\ &= \frac{1}{2}e^{-jt} - \frac{1}{2}e^{jt}\\ &= -\left(\frac{1}{2}e^{jt} - \frac{1}{2}e^{-jt}\right) \end{aligned}$$

Hence

$$y(t) = -\cos(t)$$

5 Problem 28 Chapter 3

$k = 1$

3.28. Determine the Fourier series coefficients for each of the following discrete-time periodic signals. Plot the magnitude and phase of each set of coefficients a_k.

(a) Each $x[n]$ depicted in Figure P3.28(a)–(c)

(b) $x[n] = \sin(2\pi n/3)\cos(\pi n/2)$

(c) $x[n]$ periodic with period 4 and

$$x[n] = 1 - \sin\frac{\pi n}{4} \quad \text{for } 0 \le n \le 3$$

(d) $x[n]$ periodic with period 12 and

$$x[n] = 1 - \sin\frac{\pi n}{4} \quad \text{for } 0 \le n \le 11$$

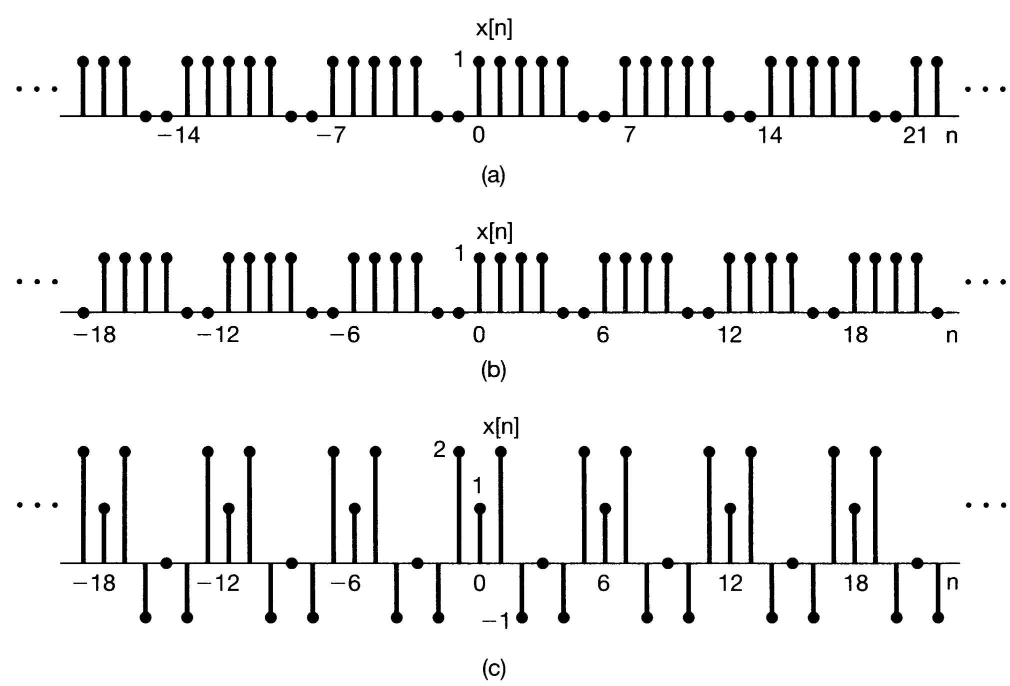

Figure P3.28

Figure 3: Problem description

Solution

5.1 Part a

First signal

The signal in P3.28(a) has period $N = 7$. Therefore $x[n] = \sum_{k=0}^{N-1} a_k e^{jn(k\omega_0)}$. We need to determine a_k. Since $\omega_0 = \frac{2\pi}{N} = \frac{2\pi}{7}$, then

$$\begin{aligned} a_k &= \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-jk\omega_0 n} \\ &= \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-jk\frac{2\pi}{N}n} \\ &= \frac{1}{7}\sum_{n=0}^{6} x[n]\, e^{-jk\frac{2\pi}{7}n} \end{aligned}$$

We first notice that $x[n] = 0$ for $n = 5, 6$ and $x[n] = 1$ otherwise. Hence the above sum simplifies to

$$a_k = \frac{1}{7}\sum_{n=0}^{4} e^{-jk\frac{2\pi}{7}n}$$

Using the relation $\sum_{n=0}^{M-1} a^n = \begin{cases} M & a = 1 \\ \frac{1-a^M}{1-a} & a \neq 1 \end{cases}$ to simplify the above where now $M = 5$ gives

$$\begin{aligned} a_k &= \frac{1}{7}\frac{1-\left(e^{-jk\frac{2\pi}{7}}\right)^5}{1-e^{-jk\frac{2\pi}{7}}} \qquad k = 0, 1, \cdots 6 \\ &= \frac{1}{7}\frac{1-e^{-jk\frac{10\pi}{7}}}{1-e^{-jk\frac{2\pi}{7}}} \end{aligned}$$

This is plot of $|a_k|$

```
In[ ]:= x[n_] := DiscreteDelta[n] + DiscreteDelta[n - 1] + DiscreteDelta[n - 2] + DiscreteDelta[n - 3] + DiscreteDelta[n - 4];
        ak[k_] := 1 / 7 Sum[x[n] Exp[-I k 2 Pi / 7 n], {n, 0, 6}]
        data = Table[ak[k], {k, 0, 6}];
        absAk = Abs[data] // N
Out[ ]= {0.714286, 0.25742, 0.17814, 0.0635774, 0.0635774, 0.17814, 0.25742}

In[ ]:= ListPlot[absAk, Mesh → All, Filling → Axis, Axes → {True, False},
           Ticks → {Range[0, 6], None}, DataRange → {0, 6},
           AxesLabel → {"k", None}, PlotLabel → "|a[k]|",
           BaseStyle → 12, PlotStyle → {Thick, Red}]
```

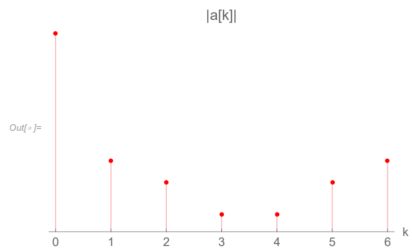

Figure 4: Plot of $|a_k|$

This is plot of the phase of a_k

```
In[ ]:= arg = Arg[akData] // N
Out[ ]= {0., -1.7952, -0.448799, 0.897598, -0.897598, 0.448799, 1.7952}

In[ ]:= ListPlot[ 180 / Pi arg, Ticks → {Range[0, 6], None},
           LabelingFunction → (Callout[Round[Last[#1], 0.1], Automatic] &),
           Axes → {Automatic, None}, AxesLabel → {"k", None},
           PlotLabel → "Phase in degrees", PlotRange → All, DataRange → {0, 6},
           Mesh → All, Filling → Axis]
```

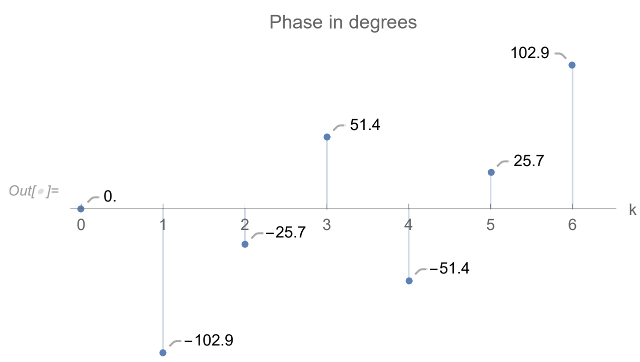

Figure 5: Plot of phase of a_k

<u>second signal</u>

The signal in P3.28(b) has period <u>$N = 6$</u>. Therefore $x[n] = \sum_{k=0}^{N-1} a_k e^{jn(k\omega_0)}$. We need to determine a_k, where $\omega_0 = \frac{2\pi}{N} = \frac{2\pi}{6}$. Hence

$$\begin{aligned} a_k &= \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-jk\omega_0 n} \\ &= \frac{1}{6}\sum_{n=0}^{5} x[n]\, e^{-jk\frac{2\pi}{6}n} \end{aligned}$$

We first notice that $x[n] = 0$ for $n = 4,5$ and $x[n] = 1$ otherwise, Hence the above sum simplifies to

$$a_k = \frac{1}{6}\sum_{n=0}^{3} e^{-jk\frac{2\pi}{6}n}$$

Using the relation $\sum_{n=0}^{M-1} a^n = \begin{cases} M & a = 1 \\ \frac{1-a^M}{1-a} & a \neq 1 \end{cases}$ to simplify the above, where now $M = 4$ gives

$$\begin{aligned} a_k &= \frac{1}{6}\frac{1-\left(e^{-jk\frac{2\pi}{6}}\right)^4}{1-e^{-jk\frac{2\pi}{6}}} \qquad k = 0,1,\cdots 5 \\ &= \frac{1}{6}\frac{1-e^{-jk\frac{8\pi}{6}}}{1-e^{-jk\frac{2\pi}{6}}} \end{aligned}$$

This is plot of $|a_k|$

```
In[]:= ak[k_] := 1/6 (1 - Exp[- I k 8 Pi/6]) / (1 - Exp[- I k 2 Pi/6])
       akData = Table[Limit[ak[n], n → k], {k, 0, 5}];
       absAk = Abs[akData] // N
Out[]= {0.666667, 0.288675, 0.166667, 0., 0.166667, 0.288675}

In[]:= ListPlot[absAk, Mesh → All, Filling → Axis, Axes → {True, False}, Ticks → {Range[0, 5], None},
         DataRange → {0, 5}, AxesLabel → {"k", None}, PlotLabel → "|a[k]|", BaseStyle → 12,
         PlotStyle → {Thick, Red}, PlotRange → All]
Out[]=
```

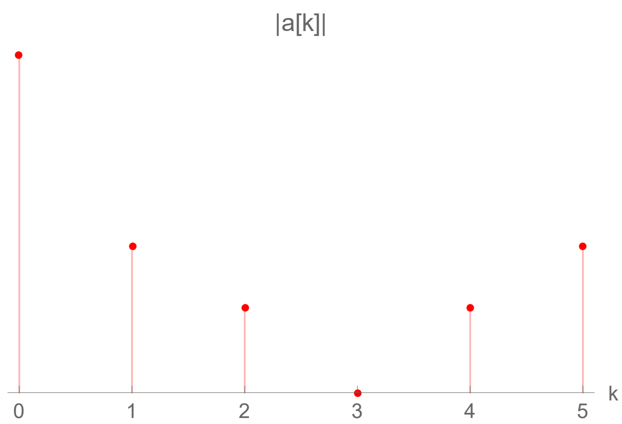

Figure 6: Plot of $|a_k|$

This is plot of the phase of a_k

```
In[]:= ak[k_] := 1/6 (1 - Exp[- I k 8 Pi/6]) / (1 - Exp[- I k 2 Pi/6])
       akData = Table[Limit[ak[n], n → k], {k, 0, 5}];
       arg = Arg[akData] // N
Out[]= {0., -1.5708, 0., 0., 0., 1.5708}

In[]:= ListPlot[ 180/Pi arg, Ticks → {Range[0, 5], None},
         LabelingFunction → (Callout[Round[Last[#1], 0.1], Automatic] &),
         Axes → {Automatic, None}, AxesLabel → {"k", None}, PlotLabel → "Phase in degrees",
         PlotRange → All, DataRange → {0, 5}, Mesh → All, Filling → Axis]
Out[]=
```

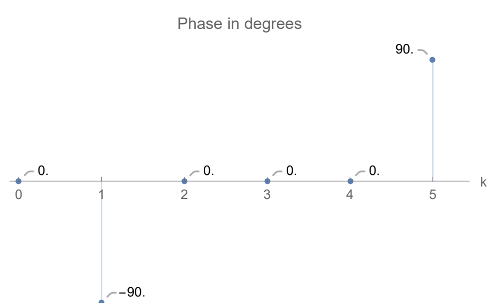

Figure 7: Plot of phase of a_k

Third signal

The signal in P3.28(c) also has period $N = 6$. Therefore $x[n] = \sum_{k=0}^{N-1} a_k e^{jn(k\omega_0)}$. We need to determine a_k. Given that $\omega_0 = \frac{2\pi}{N}$ then

$$\begin{aligned} a_k &= \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-jk\omega_0 n} \\ &= \frac{1}{6}\sum_{n=0}^{5} x[n]\, e^{-jk\frac{2\pi}{6}n} \end{aligned}$$

Where $x[0] = 1, x[1] = 2, x[2] = -1, x[3] = 0, x[4] = -1, x[5] = 2$. Hence the above sum becomes

$$\begin{aligned} a_k &= \frac{1}{6}\left(1 + 2e^{-jk\frac{2\pi}{6}} - e^{-jk\frac{2\pi}{6}2} + 0 - e^{-jk\frac{2\pi}{6}4} + 2e^{-jk\frac{2\pi}{6}5}\right) \\ &= \frac{1}{6}\left(1 + 2e^{-jk\frac{2\pi}{6}} - e^{-jk\frac{4\pi}{6}} - e^{-jk\frac{8\pi}{6}} + 2e^{-jk\frac{10\pi}{6}}\right) \end{aligned}$$

This is plot of $|a_k|$

```
In[∘]:= ak[k_] := 1 / 6 (1 + 2 Exp[-I k 2 Pi / 6] - Exp[-I k 4 Pi / 6] - Exp[-I k 8 Pi / 6] + 2 Exp[-I k 10 Pi / 6])
        akData = Table[Limit[ak[n], n → k], {k, 0, 5}];
        absAk = Abs[akData] // N
Out[∘]= {0.5, 0.666667, 7.40149 × 10^-17, 0.833333, 7.40149 × 10^-17, 0.666667}

In[∘]:= ListPlot[absAk, Mesh → All, Filling → Axis, Axes → {True, False}, Ticks → {Range[0, 5], None},
          DataRange → {0, 5}, AxesLabel → {"k", None}, PlotLabel → "|a[k]|", BaseStyle → 12,
          PlotStyle → {Thick, Red}, PlotRange → All]
```

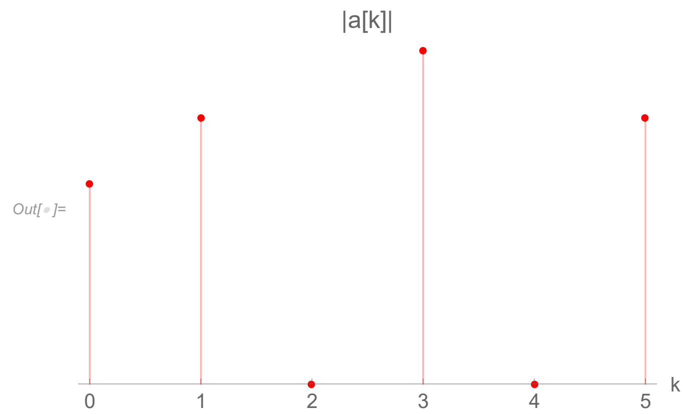

Figure 8: Plot of $|a_k|$

This is plot of the phase of a_k

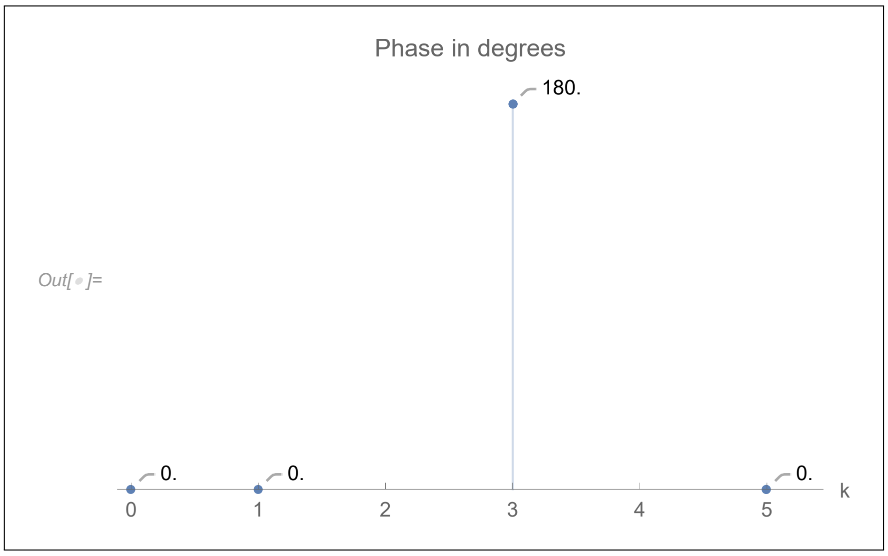

Figure 9: Plot of phase of a_k

5.2 Part b

$$x[n] = \sin\left(2\pi\frac{n}{3}\right)\cos\left(\pi\frac{n}{2}\right)$$

The first step is to find N, the fundamental period. Since $\sin(A)\cos(B) = \frac{1}{2}(\sin(A+B) + \sin(A-B))$ then

$$\begin{aligned} x[n] &= \frac{1}{2}\left(\sin\left(2\pi\frac{n}{3} + \pi\frac{n}{2}\right) + \sin\left(2\pi\frac{n}{3} - \pi\frac{n}{2}\right)\right) \\ &= \frac{1}{2}\left(\sin\left(\frac{7}{6}\pi n\right) + \sin\left(\frac{1}{6}\pi n\right)\right) \end{aligned}$$

To find the period of $\sin\left(\frac{7}{6}\pi n\right) = \sin\left(\frac{7}{6}\pi(n+N)\right)$ or $\sin\left(\frac{7}{6}\pi n\right) = \sin\left(\frac{7}{6}\pi n + \frac{7}{6}\pi N\right)$. Hence $\frac{7}{6}\pi N = 2\pi m$ which gives $\frac{m}{N} = \frac{7}{12}$. Hence $\underline{N = 12}$.

The period of $\sin\left(\frac{1}{6}\pi n\right) = \sin\left(\frac{1}{6}\pi (n+N)\right)$ or $\sin\left(\frac{1}{6}\pi n\right) = \sin\left(\frac{1}{6}\pi n + \frac{1}{6}\pi N\right)$. Hence $\frac{1}{6}\pi N = 2\pi m$ or $\frac{m}{N} = \frac{1}{12}$. Hence common period is $\underline{N = 12}$. Now that we know N then

$$a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-jk\omega_0 n}$$

Where $\omega_0 = \frac{2\pi}{12}$. The above becomes

$$\begin{aligned}
a_k &= \frac{1}{12}\sum_{n=0}^{11}\sin\left(2\pi\frac{n}{3}\right)\cos\left(\pi\frac{n}{2}\right)e^{-jk\frac{2\pi}{12}n}\\
12a_k &= 0+\sin\left(2\pi\frac{1}{3}\right)\cos\left(\pi\frac{1}{2}\right)e^{-jk\frac{2\pi}{12}}+\sin\left(2\pi\frac{2}{3}\right)\cos\left(\pi\frac{2}{2}\right)e^{-jk\frac{2\pi}{12}2}+\sin\left(2\pi\frac{3}{3}\right)\cos\left(\pi\frac{3}{2}\right)e^{-jk\frac{2\pi}{12}3}\\
&+\sin\left(2\pi\frac{4}{3}\right)\cos\left(\pi\frac{4}{2}\right)e^{-jk\frac{2\pi}{12}4}+\sin\left(2\pi\frac{5}{3}\right)\cos\left(\pi\frac{5}{2}\right)e^{-jk\frac{2\pi}{12}5}+\sin\left(2\pi\frac{6}{3}\right)\cos\left(\pi\frac{6}{2}\right)e^{-jk\frac{2\pi}{12}6}\\
&+\sin\left(2\pi\frac{7}{3}\right)\cos\left(\pi\frac{7}{2}\right)e^{-jk\frac{2\pi}{12}7}+\sin\left(2\pi\frac{8}{3}\right)\cos\left(\pi\frac{8}{2}\right)e^{-jk\frac{2\pi}{12}8}+\sin\left(2\pi\frac{9}{3}\right)\cos\left(\pi\frac{9}{2}\right)e^{-jk\frac{2\pi}{12}9}\\
&+\sin\left(2\pi\frac{10}{3}\right)\cos\left(\pi\frac{10}{2}\right)e^{-jk\frac{2\pi}{12}10}+\sin\left(2\pi\frac{11}{3}\right)\cos\left(\pi\frac{11}{2}\right)e^{-jk\frac{2\pi}{12}11}
\end{aligned}$$

Which simplifies to (many terms go to zero)

$$12a_k = \frac{1}{2}\sqrt{3}e^{-jk\frac{4\pi}{12}}+\frac{1}{2}\sqrt{3}e^{-jk\frac{8\pi}{12}}-\frac{1}{2}\sqrt{3}e^{-jk\frac{2\pi}{12}8}-\frac{1}{2}\sqrt{3}e^{-jk\frac{2\pi}{12}10}$$

Hence

$$a_k = \frac{\sqrt{3}}{24}\left(e^{-jk\frac{4\pi}{12}}+e^{-jk\frac{8\pi}{12}}-e^{-jk\frac{16\pi}{12}}-e^{-jk\frac{20\pi}{12}}\right)$$

Evaluating these for $k = 0\cdots N-1$ gives

k	a_k
0	0
1	$\frac{-j}{4}$
2	0
3	0
4	0
5	$\frac{j}{4}$
6	0
7	$\frac{-j}{4}$
8	0
9	0
10	0
11	$\frac{j}{4}$

Hence the $|a_k|$ and phase are

k	a_k	$\|a_k\|$	phase (degree)
0	0	0	0
1	$\frac{-j}{4}$	$\frac{1}{4}$	−90
2	0	0	0
3	0	0	0
4	0	0	0
5	$\frac{j}{4}$	$\frac{1}{4}$	90
6	0	0	0
7	$\frac{-j}{4}$	$\frac{1}{4}$	−90
8	0	0	0
9	0	0	0
10	0	0	0
11	$\frac{j}{4}$	$\frac{1}{4}$	90

6 Problem 47 Chapter 3

Consider the signal $x(t) = \cos(2\pi t)$ since $x(t)$ is periodic with a fundamental period of 1, it is also periodic with a period of N, where N is any positive integer. What are the Fourier series coefficients of $x(t)$ if we regard it as a periodic signal with period 3?

Solution

The Fourier series coefficients for $\cos(2\pi t)$ are found from Euler relation. Since $\omega_0 = 2\pi$ rad/sec, then

$$\cos(\omega_0 t) = \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}$$

Comparing the above to

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

Show that $a_1 = \frac{1}{2}$ and $a_{-1} = \frac{1}{2}$ and all other $a_k = 0$.

Similarly, if the period happened to be 3, then $\omega_0 = \frac{2\pi}{3}$ and now $x(t)$ can be written as $\cos(2\pi t) = \cos(3\omega_0 t)$. Therefore doing the same as above gives

$$\cos(3\omega_0 t) = \frac{1}{2}e^{j3\omega_0 t} + \frac{1}{2}e^{-j3\omega_0 t}$$

Comparing the above to $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$ shows that $a_3 = \frac{1}{2}$ and $a_{-3} = \frac{1}{2}$ and all other $a_k = 0$.