HW 2

EE 3015
Signals and Systems

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Contents

1 Problem 2.1, Chapter 2

Let $x[n] = \delta[n] + 2\delta[n-1] - \delta[n-3]$ and $h[n] = 2\delta[n+1] + 2\delta[n-1]$. Compute and plot each of the following convolutions (a) $y_1[n] = x[n] \circledast h[n]$ (b) $y_2[n] = x[n+2] \circledast h[n]$

Solution

1.1 Part a

The following is plot of $x[n], h[n]$

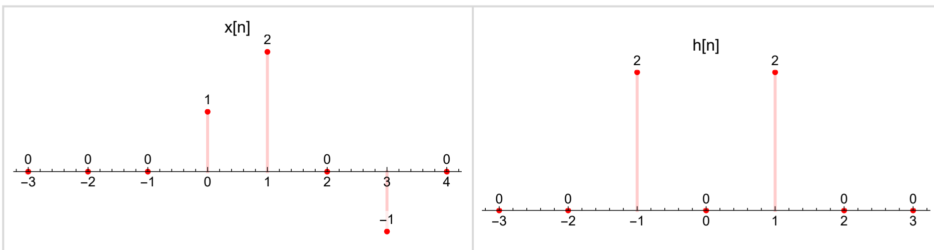


Figure 1: Plot of $x[n], h[n]$

```
x[n_] := If[n == 0, 1, 0];
p1 = DiscretePlot[x[n] + 2 x[n - 1] - x[n - 3], {n, -3, 4},
         Axes → {True, False},
         PlotRangePadding → 0.25, PlotLabel → "x[n]",
         ImageSize → 300,
         PlotStyle → {Thick, Red},
         LabelingFunction → Above,
         AspectRatio → Automatic,
         PlotRange → {Automatic, {-1, 2}}];
p2 = DiscretePlot[2 x[n + 1] + 2 x[n - 1], {n, -3, 3},
         Axes → {True, False},
         PlotRangePadding → 0.25,
         LabelingFunction → Above,
         PlotStyle → {Thick, Red},
         PlotRangePadding → 2,
         PlotLabel → "h[n]",
         ImageSize → 300,
         AspectRatio → Automatic,
         PlotRange → {Automatic, {0, 2}}];
p = Grid[{{p1, p2}}, Spacings → {1, 1}, Frame → All, FrameStyle → LightGray];
```

Figure 2: Code used for the above

Linear convolution is done by flipping $h[n]$ (reflection), then shifting the now flipped $h[n]$ one step to the right at a time. Each step the corresponding entries of $h[n]$ and $x[n]$ are multiplied and added. This is done until no overlapping between the two sequences. Mathematically this is the same as

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

Since $x[n]$ length is 3 and $x[n] = 0$ for $n < 0$ then the sum is

$$y[n] = \sum_{k=0}^{3} x[k]\, h[n-k]$$

For $\underline{n = -1}$

$$\begin{aligned} y[-1] &= \sum_{k=0}^{3} x[k]\, h[-1-k] \\ &= x[0]\, h[-1] + x[1]\, h[0] + x[2]\, h[1] + x[3]\, h[2] \\ &= (1)(2) + (2)(0) + (0)(2) + (-1)(0) \\ &= 2 \end{aligned}$$

For $\underline{n = 0}$

$$\begin{aligned} y[0] &= \sum_{k=0}^{3} x[k]\, h[-k] \\ &= x[0]\, h[0] + x[1]\, h[-1] + x[2]\, h[-2] + x[3]\, h[-3] \\ &= 0 + (2)(2) + 0 + 0 \\ &= 4 \end{aligned}$$

For $\underline{n = 1}$

$$\begin{aligned} y[1] &= \sum_{k=0}^{3} x[k]\, h[1-k] \\ &= x[0]\, h[1] + x[1]\, h[0] + x[2]\, h[-1] + x[3]\, h[-2] \\ &= (1)(2) + (2)(0) + (0)(1) + (-1)(0) \\ &= 2 \end{aligned}$$

For $\underline{n = 2}$

$$\begin{aligned} y[2] &= \sum_{k=0}^{3} x[k]\, h[2-k] \\ &= x[0]\, h[2] + x[1]\, h[1] + x[2]\, h[0] + x[3]\, h[-1] \\ &= (1)(0) + (2)(2) + (0)(0) + (-1)(2) \\ &= 2 \end{aligned}$$

For $\underline{n = 3}$

$$\begin{aligned} y[3] &= \sum_{k=0}^{3} x[k]\, h[3-k] \\ &= x[0]\, h[3] + x[1]\, h[2] + x[2]\, h[1] + x[3]\, h[0] \\ &= (1)(0) + (2)(0) + (0)(2) + (-1)(2) \\ &= 0 \end{aligned}$$

For $\underline{n = 4}$

$$\begin{aligned} y[4] &= \sum_{k=0}^{3} x[k]\, h[4-k] \\ &= x[0]\, h[4] + x[1]\, h[3] + x[2]\, h[2] + x[3]\, h[1] \\ &= (1)(0) + (2)(0) + (0)(2) + (-1)(2) \\ &= -2 \end{aligned}$$

All higher n values give $y[n] = 0$. Therefore

$$y_1[n] = 2\delta[n+1] + 4\delta[n] + 2\delta[n-1] + 2\delta[n-2] - 2\delta[n-4]$$

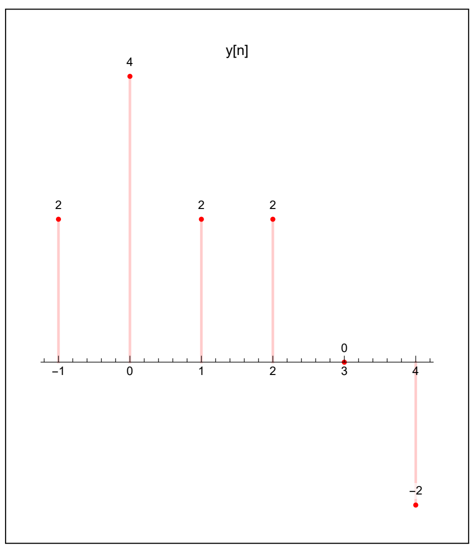


Figure 3: Plot of $y[n]$

1.2 Part b

First $x[n]$ is shifted to the left by 2 to obtain $x[n+2]$ and the result is convolved with $h[n]$

The following is plot of $x[n+2], h[n]$

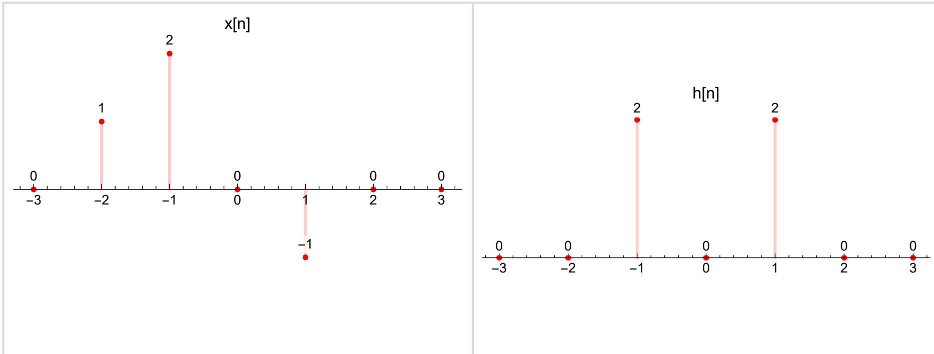


Figure 4: Plot of $x[n+2], h[n]$

Since Linear time invariant system, then shifted input convolved with $h[n]$ will give the shifted output found in part (a). Hence $y_2[n] = y_1[n+2]$. Hence

$$y_2[n] = 2\delta[n+3] + 4\delta[n+2] + 2\delta[n+1] + 2\delta[n] - 2\delta[n-2]$$

To show this explicitly, the convolution of shifted input is now computed directly. Linear convolution is

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$$

Since $x[n+2]$ length is 3 and $x[n] = 0$ for $n < -2$ then the sum is

$$y[n] = \sum_{k=-2}^{1} x[k]\,h[n-k]$$

For $\underline{n=-3}$

$$\begin{aligned} y[-3] &= \sum_{k=-2}^{1} x[k]\,h[-3-k] \\ &= x[-2]\,h[-1] + x[-1]\,h[-2] + x[0]\,h[-3] + x[1]\,h[-4] \\ &= (1)(2) + (2)(0) + (0)(0) + (-1)(0) \\ &= 2 \end{aligned}$$

For $\underline{n=-2}$

$$\begin{aligned} y[-2] &= \sum_{k=-2}^{1} x[k]\,h[-2-k] \\ &= x[-2]\,h[0] + x[-1]\,h[-1] + x[0]\,h[-2] + x[1]\,h[-3] \\ &= (1)(0) + (2)(2) + 0 + (-1)(0) \\ &= 4 \end{aligned}$$

For $\underline{n = -1}$

$$\begin{aligned} y[-1] &= \sum_{k=-2}^{1} x[k]\,h[-1-k] \\ &= x[-2]\,h[1] + x[-1]\,h[0] + x[0]\,h[-1] + x[1]\,h[-2] \\ &= (1)(2) + (2)(0) + 0 + (-1)(0) \\ &= 2 \end{aligned}$$

For $\underline{n = 0}$

$$\begin{aligned} y[0] &= \sum_{k=-2}^{1} x[k]\,h[0-k] \\ &= x[-2]\,h[2] + x[-1]\,h[1] + x[0]\,h[0] + x[1]\,h[-1] \\ &= (1)(0) + (2)(2) + 0 + (-1)(2) \\ &= 2 \end{aligned}$$

For $\underline{n = 1}$

$$\begin{aligned} y[1] &= \sum_{k=-2}^{1} x[k]\,h[1-k] \\ &= x[-2]\,h[3] + x[-1]\,h[2] + x[0]\,h[1] + x[1]\,h[0] \\ &= (1)(0) + (2)(2) + 0 + (-1)(0) \\ &= 4 \end{aligned}$$

For $\underline{n = 2}$

$$\begin{aligned} y[2] &= \sum_{k=-2}^{1} x[k]\,h[2-k] \\ &= x[-2]\,h[4] + x[-1]\,h[3] + x[0]\,h[2] + x[1]\,h[1] \\ &= (1)(0) + (2)(0) + 0 + (-1)(2) \\ &= -2 \end{aligned}$$

Hence

$$y[n] = 2\delta[n+3] + 4\delta[n+2] + 2\delta[n+1] + 2\delta[n] - 2\delta[n-2]$$

Which is the shifted output found in part (a)

2 Problem 2.6, Chapter 2

Compute and plot the convolution $y[n] = x[n] \circledast h[n]$ where $x[n] = \left(\frac{1}{3}\right)^{-n} u[-n-1]$ and $h[n] = u[n-1]$

<u>Solution</u>

It is easier to do this using graphical method. $y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$. We could either flip and shift $x[n]$ or $h[n]$. Let us flip and shift $h[n]$. This below is the result for $n = 0$ when $h[n-k]$ and $x[k]$ are plotted on top of each others

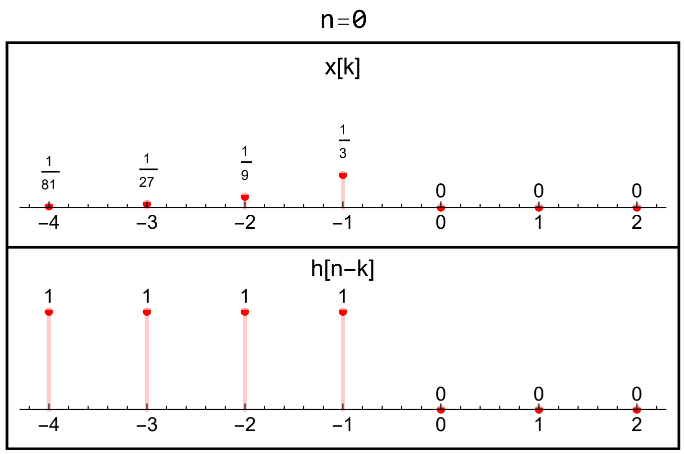


Figure 5: Convolution sum for $n = 0$

By multiplying corresponding values and summing the result can be seen to be $\sum_{k=1}^{\infty} \left(\frac{1}{3}\right)^k$. Let $r = \frac{1}{3}$ then this sum is $\left(\sum_{k=0}^{\infty} r^k\right) - 1$ But $\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$ since $r < 1$. Therefore

$$\begin{aligned}
\sum_{k=1}^{\infty} \left(\frac{1}{3}\right)^k &= \frac{1}{1-\frac{1}{3}} - 1 \\
&= \frac{3}{3-1} - 1 \\
&= \frac{3}{2} - 1 \\
&= \frac{1}{2}
\end{aligned}$$

Hence $y[0] = \frac{1}{2}$. Now, the signal $h[n-k]$ is shifted to the right by 1 then 2 then 3 and so on. This gives $y[1], y[2], \cdots$. Each time, the same sum result which is $\frac{1}{2}$. Here is a diagram for $n = 1$ and $n = 2$ for illustration

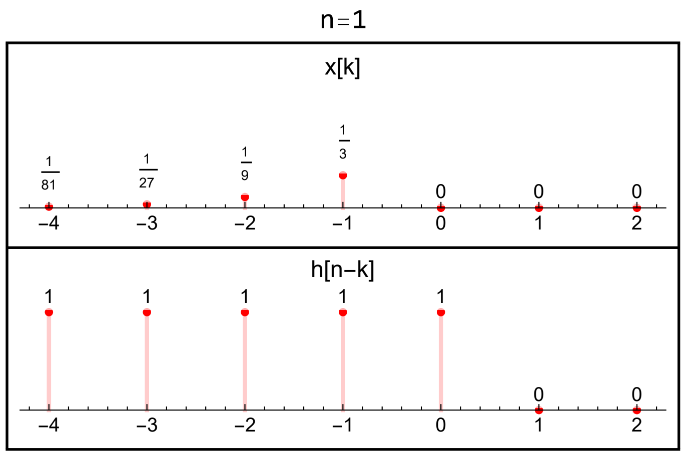


Figure 6: Convolution sum for $n = 1$

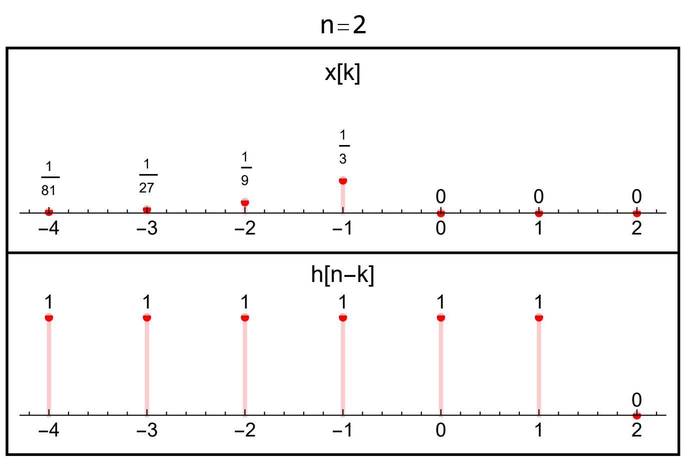


Figure 7: Convolution sum for $n = 2$

Therefore $y[n] = \frac{1}{2}$ for $n \geq 0$. Now we will look to see what happens when $h[-k]$ is shifted to the left. For $n = -1$ this is the result

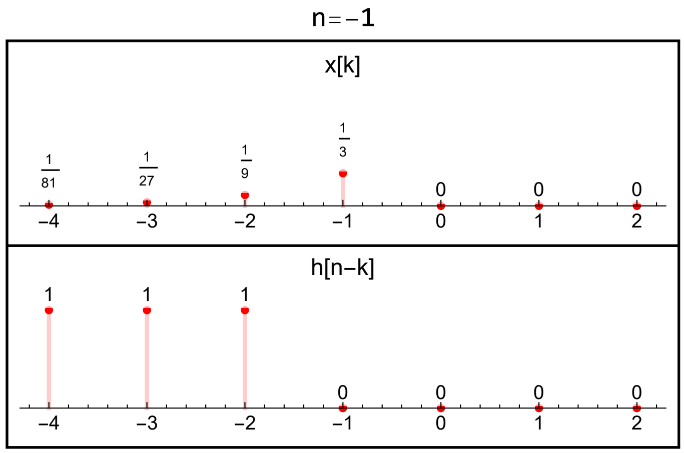


Figure 8: Convolution sum for $n = -1$

When multiplying the corresponding elements and adding, now the element $\frac{1}{3}$ is multiplied by a zero and not by 1. Hence the sum becomes $\left(\sum_{k=1}^{\infty}\left(\frac{1}{3}\right)^k\right) - \frac{1}{3}$ which is $\frac{1}{2} - \frac{1}{3} = \frac{1}{6} = \left(\frac{1}{2}\right)\left(\frac{1}{3}\right)$. Therefore $y[-1] = \frac{1}{6}$. When $h[-k]$ is shifted to the left one more step, it gives $y[-2]$ which is

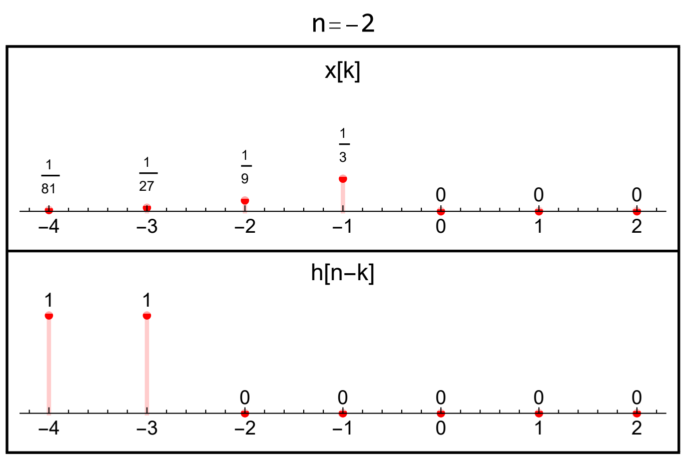


Figure 9: Convolution sum for $n = -2$

We see from the above diagram that now $\frac{1}{3}$ and $\frac{1}{9}$ do not contribute to the sum since both are multiplied by zero. This means $y[-2] = \left(\sum_{k=1}^{\infty}\left(\frac{1}{3}\right)^k\right) - \left(\frac{1}{3} + \frac{1}{9}\right) = \frac{1}{2} - \left(\frac{1}{3} + \frac{1}{9}\right) = \frac{1}{18} = \left(\frac{1}{2}\right)\left(\frac{1}{9}\right)$.

Each time $h[-k]$ is shifted to the left by one, the sum reduces. From the above we see that

$$y[-1] = \left(\frac{1}{2}\right)\left(\frac{1}{3}\right)$$
$$y[-2] = \left(\frac{1}{2}\right)\left(\frac{1}{3^2}\right)$$

Hence by extrapolation the pattern is

$$\begin{aligned} y[-n] &= \left(\frac{1}{2}\right)\left(\frac{1}{3^{-n}}\right) \\ &= \frac{3^n}{2} \end{aligned}$$

Therefore the final result is

$$y[n] = \begin{cases} \frac{1}{2} & n \geq 0 \\ \frac{3^n}{2} & n < 0 \end{cases}$$

Here is plot of $y[n] = x[n] \circledast h[n]$ given by the above

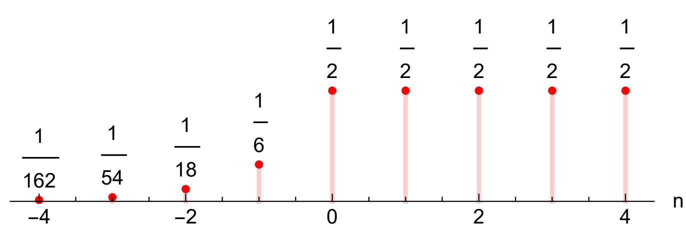


Figure 10: Plot of $y[n]$

3 Problem 2.11, Chapter 2

Let $x(t) = u(t-3) - u(t-5)$ and $h(t) = e^{-3t}u(t)$. (a) compute $y(t) = x(t) \circledast h(t)$. (b) Compute $g(t) = \frac{dx}{dt} \circledast h(t)$. (c) How is $g(t)$ related to $y(t)$?

Solution

3.1 Part (a)

It is easier to do this using graphical method. This is plot of $x(t)$ and $h(t)$.

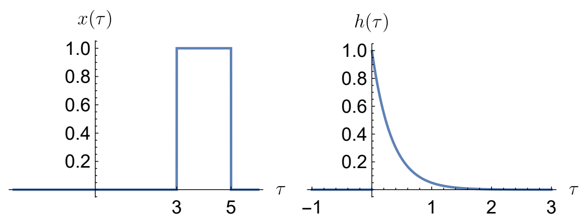


Figure 11: Plot $x(t)$ and $h(t)$

```
p1 = Plot[(UnitStep[t - 3] - UnitStep[t - 5]), {t, -3, 6},
        Exclusions → None, AxesLabel → {MaTeX["\\tau"], MaTeX["x(\\tau)"]},
        BaseStyle → 12, Ticks → {{3, 5}, Automatic}];
p2 = Plot[Exp[-3 t] UnitStep[t], {t, -1, 3}, AxesLabel → {MaTeX["\\tau"], MaTeX["h(\\tau)"]},
        BaseStyle → 12, PlotRange → All];
p = Grid[{{p1, p2}}];
```

Figure 12: Code used for the above plot

The next step is to fold one of the signals and then slide it to the right. We can folder either $x(t)$ or $h(t)$. Let us fold $x(t)$. Hence the integral is

$$y(t) = \int_{-\infty}^{\infty} x(t-\tau)\,h(\tau)\,d\tau$$

If we have chosen to fold $h(t)$ instead, then the integral would have been

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$$

This is the result after folding (reflection) of $x(t)$

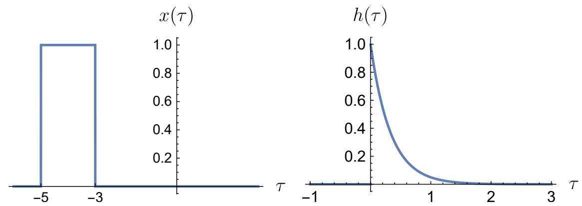


Figure 13: Folding $x(t)$

Next we label each edge of the folded signal before shifting it to the right as follows

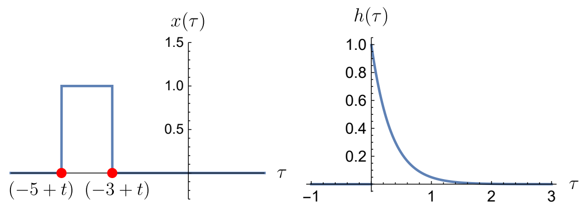


Figure 14: Folding $x(t)$ and labeling the edges

We see from the above that for $t-3<0$ or for $t<3$ the integral is zero since there is no overlapping between the folded $x(\tau)$ and $h(\tau)$. As we slide the folded $x(\tau)$ more to the right, we end up with $x(\tau)$ partially under $h(\tau)$ like this

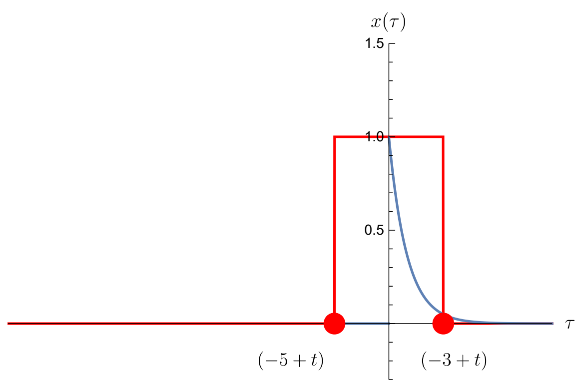


Figure 15: Shifting $x(\tau)$ to the right, partially inside

From the above, we see that for $0<t-3<2$ (since 2 is the width of $x(\tau)$) or for $3<t<5$, then the overlap is partial. Hence the integral now becomes

$$\begin{aligned} y(t) &= \int_0^{t-3} x(t-\tau)\,h(\tau)\,d\tau \qquad 3 < t \leq 5 \\ &= \int_0^{t-3} e^{-3\tau} d\tau \\ &= \frac{-1}{3}\left[e^{-3\tau}\right]_0^{t-3} \\ &= \frac{-1}{3}\left[e^{-3(t-3)} - 1\right] \\ &= \frac{1}{3}\left(1 - e^{-3(t-3)}\right) \end{aligned}$$

The next step is when folded $x(\tau)$ is fully inside $h(\tau)$ as follows

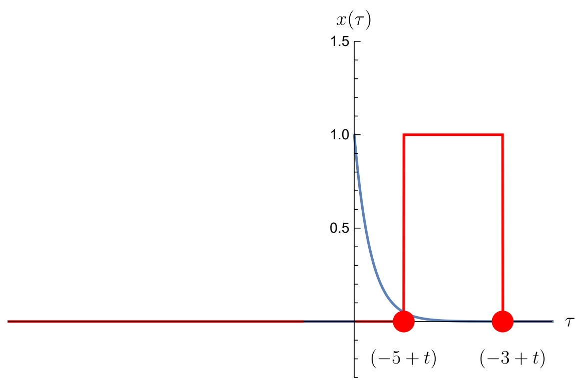


Figure 16: Shifting $x(\tau)$ to the right, fully inside

From the above, we see that for $0 < t - 5$ or $t > 5$, then the overlap is complete. Hence the integral now becomes

$$\begin{aligned} y(t) &= \int_{t-5}^{t-3} x(t-\tau)\,h(\tau)\,d\tau \qquad 5 < t \leq \infty \\ &= \int_{t-5}^{t-3} e^{-3\tau} d\tau \\ &= \frac{-1}{3}\left[e^{-3\tau}\right]_{t-5}^{t-3} \\ &= \frac{-1}{3}\left(e^{-3(t-3)} - e^{-3(t-5)}\right) \\ &= \frac{1}{3}\left(e^{-3(t-5)} - e^{-3(t-3)}\right) \end{aligned}$$

The above result $y(t) = \frac{1}{3}\left[e^{-3(t-5)} - e^{-3(t-3)}\right]$ can be rewritten as $\frac{1}{3}\left[\left(1 - e^{-6}\right)e^{-3(t-5)}\right]$ if needed to match the book. Therefore the final answer is

$$y(t)=\begin{cases} 0 & -\infty < t \leq 3 \\ \frac{1}{3}\left(1-e^{-3(t-3)}\right) & 3 < t \leq 5 \\ \frac{1}{3}\left(e^{-3(t-5)}-e^{-3(t-3)}\right) & 5 < t \leq \infty \end{cases}$$

Here is a plot of the above

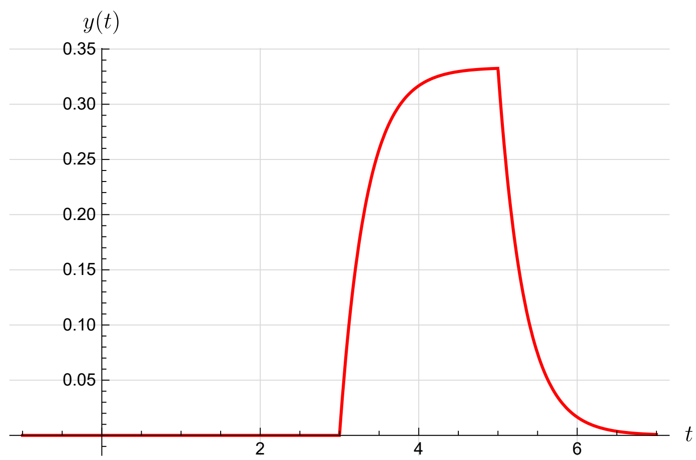


Figure 17: $y(t)$

```
y[t_] := Piecewise[{{0, t < 3}, {1/3 (1 - Exp[-3 (t - 3)]), 3 < t < 5},
        {1/3 (Exp[-3 (t - 5)] - Exp[-3 (t - 3)]), t > 5}}];
p = Plot[y[t], {t, -1, 7}, AxesLabel → {MaTeX["t"], MaTeX["y(t)"]},
      PlotStyle → Red, GridLines → Automatic, GridLinesStyle → LightGray];
```

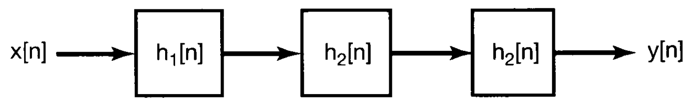
Figure 18: Code for the above

4 Problem 2.24, Chapter 2

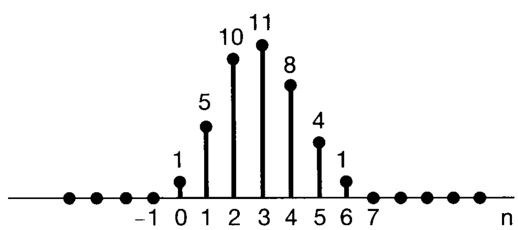
2.24. Consider the cascade interconnection of three causal LTI systems, illustrated in Figure P2.24(a). The impulse response $h_2[n]$ is

$$h_2[n] = u[n] - u[n-2],$$

and the overall impulse response is as shown in Figure P2.24(b).



(a)



(b)

Figure P2.24

(a) Find the impulse response $h_1[n]$.
(b) Find the response of the overall system to the input

$$x[n] = \delta[n] - \delta[n-1].$$

Figure 19: Problem description

Solution

4.1 Part a

The impulse response $h[n]$ is given. This is the response when the input is $x[n] = \delta[0]$. Hence

$$h[n] = h_1[n] \circledast (h_2[n] \circledast h_2[n])$$

But $h_2[n]$ is given as $h_2[n] = \delta[0] + \delta[1]$. Hence, let $H[n] = h_2[n] \circledast h_2[n]$, therefore

$$\begin{aligned} H[n] &= \sum_{k=-\infty}^{\infty} h_2[k]\, h_2[n-k] \\ &= \sum_{k=-1}^{2} h_2[k]\, h_2[n-k] \end{aligned}$$

For $n = 0$.

$$\begin{aligned} H[0] &= \sum_{k=-1}^{0} h_2[k]\, h_2[-k] \\ &= h_2[-1]\, h_2[1] + h_2[0]\, h_2[0] \\ &= 0 + 1 \\ &= 1 \end{aligned}$$

For $n = 1$.

$$\begin{aligned} H[1] &= \sum_{k=-1}^{0} h_2[k]\, h_2[1-k] \\ &= h_2[-1]\, h_2[0] + h_2[0]\, h_2[1] \\ &= 0 + 2 \\ &= 2 \end{aligned}$$

For $n = 2$.

$$\begin{aligned} H[2] &= \sum_{k=-1}^{0} h_2[k]\, h_2[2-k] \\ &= h_2[-1]\, h_2[3] + h_2[0]\, h_2[2] \\ &= 0 + 1 \\ &= 1 \end{aligned}$$

And zero for all other n. Hence

$$\begin{aligned} H[n] &= h_2[n] \circledast h_2[n] \\ &= \delta[n] + 2\delta[n-1] + \delta[n-2] \end{aligned}$$

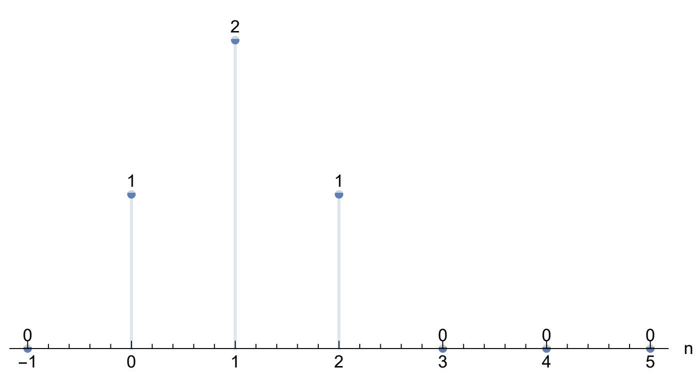


Figure 20: Plot of $h_2[n] \circledast h_2[n]$

```
h[n_] := DiscreteDelta[n] + 2 DiscreteDelta[n - 1] + DiscreteDelta[n - 2];
p = DiscretePlot[h[n], {n, -1, 5}, LabelingFunction → Above,
        Axes → {True, False}, AxesLabel → {"n", None}];
```

Figure 21: Code for the above

Now we need to find $h_1[n]$ given that $h_1[n] \circledast H[n]$ is what is shown in the problem. We do not know $h_1[n]$. so let us assume it is the sequence $\{h_1[0], h_2[0], \cdots\}$. Then by doing convolution by folding $h_1[n]$ and then sliding it to the right one step at a time, we obtain the following relations for each n.

$\underline{n=0}$ $h_1[0] H_1[0] = 1$ and since $H_1[0] = 1$ then $h_1[0] = 1$

$\underline{n=1}$ $h_1[1] H_1[0] + h_1[0] H_1[1] = 5$ and since $H_1[0] = 1, H_1[1] = 2$ then $h_1[1] + 2h_1[0] = 5$. But $h_1[0] = 1$ found above. Hence $h_1[1] + 2 = 5$ or $h_1[1] = 3$

$\underline{n=2}$ $h_1[2] H_1[0] + h_1[1] H_1[1] + h_1[0] H_1[2] = 10$ and since $H_1[0] = 1, H_1[1] = 2, H_1[2] = 1$ then $h_1[2] + 2h_1[1] + h_1[0] = 10$. But $h_1[0] = 1, h_1[1] = 3$ found above. Hence $h_1[2] + (2)(3) + 1 = 10$ or $h_1[2] = 3$

$\underline{n=3}$ $h_1[3] H_1[0] + h_1[2] H_1[1] + h_1[1] H_1[2] = 11$ and since $H_1[0] = 1, H_1[1] = 2, H_1[2] = 1$ then $h_1[3] + 2h_1[2] + h_1[1] = 11$. But $h_1[2] = 3, h_1[1] = 3$ found above. Hence $h_1[3] + (2)(3) + 3 = 11$ or $h_1[3] = 2$

$\underline{n=4}$ $h_1[4] H_1[0] + h_1[3] H_1[1] + h_1[2] H_1[2] = 8$ and since $H_1[0] = 1, H_1[1] = 2, H_1[2] = 1$ then $h_1[4] + 2h_1[3] + h_1[2] = 8$. But $h_1[3] = 2, h_1[2] = 3$ found above. Hence $h_1[4] + 2(2) + 3 = 8$ or $h_1[4] = 1$

$\underline{n=5}$ $h_1[5] H_1[0] + h_1[4] H_1[1] + h_1[3] H_1[2] = 4$ and since $H_1[0] = 1, H_1[1] = 2, H_1[2] = 1$ then $h_1[5] + 2h_1[4] + h_1[3] = 4$. But $h_1[4] = 1, h_1[3] = 2$ found above. Hence $h_1[5] + 2(1) + 2 = 4$ or $h_1[5] = 0$

And since the output is zero for $n > 5$ then $h_1[n] = 0$ for all $n > 5$. Therefore

$$h_1[n] = \delta[n] + 3\delta[n-1] + 3\delta[n-2] + 2\delta[n-3] + \delta[n-4]$$

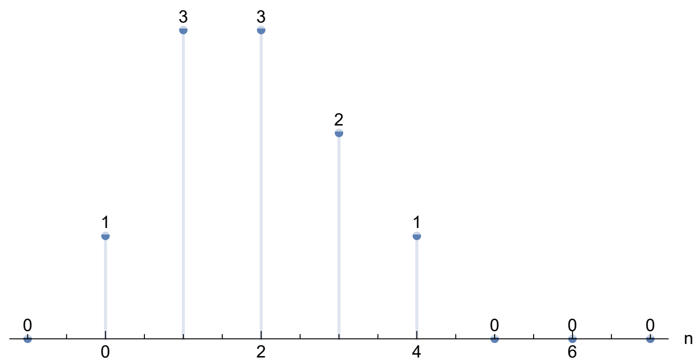


Figure 22: Plot of $h_1[n]$

```
h[n_] := DiscreteDelta[n] + 3 DiscreteDelta[n - 1] +
        3 DiscreteDelta[n - 2] + 2 DiscreteDelta[n - 3] + DiscreteDelta[n - 4];
p = DiscretePlot[h[n], {n, -1, 7}, LabelingFunction → Above,
        Axes → {True, False}, AxesLabel → {"n", None}];
```

Figure 23: Code for the above

4.2 Part b

When the input is $x[n] = \delta[n] - \delta[n-1]$ then response is given by $y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$ where $h[n]$ is the impulse response given in the problem P2.24 diagram. Hence we need to convolve the following two signals

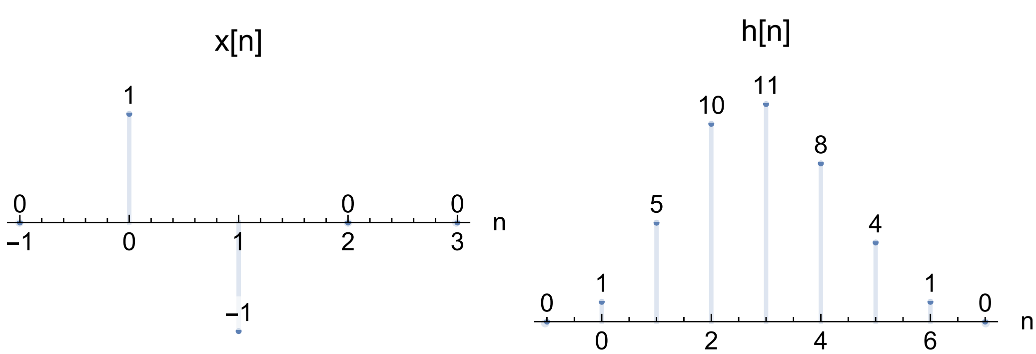


Figure 24: Plot of $x[n], h[n]$

By folding $x[n]$ and then shift it one step at a time, we see that we obtain the following

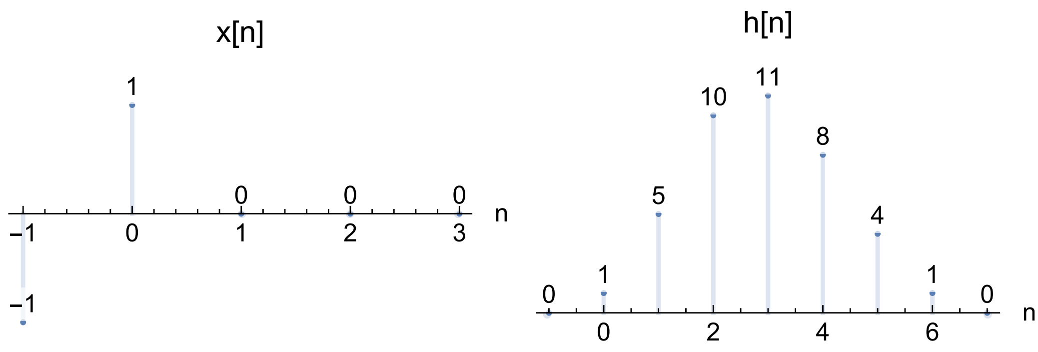


Figure 25: Plot of $x[n], h[n]$

$\underline{n = 0}$ $(1)(1) = 1$

$\underline{n = 1}$ $(-1)(1) + (1)(5) = 4$

$\underline{n = 2}$ $(-1)(5) + (1)(10) = 5$

$\underline{n = 3}$ $(-1)(10) + (1)(11) = 1$

$\underline{n = 4}$ $(-1)(11) + (1)(8) = -3$

$\underline{n = 5}$ $(-1)(8) + (1)(4) = -4$

$\underline{n = 6}$ $(-1)(4) + (1)(1) = -3$

$\underline{n = 7}$ $(-1)(1) + (1)(0) = -1$

$\underline{n = 8}$ $(-1)(0) + (1)(0) = 0$

And zero for all $n > 7$. This is plot of $y[n]$

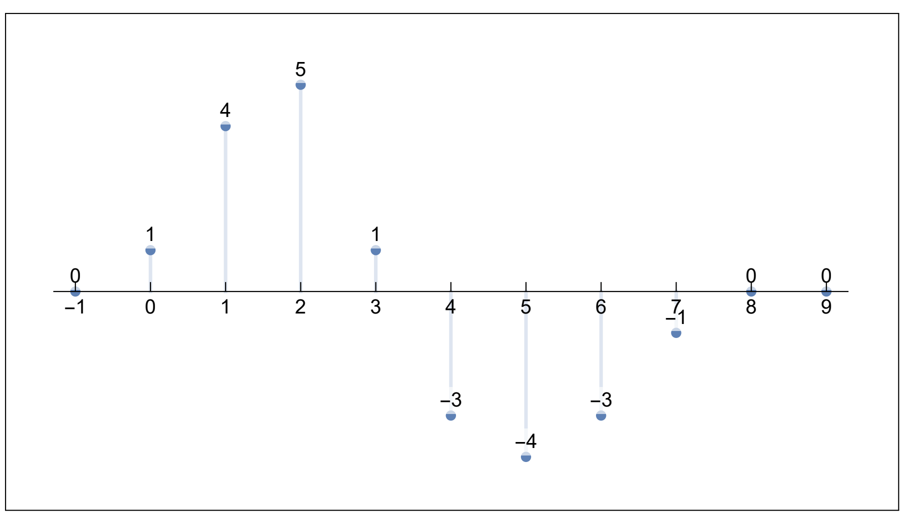


Figure 26: Plot of $y[n]$

5 Problem 2.32, Chapter 2

Solution

−2 −1 0 1 2 3 4 n **Figure P2.31**

2.32. Consider the difference equation

$$y[n] - \frac{1}{2}y[n-1] = x[n], \qquad \text{(P2.32–1)}$$

and suppose that

$$x[n] = \left(\frac{1}{3}\right)^n u[n]. \qquad \text{(P2.32–2)}$$

Assume that the solution $y[n]$ consists of the sum of a particular solution $y_p[n]$ to eq. (P2.32–1) and a homogeneous solution $y_h[n]$ satisfying the equation

$$y_h[n] - \frac{1}{2}y_h[n-1] = 0.$$

(a) Verify that the homogeneous solution is given by

$$y_h[n] = A\left(\frac{1}{2}\right)^n$$

(b) Let us consider obtaining a particular solution $y_p[n]$ such that

$$y_p[n] - \frac{1}{2}y_p[n-1] = \left(\frac{1}{3}\right)^n u[n].$$

By assuming that $y_p[n]$ is of the form $B(\frac{1}{3})^n$ for $n \geq 0$, and substituting this in the above difference equation, determine the value of B.

(c) Suppose that the LTI system described by eq. (P2.32–1) and initially at rest has as its input the signal specified by eq. (P2.32–2). Since $x[n] = 0$ for $n < 0$, we have that $y[n] = 0$ for $n < 0$. Also, from parts (a) and (b) we have that $y[n]$ has the form

$$y[n] = A\left(\frac{1}{2}\right)^n + B\left(\frac{1}{3}\right)^n$$

for $n \geq 0$. In order to solve for the unknown constant A, we must specify a value for $y[n]$ for some $n \geq 0$. Use the condition of initial rest and eqs. (P2.32–1) and (P2.32–2) to determine $y[0]$. From this value determine the constant A. The result of this calculation yields the solution to the difference equation (P2.32–1) under the condition of initial rest, when the input is given by eq. (P2.32–2).

2.33. Consider a system whose input $x(t)$ and output $y(t)$ satisfy the first-order differential

Figure 27: Problem description

5.1 Part a

Substituting $y_h[n] = A\left(\frac{1}{2}\right)^n$ into the difference equation $y_h[n] - \frac{1}{2}y_h[n-1] = 0$ gives

$$A\left(\frac{1}{2}\right)^n - \frac{1}{2}A\left(\frac{1}{2}\right)^{n-1} = 0$$

Since $A \neq 0$, the above simplifies to

$$\begin{aligned}
\frac{1}{2^n} - \frac{1}{2}\left(\frac{1}{2^{n-1}}\right) &= 0 \\
\frac{1}{2^n} - \frac{1}{2^n} &= 0 \\
0 &= 0
\end{aligned}$$

Verified OK.

5.2 Part b

Substituting $y_p[n] = B\left(\frac{1}{3^n}\right)$ into $y_p[n] - \frac{1}{2}y_p[n-1] = \frac{1}{3^n}u[n]$ gives

$$\begin{aligned}
B\left(\frac{1}{3^n}\right) - \frac{1}{2}B\left(\frac{1}{3^{n-1}}\right) &= \frac{1}{3^n}u[n] \\
B\left(\frac{1}{3^n} - \frac{1}{2}\frac{1}{3^{n-1}}\right) &= \frac{1}{3^n}u[n] \\
B\left(\frac{1}{3^n}\left(1 - \frac{1}{2}\frac{1}{3^{-1}}\right)\right) &= \frac{1}{3^n}u[n] \\
B\left(\frac{1}{3^n}\left(1 - \frac{3}{2}\right)\right) &= \frac{1}{3^n}u[n] \\
B\left(\frac{1}{3^n}\left(\frac{-1}{2}\right)\right) &= \frac{1}{3^n}u[n] \\
\frac{-1}{2}B &= u[n] \\
B &= -2u[n]
\end{aligned}$$

Hence for $n \geq 0$

$$B = -2$$

Therefore

$$y_p[n] = -2\left(\frac{1}{3^n}\right)$$

5.3 Part c

The solution is given by the sum of the homogenous and particular solutions. Hence

$$\begin{aligned}
y[n] &= y_h[n] + y_p[n] \\
&= A\left(\frac{1}{2}\right)^n - 2\left(\frac{1}{3^n}\right) \qquad (1)
\end{aligned}$$

Since system initially at rest, then $y[-1] = 0$. The recurrence equation is given as

$$y[n] - \frac{1}{2}y[n-1] = x[n]$$

Substituting (1) into the above and using $x[n] = \frac{1}{3^n}u[n]$ gives

$$y[n] - \frac{1}{2}y[n-1] = \frac{1}{3^n}u[n]$$

At $n = 0$ the above becomes

$$y[0] - \frac{1}{2}y[-1] = 1$$

But $y[-1] = 0$ and $y[0] = \left(A \left(\frac{1}{2}\right)^n - 2\left(\frac{1}{3^n}\right)\right)_{n=0} = A - 2$. Hence $A - 2 = 1$ or

$$A = 3$$

Therefore the solution (1) becomes

$$y[n] = 3\left(\frac{1}{2}\right)^n - 2\left(\frac{1}{3^n}\right)$$

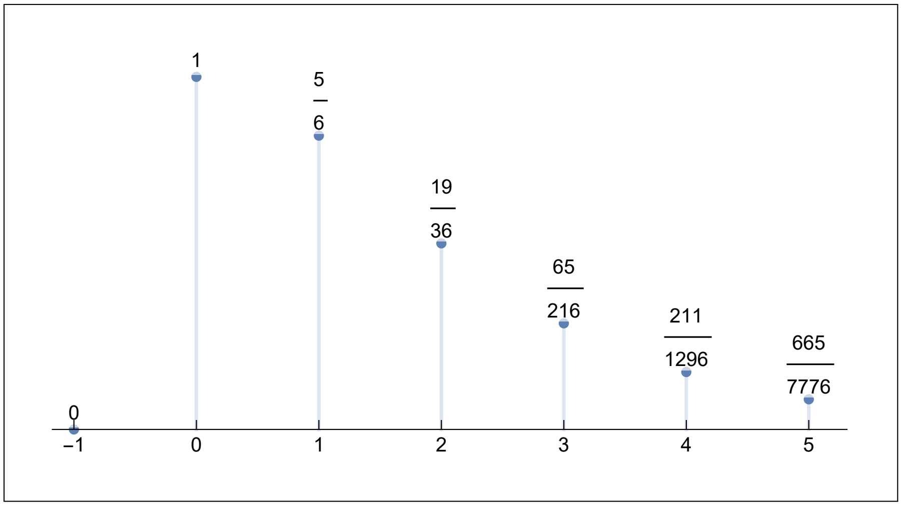


Figure 28: Plot of $y[n]$

```
y[n_] := 3 (1/2)^n - 2 (1/3)^n
p = DiscretePlot[y[n], {n, -1, 5}, LabelingFunction → Above,
        Axes → {True, False}, Ticks → {Range[-1, 9], None}];
```

Figure 29: Code used for the above

6 Problem 2.42, Chapter 2

Suppose the signal $x(t) = u\left(t+\frac{1}{2}\right) - u\left(t-\frac{1}{2}\right)$ is convolved with the signal $h(t) = e^{j\omega_0 t}$. (a) Determine the value of ω_0 which insures that $y(0) = 0$. Where $y(t) = x(t) \circledast h(t)$. (b) Is your answer to previous part unique?

Solution

6.1 Part a

$$x(t) \circledast h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

Since $x(t)$ is box function from $t = -\frac{1}{2}$ to $t = \frac{1}{2}$

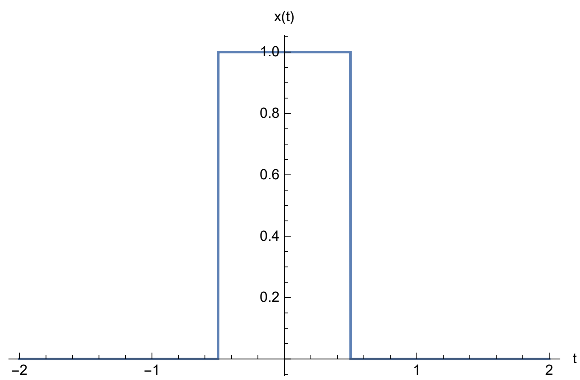


Figure 30: Plot of $x(n)$

```
x[t_] := UnitStep[t + 1 / 2] - UnitStep[t - 1 / 2]
p = Plot[x[t], {t, -2, 2}, Exclusions → None, AxesLabel → {"t", "x(t)"}];
```

Figure 31: Code used for the above

Then by folding $h(t)$ and shifting it over $x(t)$ it is clear that only the region between $\tau = -\frac{1}{2}$ to $\tau = \frac{1}{2}$ will contribute to the integral above since $x(\tau)$ is zero everywhere else. Hence the integral simplifies to

$$\begin{aligned}
y(t) &= x(t) \circledast h(t) \\
&= \int_{\frac{-1}{2}}^{\frac{1}{2}} h(t-\tau)\, d\tau \\
&= \int_{\frac{-1}{2}}^{\frac{1}{2}} e^{j\omega_0(t-\tau)} d\tau \\
&= e^{j\omega_0 t} \int_{\frac{-1}{2}}^{\frac{1}{2}} e^{-j\omega_0 \tau} d\tau \\
&= e^{j\omega_0 t} \left[\frac{e^{-j\omega_0 \tau}}{-j\omega_0} \right]_{-\frac{1}{2}}^{\frac{1}{2}} \\
&= e^{j\omega_0 t} \left(\frac{e^{-\frac{1}{2}j\omega_0} - e^{\frac{1}{2}j\omega_0}}{-j\omega_0} \right) \\
&= e^{j\omega_0 t} \left(\frac{e^{\frac{1}{2}j\omega_0} - e^{-\frac{1}{2}j\omega_0}}{j\omega_0} \right) \\
&= 2\frac{e^{j\omega_0 t}}{\omega_0} \left(\frac{e^{\frac{1}{2}j\omega_0} - e^{-\frac{1}{2}j\omega_0}}{2j} \right)
\end{aligned}$$

But $\frac{e^{\frac{1}{2}j\omega_0} - e^{-\frac{1}{2}j\omega_0}}{2j} = \sin\left(\frac{\omega_0}{2}\right)$ using Euler relation. Hence the above becomes

$$y(t) = 2\frac{e^{j\omega_0 t}}{\omega_0} \sin\left(\frac{\omega_0}{2}\right)$$

When $t = 0$ we are told $y(0) = 0$. The above becomes

$$0 = \frac{2}{\omega_0} \sin\left(\frac{\omega_0}{2}\right)$$

A value of ω_0 which will satisfy the above is $\underline{\omega_0 = 2\pi}$

6.2 Part b

The value ω_0 found in part (a) is not unique, since any nonzero integer multiple of 2π will also satisfy $y(0) = 0$