HW 10

EE 3015
Signals and Systems

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University of Minnesota, Twin Cities

Nasser M. Abbasi

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Contents

1 Problem 11.1

11.1. Consider the interconnection of discrete-time LTI systems shown in Figure P11.1. Express the overall system function for this interconnection in terms of $H_0(z)$, $H_1(z)$, and $G(z)$.

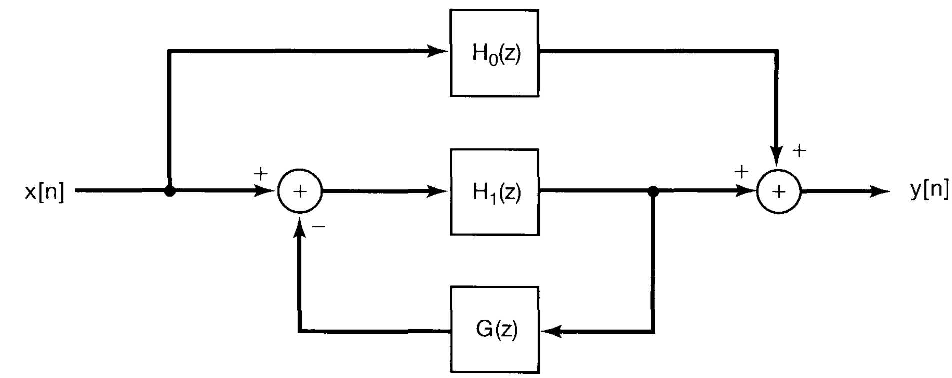


Figure P11.1

Figure 1: Problem description

<u>solution</u>

Adding the following notations on the diagram to make it easy to do the computation

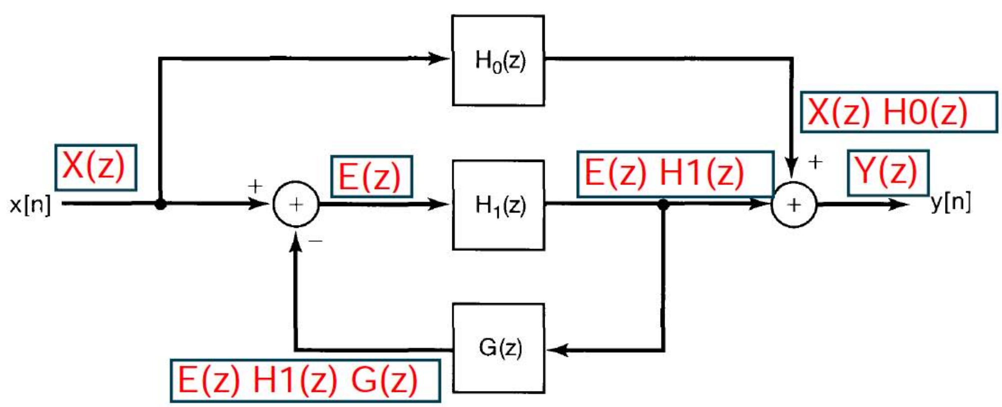


Figure 2: Annotations added

Therefore we see that

$$Y(z) = X(z)\,H_0(z) + E(z)\,H_1(z) \tag{1}$$

So we just need to determine $E(z)$. But $E(z) = X(z) - E(z)\,H_1(z)\,G(z)$. Hence $E(z)\,(1 + H_1(z)\,G(z)) = X(z)$ or

$$E(z) = \frac{X(z)}{1 + H_1(z)\,G(z)}$$

Substituting this into (1) gives

$$Y(z) = X(z)\,H_0(z) + \frac{X(z)}{1 + H_1(z)\,G(z)}H_1(z)$$
$$Y(z) = X(z)\left(H_0(z) + \frac{H_1(z)}{1 + H_1(z)\,G(z)}\right)$$

Hence

$$\frac{Y(z)}{X(z)} = H_0(z) + \frac{H_1(z)}{1 + H_1(z)\,G(z)}$$

2 Problem 11.2

11.2. Consider the interconnection of discrete-time LTI systems shown in Figure P11.2. Express the overall system function for this interconnection in terms of $H_1(s)$, $H_2(s)$, $G_1(s)$, and $G_2(s)$.

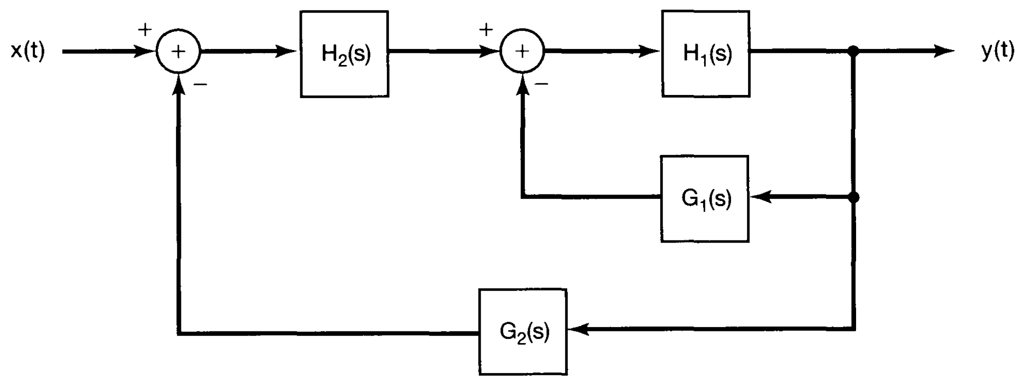


Figure P11.2

Figure 3: Problem description

solution

Adding the following notations on the diagram to make it easy to do the computation

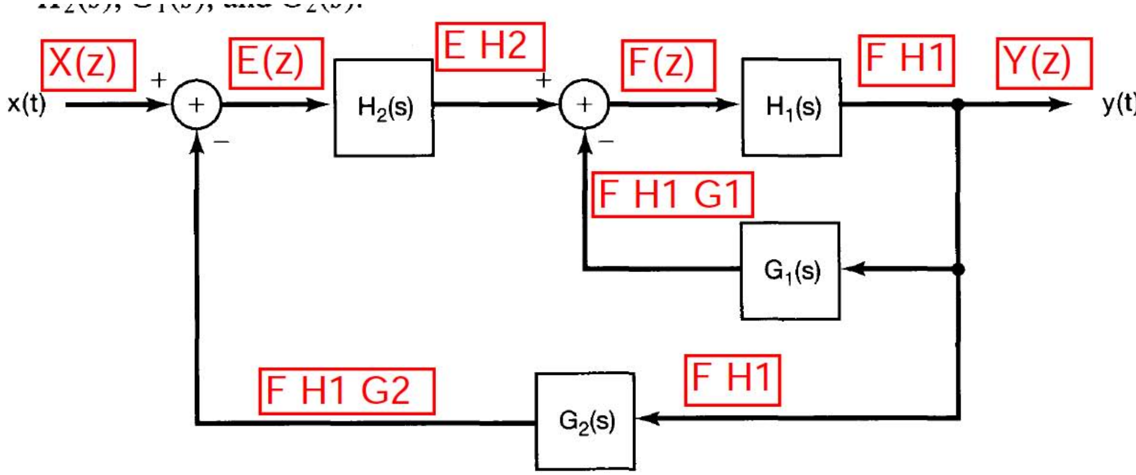


Figure 4: Annotations added

Therefore we see that

$$E = X - FH_1G_2 \tag{1}$$
$$F = EH_2 - FH_1G_1 \tag{2}$$

We have 2 equations with 2 unknowns E, F. Substituting first equation into the second gives

$$\begin{aligned}
F &= (X - FH_1G_2)\,H_2 - FH_1G_1\\
F &= XH_2 - FH_1G_2H_2 - FH_1G_1\\
F\,(1 + H_1G_2H_2 + H_1G_1) &= XH_2\\
F &= \frac{XH_2}{1 + H_1G_2H_2 + H_1G_1} \qquad (3)
\end{aligned}$$

But

$$Y(z) = F(z)\,H_1(z)$$

Hence using (3) into the above gives

$$\begin{aligned}
Y(z) &= \frac{XH_2}{1 + H_1G_2H_2 + H_1G_1}H_1\\
\frac{Y(z)}{X(z)} &= \frac{H_2H_1}{1 + H_1G_2H_2 + H_1G_1}
\end{aligned}$$

3 Problem 11.4

For what real values of b is the feedback system stable?

11.4. A causal LTI system S with input $x(t)$ and output $y(t)$ is represented by the differential equation

$$\frac{d^2y(t)}{dt^2} + \frac{dy(t)}{dt} + y(t) = \frac{dx(t)}{dt}.$$

S is to be implemented using the feedback configuration of Figure 11.3(a) with $H(s) = 1/(s+1)$. Determine $G(s)$.

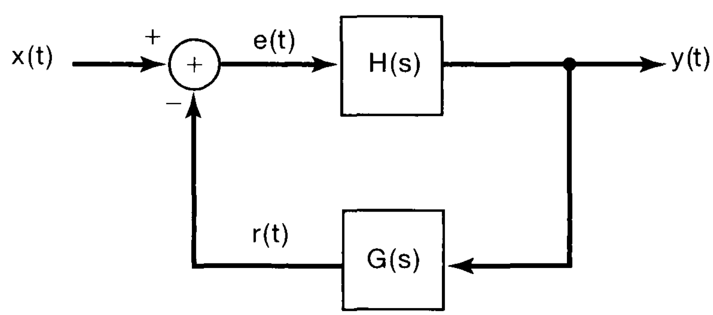
11.5. Consider the discrete-time feedback system depicted in Figure 11.3(b) with

Figure 5: Problem description

<u>solution</u>

Figure 11.3 a is the following

ure 11.3(a) and that of a discrete-time LTI feedback system ir



(a)

Figure 6: figure from book 11.3(a)

Taking the Laplace transform of the ODE gives (assuming zero initial conditions)

$$s^2Y(s) + sY(s) + Y(s) = sX(x)$$
$$\frac{Y(s)}{X(s)} = \frac{s}{s^2+s+1} \tag{1}$$

From the diagram, we see that

$$Y(s) = E(s)H(s) \tag{2}$$

But $E(s) = X(s) - R(s)$ and $R(s) = E(s)H(s)G(s)$. Hence

$$\begin{aligned}E(s) &= X(s) - (E(s)H(s)G(s)) \\ E(s)(1 + H(s)G(s)) &= X(s) \\ E(s) &= \frac{X(s)}{1 + H(s)G(s)}\end{aligned}$$

Substituting the above in (2) gives

$$\begin{aligned}Y(s) &= \frac{X(s)}{1 + H(s)G(s)}H(s) \\ \frac{Y(s)}{X(s)} &= \frac{H(s)}{1 + H(s)G(s)} \qquad (3)\end{aligned}$$

Comparing (3) and (1) shows that

$$\frac{H(s)}{1 + H(s)G(s)} = \frac{s}{s^2 + s + 1}$$

But we are given that $H(s) = \frac{1}{s+1}$. Hence the above becomes

$$\frac{\frac{1}{s+1}}{1 + \frac{1}{s+1}G(s)} = \frac{s}{s^2 + s + 1}$$

Now we solve for $G(s)$

$$\begin{aligned}\frac{\frac{1}{s+1}}{\frac{s+1+G(s)}{s+1}} &= \frac{s}{s^2 + s + 1} \\ \frac{1}{s + 1 + G(s)} &= \frac{s}{s^2 + s + 1} \\ s^2 + s + sG(s) &= s^2 + s + 1 \\ sG(s) &= s^2 + s + 1 - s^2 - s \\ G(s) &= \frac{1}{s}\end{aligned}$$

4 Problem 11.5

$H(s) = 1/(s+1)$. Determine $G(s)$.

11.5. Consider the discrete-time feedback system depicted in Figure 11.3(b) with

$$H(z) = \frac{1}{1-\frac{1}{2}z^{-1}} \quad \text{and} \quad G(z) = 1 - bz^{-1}.$$

For what real values of b is the feedback system stable?

11.6. Consider the discrete-time feedback system depicted in Figure 11.3(b) with

Figure 7: Problem description

<u>solution</u>

Figure 11.3 b is the following

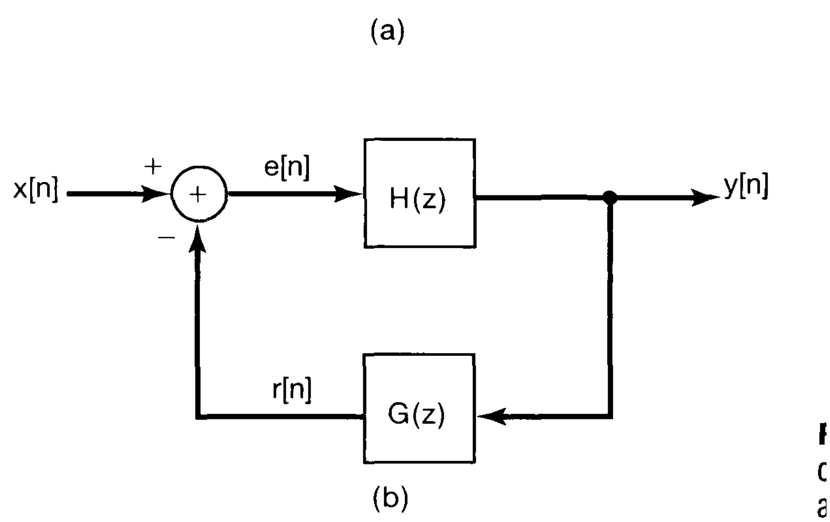


Figure 8: figure from book 11.3(b)

From the diagram $Y(z) = E(z)H(z)$ but $E(z) = X(z) - R(z)$ and $R(z) = E(z)H(z)G(z)$, hence

$$\begin{aligned} E(z) &= X(z) - E(z)H(z)G(z) \\ E(z)(1 + H(z)G(z)) &= X(z) \\ E(z) &= \frac{X(z)}{(1 + H(z)G(z))} \end{aligned}$$

Therefore

$$
\begin{aligned}
Y(z) &= E(z)\,H(z) \\
&= \frac{X(z)}{(1+H(z)\,G(z))}H(z) \\
\frac{Y(z)}{X(z)} &= \frac{H(z)}{1+H(z)\,G(z)}
\end{aligned}
$$

But $H(z) = \frac{1}{1-\frac{1}{2}z^{-1}}$ and $G(z) = 1 - bz^{-1}$. Hence the above becomes

$$
\begin{aligned}
\frac{Y(z)}{X(z)} &= \frac{\frac{1}{1-\frac{1}{2}z^{-1}}}{1+\frac{1}{1-\frac{1}{2}z^{-1}}\left(1-bz^{-1}\right)} \\
&= \frac{1}{1-\frac{1}{2}z^{-1}+1-bz^{-1}} \\
&= \frac{1}{2-\frac{1}{2}z^{-1}-bz^{-1}} \\
&= \frac{1}{2-\left(\frac{1}{2}+b\right)z^{-1}} \\
&= \frac{1}{2}\frac{1}{1-\left(\frac{1}{4}+\frac{b}{2}\right)z^{-1}}
\end{aligned}
$$

The pole is $\left(\frac{1}{4}+\frac{b}{2}\right)z^{-1} = 1$ or $z = \frac{1}{4}+\frac{b}{2}$. For causal system the pole should be inside the unit circle for stable system (so that it has a DFT). Therefore

$$
\begin{aligned}
\left|\frac{1}{4}+\frac{b}{2}\right| &< 1 \\
-1 < \frac{1}{4}+\frac{b}{2} &< 1 \\
-1-\frac{1}{4} < \frac{b}{2} &< 1-\frac{1}{4} \\
-\frac{5}{4} < \frac{b}{2} &< \frac{3}{4} \\
-\frac{10}{4} < b &< \frac{6}{4} \\
-\frac{5}{2} < b &< \frac{3}{2}
\end{aligned}
$$