① $x = \frac{a}{2}\cosh\mu \cos\theta \qquad y = \frac{a}{2}\sinh\mu \sin\theta$

Using notation in lecture $\xi^1 = x,\ \xi^2 = y,\ x^1 = \mu,\ x^2 = \theta$

$$g_{kl} = \frac{\partial \xi^i}{\partial x^k}\frac{\partial \xi^j}{\partial x^l}\delta_{ij}$$

$$g_{\mu\mu} = \left(\frac{\partial x}{\partial \mu}\right)^2 + \left(\frac{\partial y}{\partial \mu}\right)^2 = \frac{a^2}{4}\left[\sinh^2\mu \cos^2\theta + \underset{1+\sinh^2\mu}{\underset{\shortparallel}{\cosh^2\mu}} \sin^2\theta\right]$$

$$= \frac{a^2}{4}\left[\sinh^2\mu + \sin^2\theta\right] = \frac{a^2}{4}\left[\cosh^2\mu - \cos^2\theta\right]$$

$$g_{\theta\theta} = \left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 = \frac{a^2}{4}\left[\cosh^2\mu \sin^2\theta + \sinh^2\mu \cos^2\theta\right] = g_{\theta\theta}$$

$$g_{\mu\theta} = g_{\theta\mu} = \frac{\partial x}{\partial \mu}\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial \mu}\frac{\partial y}{\partial \theta} = \frac{a^2}{4}\Big[(\sinh\mu \cos\theta)(-\cosh\mu \sin\theta)$$

$$+ (\cosh\mu \sin\theta)(\sinh\mu \cos\theta)\Big] = 0$$

$$\boxed{\begin{aligned} g_{\mu\mu} &= g_{\theta\theta} = \frac{a^2}{4}\left[\sinh^2\mu + \sin^2\theta\right] = \frac{a^2}{4}\left[\cosh^2\mu - \cos^2\theta\right] \\ g_{\mu\theta} &= g_{\theta\mu} = 0 \end{aligned}}$$

(2) $\Gamma^i_{jk} = \frac{1}{2} g^{li}\left[\frac{\partial g_{kl}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\right]$

From lecture we know that $g_{rr} = 1$, $g_{\theta\theta} = r^2$, $g_{r\theta} = 0$.

Here $x^1 = r$ and $x^2 = \theta$. Since g_{ij} is diagonal we have $g^{rr} = 1$, $g^{\theta\theta} = \frac{1}{r^2}$, $g^{r\theta} = 0$.

Hence $\Gamma^i_{jk} = \frac{1}{2} g^{ii}\left[\frac{\partial g_{ki}}{\partial x^j} + \frac{\partial g_{ij}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^i}\right]$ no sum on i

$\Gamma^r_{rr} = \frac{1}{2}\left[0 + 0 - 0\right] = 0$ $\boxed{\Gamma^r_{rr} = 0}$

$\Gamma^r_{\theta r} = \frac{1}{2}\left[0 + 0 - 0\right] = 0$ $\boxed{\Gamma^r_{\theta r} = \Gamma^r_{r\theta} = 0}$

$\Gamma^r_{\theta\theta} = \frac{1}{2}\left[0 + 0 - 2r\right] = -r$ $\boxed{\Gamma^r_{\theta\theta} = -r}$

$\Gamma^\theta_{\theta\theta} = \frac{1}{2}\frac{1}{r^2}\left[0 + 0 - 0\right] = 0$ $\boxed{\Gamma^\theta_{\theta\theta} = 0}$

$\Gamma^\theta_{rr} = \frac{1}{2}\frac{1}{r^2}\left[0 + 0 - 0\right] = 0$ $\boxed{\Gamma^\theta_{rr} = 0}$

$\Gamma^\theta_{\theta r} = \frac{1}{2}\frac{1}{r^2}\left[0 + 2r - 0\right] = \frac{1}{r}$ $\boxed{\Gamma^\theta_{\theta r} = \Gamma^\theta_{r\theta} = \frac{1}{r}}$

③ $\epsilon_{i_1 \cdots i_n} = g_{i_1 i_1'} g_{i_2 i_2'} \cdots g_{i_n i_n'} \epsilon^{i_1' i_2' \cdots i_n'}$

which convert contravariant to covariant.

By definition $\epsilon^{i_1 i_2 \cdots i_n} = \begin{cases} 1 & \text{if even permutation} \\ -1 & \text{if odd permutation} \\ 0 & \text{if any pair of indices are equal} \end{cases}$

Then the same is true of $\epsilon_{i_1 i_2 \cdots i_n}$, because $g_{ij} = g_{ji}$.

Thus $\epsilon_{i_1 \cdots i_n} = \text{constant} \cdot \epsilon^{i_1 \cdots i_n}$

To find the constant choose $i_1 \cdots i_n = 12 \cdots n$.

$\epsilon_{12 \cdots n} = \text{constant}\ \underbrace{\epsilon^{12 \cdots n}}_{1} = \text{constant}$

$\shortparallel$

$g_{1 i_1} g_{2 i_2} \cdots g_{n i_n} \epsilon^{i_1 i_2 \cdots i_n} = \det g_{ij} = g$

$\Rightarrow \boxed{\epsilon_{i_1 i_2 \cdots i_n} = g\, \epsilon^{i_1 i_2 \cdots i_n}}$

(4) Spherical coordinates are orthogonal so g_{ij} is diagonal.

$x = r\cos\phi\sin\theta \qquad y = r\sin\phi\sin\theta \qquad z = r\cos\theta$

$$h_r^2 = g_{rr} = \left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2 = 1$$

$$h_\theta^2 = g_{\theta\theta} = \left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2 = r^2$$

$$h_\phi^2 = g_{\phi\phi} = \left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2 + \left(\frac{\partial z}{\partial \phi}\right)^2 = r^2\sin^2\theta$$

Ordinary vectors with components ordered by $\hat{r}, \hat{\theta}, \hat{\phi}$.

gradient components are $\frac{1}{h_i}\frac{\partial}{\partial x^i}$

$$\boxed{\vec{\nabla}\Phi = \left(\frac{\partial \Phi}{\partial r}, \frac{1}{r}\frac{\partial \Phi}{\partial \theta}, \frac{1}{r\sin\theta}\frac{\partial \Phi}{\partial \phi}\right)}$$

divergence $\vec{\nabla}\cdot\vec{V} = \frac{1}{h_1h_2h_3}\sum_i \frac{\partial}{\partial x^i}\left(\frac{h_1h_2h_3}{h_i}\bar{V}_i\right)$

$$\boxed{\vec{\nabla}\cdot\vec{V} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\bar{V}_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\bar{V}_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial \bar{V}_\phi}{\partial \phi}}$$

curl $(\vec{\nabla}\times\vec{V})_k = \frac{h_k}{h_1h_2h_3}\sum_{ij}\epsilon^{ijk}\frac{\partial}{\partial x^j}\left(h_i\bar{V}_i\right)$

$$\boxed{\vec{\nabla} \times \vec{\bar{V}} = \left(\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\, \bar{V}_\phi\right) - \frac{\partial \bar{V}_\theta}{\partial\phi}\right], \frac{1}{r\sin\theta}\left[\frac{\partial \bar{V}_r}{\partial\phi} - \sin\theta\frac{\partial}{\partial r}\left(r\bar{V}_\phi\right)\right], \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\bar{V}_\theta\right) - \frac{\partial \bar{V}_r}{\partial\theta}\right]\right)}$$

Laplacian $\nabla^2 \Phi = \frac{1}{h_1 h_2 h_3} \sum_i \frac{\partial}{\partial x^i}\left(\frac{h_1 h_2 h_3}{h_i^2}\frac{\partial\Phi}{\partial x^i}\right)$

$$\boxed{\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}}$$