

HW 9  
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# 1 Problem 1

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Problem Calculate the metric in elliptical coordinates

$$x = \frac{a}{2} \cosh \mu \cos \theta$$

$$y = \frac{a}{2} \sinh \mu \sin \theta$$

Solution

The coordinates in the Cartesian system are  $\zeta^1 = x, \zeta^2 = y$  and the coordinates in the other system (Elliptic) are  $x^1 = \mu, x^2 = \theta$ . The relation between these must be known and invertible also, meaning  $\zeta \equiv \zeta(x)$  and  $x \equiv x(\zeta)$ . This relation is given to use above as

$$\zeta^1 = \frac{a}{2} \cosh \mu \cos \theta$$

$$\zeta^2 = \frac{a}{2} \sinh \mu \sin \theta$$

The first step is to determine the metric tensor  $g_{ij}$  for the Polar coordinates. This is given by

$$g_{kl} = \delta_{ij} \frac{\partial \zeta^i}{\partial x^k} \frac{\partial \zeta^j}{\partial x^l}$$

The above using Einstein summation notation.

$$\begin{aligned} g_{11} &= \frac{\partial \zeta^1}{\partial x^1} \frac{\partial \zeta^1}{\partial x^1} + \frac{\partial \zeta^2}{\partial x^1} \frac{\partial \zeta^2}{\partial x^1} \\ &= \frac{\partial \zeta^1}{\partial \mu} \frac{\partial \zeta^1}{\partial \mu} + \frac{\partial \zeta^2}{\partial \mu} \frac{\partial \zeta^2}{\partial \mu} \\ &= \left( \frac{\partial \zeta^1}{\partial \mu} \right)^2 + \left( \frac{\partial \zeta^2}{\partial \mu} \right)^2 \\ &= \left( \frac{a}{2} \sinh \mu \cos \theta \right)^2 + \left( \frac{a}{2} \cosh \mu \sin \theta \right)^2 \\ &= \frac{a^2}{4} (\sinh^2 \mu \cos^2 \theta + \cosh^2 \mu \sin^2 \theta) \\ &= \frac{a^2}{4} ((\cosh^2 \mu - 1) \cos^2 \theta + \cosh^2 \mu (1 - \cos^2 \theta)) \\ &= \frac{a^2}{4} (\cosh^2 \mu \cos^2 \theta - \cos^2 \theta + \cosh^2 \mu - \cosh^2 \mu \cos^2 \theta) \\ &= \frac{a^2}{4} (\cosh^2 \mu - \cos^2 \theta) \end{aligned}$$

And

$$\begin{aligned}
g_{12} &= \frac{\partial \zeta^1}{\partial x^1} \frac{\partial \zeta^1}{\partial x^2} + \frac{\partial \zeta^2}{\partial x^1} \frac{\partial \zeta^2}{\partial x^2} \\
&= \frac{\partial \zeta^1}{\partial \mu} \frac{\partial \zeta^1}{\partial \theta} + \frac{\partial \zeta^2}{\partial \mu} \frac{\partial \zeta^2}{\partial \theta} \\
&= \left( \frac{a}{2} \sinh \mu \cos \theta \right) \left( -\frac{a}{2} \cosh \mu \sin \theta \right) + \left( \frac{a}{2} \cosh \mu \sin \theta \right) \left( \frac{a}{2} \sinh \mu \cos \theta \right) \\
&= 0
\end{aligned}$$

The above is as expected since the coordinate system is orthogonal. And

$$\begin{aligned}
g_{21} &= \frac{\partial \zeta^1}{\partial x^2} \frac{\partial \zeta^1}{\partial x^1} + \frac{\partial \zeta^2}{\partial x^2} \frac{\partial \zeta^2}{\partial x^1} \\
&= \frac{\partial \zeta^1}{\partial \theta} \frac{\partial \zeta^1}{\partial \mu} + \frac{\partial \zeta^2}{\partial \theta} \frac{\partial \zeta^2}{\partial \mu} \\
&= \left( -\frac{a}{2} \cosh \mu \sin \theta \right) \left( \frac{a}{2} \sinh \mu \cos \theta \right) + \left( \frac{a}{2} \sinh \mu \cos \theta \right) \left( \frac{a}{2} \cosh \mu \sin \theta \right) \\
&= 0
\end{aligned}$$

The above is as expected since the coordinate system is orthogonal. It is also because  $g_{ij}$  is symmetric and we already found that  $g_{12} = 0$ . And finally

$$\begin{aligned}
g_{22} &= \frac{\partial \zeta^1}{\partial x^2} \frac{\partial \zeta^1}{\partial x^2} + \frac{\partial \zeta^2}{\partial x^2} \frac{\partial \zeta^2}{\partial x^2} \\
&= \frac{\partial \zeta^1}{\partial \theta} \frac{\partial \zeta^1}{\partial \theta} + \frac{\partial \zeta^2}{\partial \theta} \frac{\partial \zeta^2}{\partial \theta} \\
&= \left( \frac{\partial \zeta^1}{\partial \theta} \right)^2 + \left( \frac{\partial \zeta^2}{\partial \theta} \right)^2 \\
&= \left( -\frac{a}{2} \cosh \mu \sin \theta \right)^2 + \left( \frac{a}{2} \sinh \mu \cos \theta \right)^2 \\
&= \frac{a^2}{4} \left( \cosh^2 \mu \sin^2 \theta + \sinh^2 \mu \cos^2 \theta \right) \\
&= \frac{a^2}{4} \left( \cosh^2 \mu (1 - \cos^2 \theta) + (\cosh^2 \mu - 1) \cos^2 \theta \right) \\
&= \frac{a^2}{4} \left( \cosh^2 \mu - \cosh^2 \mu \cos^2 \theta + \cosh^2 \mu \cos^2 \theta - \cos^2 \theta \right) \\
&= \frac{a^2}{4} \left( \cosh^2 \mu - \cos^2 \theta \right)
\end{aligned}$$

From the above we see that

$$\begin{aligned} g_{ij} &= \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \\ &= \frac{a^2}{4} \begin{pmatrix} \cosh^2 \mu - \cos^2 \theta & 0 \\ 0 & \cosh^2 \mu - \cos^2 \theta \end{pmatrix} \end{aligned}$$

That there are different ways to write the above, and they are all the same. For example, we can write

$$\begin{aligned} g_{ij} &= \frac{a^2}{4} \begin{pmatrix} (1 + \sinh^2 \mu) - (1 - \sin^2 \theta) & 0 \\ 0 & (1 + \sin^2 \mu) - (1 - \sin^2 \theta) \end{pmatrix} \\ &= \frac{a^2}{4} \begin{pmatrix} \sinh^2 \mu + \sin^2 \theta & 0 \\ 0 & \sinh^2 \mu + \sin^2 \theta \end{pmatrix} \end{aligned}$$

Or we could use the double angle relations  $\cos^2 \theta = \frac{1}{2}(1 + \cos(2\theta))$  and  $\cosh^2 \mu = \frac{1}{2}(1 + \cosh(2\theta))$  to obtain

$$\begin{aligned} g_{ij} &= \frac{a^2}{4} \begin{pmatrix} \frac{1}{2}(1 + \cosh(2\theta)) - \frac{1}{2}(1 + \cos(2\theta)) & 0 \\ 0 & \frac{1}{2}(1 + \cosh(2\theta)) - \frac{1}{2}(1 + \cos(2\theta)) \end{pmatrix} \\ &= \frac{a^2}{8} \begin{pmatrix} \cosh(2\theta) - \cos(2\theta) & 0 \\ 0 & \cosh(2\theta) - \cos(2\theta) \end{pmatrix} \end{aligned}$$

## 2 Problem 2

---

Problem Show that in a general coordinates system  $\epsilon_{i_1 \dots i_N} = g \epsilon^{i_1 \dots i_N}$  where the covariant form is obtained by lowering the indices on the contravariant form.

Solution

In tensor analysis, contravariant components of a tensor uses upper indices and covariant components uses lower indices. Given a tensor in contravariant form  $\epsilon^i$  then the covariant form  $\epsilon_i$  is obtained using

$$\epsilon_i = g_{ij} \epsilon^j$$

Where on the right side the sum is taken over  $j$  since it is the repeated index. This operation is called index contracting.

Therefore extending the above to all indices in  $\epsilon_{i_1 \dots i_N}$  results in

$$\epsilon_{i_1 i_2 \dots i_N} = g_{i_1 j_1} g_{i_2 j_2} \dots g_{i_N j_N} \epsilon^{j_1 j_2 \dots j_N} \quad (1)$$

But we know that, from page 123 in the Matrices notes, that the determinant of the metric can be written using Levi-Civita tensor as

$$g = \sum_{i_1 i_2 \dots i_N} g_{1 i_1} g_{2 i_2} \dots g_{N i_N} \epsilon^{i_1 i_2 \dots i_N} \quad (2)$$

Comparing (1) and (2) shows that

$$\begin{aligned} \epsilon_{123 \dots N} &= g_{1 i_1} g_{2 i_2} \dots g_{N i_N} \epsilon^{i_1 i_2 \dots i_N} \\ &= k \epsilon^{i_1 i_2 \dots i_N} \end{aligned}$$

Where  $k$  is constant, which in the case of  $\epsilon_{123 \dots N}$ , this constant is  $g$ . Now need to show that the constant is  $g$  for all cases of indices in  $\epsilon_{i_1 i_2 \dots i_N}$  and not for the case  $\epsilon_{123 \dots N}$ .

Looking at the case of  $N = 2$ , and let us see what happens if we change the order of the indices.

$$\epsilon_{i_1 i_2} = g_{i_1 j_1} g_{i_2 j_2} \epsilon^{j_1 j_2}$$

And

$$\epsilon_{i_2 i_1} = g_{i_2 j_2} g_{i_1 j_1} \epsilon^{j_2 j_1}$$

But  $g_{i_1 j_1} g_{i_2 j_2}$  is the same as  $g_{i_2 j_2} g_{i_1 j_1}$ . So the ordering of indices does not change the constant  $k$ . And since we found that this constant is  $g$  from above, therefore we conclude that

$$\epsilon_{i_1 i_2 \dots i_N} = g \epsilon^{j_1 j_2 \dots j_N} \quad (3)$$

### 3 Problem 3

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Problem Compute all components of the affine connection in polar coordinates.

Solution

In polar coordinates  $x^1 = r, x^2 = \theta$ , the relation to the Cartesian coordinates is

$$\begin{aligned}x &= r \cos \theta \\y &= r \sin \theta\end{aligned}$$

Using

$$\Gamma_{jk}^i = \frac{1}{2} g^{li} \left( \frac{\partial g_{kl}}{\partial x^i} + \frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l} \right) \quad (1)$$

We know that in polar coordinates the metric tensor is  $g_{11} = g_{rr} = 1$ , and  $g_{12} = g_{r\theta} = 0$ , and  $g_{21} = g_{\theta r} = 0$ , and  $g_{22} = g_{\theta\theta} = r^2$  or in matrix form

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

Hence  $g^{ij}$  is its inverse

$$g^{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{r^2} \end{pmatrix}$$

Using (1), let  $i = r, j = r, k = r$  then

$$\Gamma_{rr}^r = \frac{1}{2} g^{lr} \left( \frac{\partial g_{rl}}{\partial r} + \frac{\partial g_{rl}}{\partial r} - \frac{\partial g_{rr}}{\partial x^l} \right)$$

The sum is now over  $l$ , which goes from  $r, \theta$  since these are the only coordinates. Hence the above becomes

$$\begin{aligned}\Gamma_{rr}^r &= \frac{1}{2} g^{rr} \left( \frac{\partial g_{rr}}{\partial r} + \frac{\partial g_{rr}}{\partial r} - \frac{\partial g_{rr}}{\partial r} \right) + \frac{1}{2} g^{\theta r} \left( \frac{\partial g_{rr}}{\partial r} + \frac{\partial g_{rr}}{\partial r} - \frac{\partial g_{rr}}{\partial \theta} \right) \\ &= \frac{1}{2} (1) (0 + 0 - 0) + \frac{1}{2} (0) \left( \frac{\partial g_{rr}}{\partial r} + \frac{\partial g_{rr}}{\partial r} - \frac{\partial g_{rr}}{\partial \theta} \right) \\ &= 0\end{aligned} \quad (2)$$

Using (1), let  $i = r, j = \theta, k = r$  then

$$\Gamma_{\theta r}^r = \frac{1}{2} g^{lr} \left( \frac{\partial g_{rl}}{\partial r} + \frac{\partial g_{\theta l}}{\partial r} - \frac{\partial g_{\theta r}}{\partial x^l} \right)$$

The sum is now over  $l$ , which goes from  $r, \theta$  since these are the only coordinates. Hence

the above becomes

$$\begin{aligned}
\Gamma_{\theta r}^r &= \frac{1}{2}g^{rr} \left( \frac{\partial g_{rr}}{\partial r} + \frac{\partial g_{\theta r}}{\partial r} - \frac{\partial g_{\theta r}}{\partial r} \right) + \frac{1}{2}g^{\theta r} \left( \frac{\partial g_{r\theta}}{\partial r} + \frac{\partial g_{\theta\theta}}{\partial r} - \frac{\partial g_{\theta r}}{\partial \theta} \right) \\
&= \frac{1}{2}(1)(0+0-0) + \frac{1}{2}(0) \left( \frac{\partial g_{r\theta}}{\partial r} + \frac{\partial g_{\theta\theta}}{\partial r} - \frac{\partial g_{\theta r}}{\partial \theta} \right) \\
&= 0
\end{aligned} \tag{3}$$

Using (1), now let  $i = r, j = \theta, k = \theta$  then

$$\Gamma_{\theta\theta}^r = \frac{1}{2}g^{lr} \left( \frac{\partial g_{\theta l}}{\partial \theta} + \frac{\partial g_{\theta l}}{\partial \theta} - \frac{\partial g_{\theta\theta}}{\partial x^l} \right)$$

The sum is now over  $l$ , which goes from  $r, \theta$  since these are the only coordinates. Hence the above becomes

$$\begin{aligned}
\Gamma_{\theta\theta}^r &= \frac{1}{2}g^{rr} \left( \frac{\partial g_{\theta r}}{\partial \theta} + \frac{\partial g_{\theta r}}{\partial \theta} - \frac{\partial g_{\theta\theta}}{\partial r} \right) + \frac{1}{2}g^{\theta r} \left( \frac{\partial g_{\theta\theta}}{\partial \theta} + \frac{\partial g_{\theta\theta}}{\partial \theta} - \frac{\partial g_{\theta\theta}}{\partial \theta} \right) \\
&= \frac{1}{2}(1) \left( (0) + (0) - \frac{\partial r^2}{\partial r} \right) + \frac{1}{2}(0) \left( \frac{\partial g_{r\theta}}{\partial r} + \frac{\partial g_{\theta\theta}}{\partial r} - \frac{\partial g_{\theta r}}{\partial \theta} \right) \\
&= \frac{1}{2}(-2r) \\
&= -r
\end{aligned} \tag{4}$$

Using (1), now let  $i = r, j = r, k = \theta$ . Hence we need to find  $\Gamma_{r\theta}^r$ . But due to symmetry in lower indices, then  $\Gamma_{r\theta}^r = \Gamma_{\theta r}^r$  which we found in (3) to be zero. Hence

$$\Gamma_{r\theta}^r = 0 \tag{5}$$

Using (1), now let  $i = \theta, j = r, k = r$  then

$$\Gamma_{rr}^\theta = \frac{1}{2}g^{l\theta} \left( \frac{\partial g_{rl}}{\partial \theta} + \frac{\partial g_{rl}}{\partial r} - \frac{\partial g_{rr}}{\partial x^l} \right)$$

The sum is now over  $l$ , which goes from  $r, \theta$  since these are the only coordinates. Hence the above becomes

$$\begin{aligned}
\Gamma_{rr}^\theta &= \frac{1}{2}g^{r\theta} \left( \frac{\partial g_{rr}}{\partial \theta} + \frac{\partial g_{rr}}{\partial r} - \frac{\partial g_{rr}}{\partial r} \right) + \frac{1}{2}g^{\theta\theta} \left( \frac{\partial g_{r\theta}}{\partial \theta} + \frac{\partial g_{r\theta}}{\partial r} - \frac{\partial g_{rr}}{\partial \theta} \right) \\
&= \frac{1}{2}(0) \left( \frac{\partial g_{rr}}{\partial \theta} + \frac{\partial g_{rr}}{\partial r} - \frac{\partial g_{rr}}{\partial r} \right) + \frac{1}{2} \left( \frac{1}{r^2} \right) (0+0-0) \\
&= 0
\end{aligned} \tag{6}$$

Using (1), now let  $i = \theta, j = \theta, k = r$  then

$$\Gamma_{\theta r}^\theta = \frac{1}{2}g^{l\theta} \left( \frac{\partial g_{rl}}{\partial \theta} + \frac{\partial g_{\theta l}}{\partial r} - \frac{\partial g_{\theta r}}{\partial x^l} \right)$$

The sum is now over  $l$ , which goes from  $r, \theta$  since these are the only coordinates. Hence



the above becomes

$$\begin{aligned}
\Gamma_{\theta r}^{\theta} &= \frac{1}{2} g^{r\theta} \left( \frac{\partial g_{rr}}{\partial \theta} + \frac{\partial g_{\theta r}}{\partial r} - \frac{\partial g_{\theta r}}{\partial r} \right) + \frac{1}{2} g^{\theta\theta} \left( \frac{\partial g_{r\theta}}{\partial \theta} + \frac{\partial g_{\theta\theta}}{\partial r} - \frac{\partial g_{\theta r}}{\partial \theta} \right) \\
&= \frac{1}{2} (0) \left( \frac{\partial g_{rr}}{\partial \theta} + \frac{\partial g_{\theta r}}{\partial r} - \frac{\partial g_{\theta r}}{\partial r} \right) + \frac{1}{2} \frac{1}{r^2} \left( 0 + \frac{\partial r^2}{\partial r} - 0 \right) \\
&= \frac{1}{2} \frac{1}{r^2} (2r) \\
&= \frac{1}{r}
\end{aligned} \tag{7}$$

Using (1), now let  $i = \theta, j = r, k = \theta$  which finds  $\Gamma_{r\theta}^{\theta}$  but due to symmetry this is the same as  $\Gamma_{\theta r}^{\theta}$  which is found above. Hence

$$\Gamma_{r\theta}^{\theta} = \frac{1}{r} \tag{8}$$

Using (1), now let  $i = \theta, j = \theta, k = \theta$  then

$$\Gamma_{\theta\theta}^{\theta} = \frac{1}{2} g^{l\theta} \left( \frac{\partial g_{\theta l}}{\partial \theta} + \frac{\partial g_{\theta l}}{\partial \theta} - \frac{\partial g_{\theta\theta}}{\partial x^l} \right)$$

The sum is now over  $l$ , which goes from  $r, \theta$  since these are the only coordinates. Hence the above becomes

$$\begin{aligned}
\Gamma_{\theta\theta}^{\theta} &= \frac{1}{2} g^{r\theta} \left( \frac{\partial g_{\theta r}}{\partial \theta} + \frac{\partial g_{\theta r}}{\partial \theta} - \frac{\partial g_{\theta\theta}}{\partial r} \right) + \frac{1}{2} g^{\theta\theta} \left( \frac{\partial g_{\theta\theta}}{\partial \theta} + \frac{\partial g_{\theta\theta}}{\partial \theta} - \frac{\partial g_{\theta\theta}}{\partial \theta} \right) \\
&= \frac{1}{2} (0) \left( \frac{\partial g_{\theta r}}{\partial \theta} + \frac{\partial g_{\theta r}}{\partial \theta} - \frac{\partial g_{\theta\theta}}{\partial r} \right) + \frac{1}{2} \frac{1}{r^2} (0 + 0 - 0) \\
&= 0
\end{aligned} \tag{9}$$

This completes the computation. In summary

$$\begin{aligned}
\Gamma_{rr}^r &= 0 \\
\Gamma_{\theta r}^r &= 0 \\
\Gamma_{\theta\theta}^r &= -r \\
\Gamma_{r\theta}^r &= 0 \\
\Gamma_{rr}^{\theta} &= 0 \\
\Gamma_{\theta r}^{\theta} &= \frac{1}{r} \\
\Gamma_{r\theta}^{\theta} &= \frac{1}{r} \\
\Gamma_{\theta\theta}^{\theta} &= 0
\end{aligned}$$

## 4 Problem 4

Problem Calculate the gradient curl and divergence and Laplacian in spherical coordinates using tensor analysis.

Solution

The following coordinates system convention is used

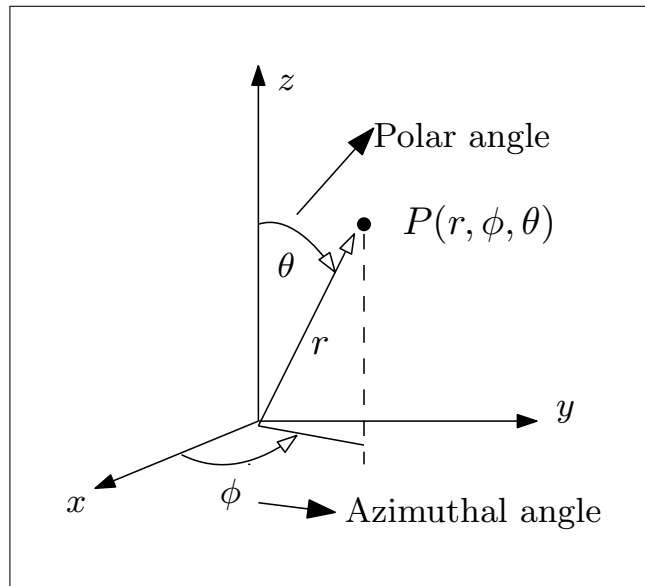


Figure 1: Spherical Coordinates system

### 4.1 Finding metric tensor $g_{ij}$

The coordinates in the Cartesian system are  $\zeta^1 = x, \zeta^2 = y, \zeta^3 = z$ . And the coordinates in the Spherical system are  $x^1 = \phi, x^2 = r, x^3 = \theta$ . The relation between these is known as (Note that the following depends on convention used for which is  $\theta$  and which is  $\phi$ . Physics convention as shown in the diagram above is used here).

$$\zeta^1 = r \sin \theta \cos \phi$$

$$\zeta^2 = r \sin \theta \sin \phi$$

$$\zeta^3 = r \cos \theta$$

The first step is to determine the metric tensor  $g$  for the Spherical coordinates. This is given by

$$g_{kl} = \delta_{ij} \frac{\partial \zeta^i}{\partial x^k} \frac{\partial \zeta^j}{\partial x^l}$$

Since the coordinate system are orthogonal,  $g_{kl}$  will be diagonal. Hence only  $g_{11}, g_{22}, g_{33}$  are non zero.

$$\begin{aligned}
g_{11} &= g_{\phi\phi} \\
&= \frac{\partial \zeta^1}{\partial x^1} \frac{\partial \zeta^1}{\partial x^1} + \frac{\partial \zeta^2}{\partial x^1} \frac{\partial \zeta^2}{\partial x^1} + \frac{\partial \zeta^3}{\partial x^1} \frac{\partial \zeta^3}{\partial x^1} \\
&= \frac{\partial \zeta^1}{\partial \phi} \frac{\partial \zeta^1}{\partial \phi} + \frac{\partial \zeta^2}{\partial \phi} \frac{\partial \zeta^2}{\partial \phi} + \frac{\partial \zeta^3}{\partial \phi} \frac{\partial \zeta^3}{\partial \phi} \\
&= \left( \frac{\partial \zeta^1}{\partial \phi} \right)^2 + \left( \frac{\partial \zeta^2}{\partial \phi} \right)^2 + \left( \frac{\partial \zeta^3}{\partial \phi} \right)^2 \\
&= (-r \sin \theta \sin \phi)^2 + (r \sin \theta \cos \phi)^2 + (0)^2 \\
&= r^2 \sin^2 \theta \sin^2 \phi + r^2 \sin^2 \theta \cos^2 \phi \\
&= r^2 \sin^2 \theta (\sin^2 \phi + \cos^2 \phi) \\
&= r^2 \sin^2 \theta
\end{aligned}$$

And

$$\begin{aligned}
g_{22} &= g_{rr} \\
&= \frac{\partial \zeta^1}{\partial x^2} \frac{\partial \zeta^1}{\partial x^2} + \frac{\partial \zeta^2}{\partial x^2} \frac{\partial \zeta^2}{\partial x^2} + \frac{\partial \zeta^3}{\partial x^2} \frac{\partial \zeta^3}{\partial x^2} \\
&= \frac{\partial \zeta^1}{\partial r} \frac{\partial \zeta^1}{\partial r} + \frac{\partial \zeta^2}{\partial r} \frac{\partial \zeta^2}{\partial r} + \frac{\partial \zeta^3}{\partial r} \frac{\partial \zeta^3}{\partial r} \\
&= \left( \frac{\partial \zeta^1}{\partial r} \right)^2 + \left( \frac{\partial \zeta^2}{\partial r} \right)^2 + \left( \frac{\partial \zeta^3}{\partial r} \right)^2 \\
&= (\sin \theta \cos \phi)^2 + (\sin \theta \sin \phi)^2 + (\cos \theta)^2 \\
&= \sin^2 \theta \cos^2 \phi + \sin^2 \theta \sin^2 \phi + \cos^2 \theta \\
&= \sin^2 \theta (\cos^2 \phi + \sin^2 \phi) + \cos^2 \theta \\
&= \sin^2 \theta + \cos^2 \theta \\
&= 1
\end{aligned}$$

And

$$\begin{aligned}
g_{33} &= g_{\theta\theta} \\
&= \frac{\partial \zeta^1}{\partial x^3} \frac{\partial \zeta^1}{\partial x^3} + \frac{\partial \zeta^2}{\partial x^3} \frac{\partial \zeta^2}{\partial x^3} + \frac{\partial \zeta^3}{\partial x^3} \frac{\partial \zeta^3}{\partial x^3} \\
&= \frac{\partial \zeta^1}{\partial \theta} \frac{\partial \zeta^1}{\partial \theta} + \frac{\partial \zeta^2}{\partial \theta} \frac{\partial \zeta^2}{\partial \theta} + \frac{\partial \zeta^3}{\partial \theta} \frac{\partial \zeta^3}{\partial \theta} \\
&= \left( \frac{\partial \zeta^1}{\partial \theta} \right)^2 + \left( \frac{\partial \zeta^2}{\partial \theta} \right)^2 + \left( \frac{\partial \zeta^3}{\partial \theta} \right)^2 \\
&= (r \cos \theta \cos \phi)^2 + (r \cos \theta \sin \phi)^2 + (-r \sin \theta)^2 \\
&= r^2 \cos^2 \theta (\cos^2 \phi + \sin^2 \phi) + r^2 \sin^2 \theta \\
&= r^2 \cos^2 \theta + r^2 \sin^2 \theta \\
&= r^2
\end{aligned}$$

Hence  $ds^2$  in Spherical coordinates is

$$\begin{aligned}
ds^2 &= g_{kl} dx^k dx^l \\
&= g_{11} (dx^1)^2 + g_{22} (dx^2)^2 + g_{33} (dx^3)^2 \\
&= g_{11} (d\phi)^2 + g_{22} (dr)^2 + g_{33} (d\theta)^2 \\
&= r^2 \sin^2 \theta (d\phi)^2 + (dr)^2 + r^2 (d\theta)^2
\end{aligned}$$

From the above we see that, using the order  $\phi, r, \theta$  for the rows and columns

$$\begin{aligned}
g_{ij} &= \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} \\
&= \begin{pmatrix} r^2 \sin^2 \theta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & r^2 \end{pmatrix}
\end{aligned}$$

Therefore the determinant is  $g = r^4 \sin^2 \theta$  and  $h_i$  are given by the square root of the diagonal elements of  $g_{ij}$

$$\begin{aligned}
h_1 &= r \sin \theta \\
h_2 &= 1 \\
h_3 &= r
\end{aligned} \tag{A}$$

## 4.2 Finding Gradient

$$\nabla = \left( \frac{1}{h_1} \frac{\partial}{\partial x^1}, \frac{1}{h_2} \frac{\partial}{\partial x^2}, \frac{1}{h_3} \frac{\partial}{\partial x^3} \right)$$

Where  $h_i$  are given in (A) and  $x^1 = \phi, x^2 = r, x^3 = \theta$ . Therefore

$$\nabla = \left( \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi}, \frac{\partial}{\partial r}, \frac{1}{r} \frac{\partial}{\partial \theta} \right)$$

Hence given a function scalar  $f(\phi, r, \theta)$  then

$$\nabla f = \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \hat{e}_\phi + \frac{\partial f}{\partial r} \hat{e}_r + \frac{1}{r} \frac{\partial f}{\partial \theta} \hat{e}_\theta$$

### 4.3 Finding Curl

Using  $h_i$  in (A) and  $x^1 = \phi, x^2 = r, x^3 = \theta$  then

$$(\vec{\nabla} \times \vec{V})_1 = \frac{1}{h_2 h_3} \left( \frac{\partial}{\partial x^2} (h_3 V_3) - \frac{\partial}{\partial x^3} (h_2 V_2) \right)$$

$$(\vec{\nabla} \times \vec{v})_\phi = \frac{1}{r} \left( \frac{\partial (r V_\theta)}{\partial r} - \frac{\partial V_r}{\partial \theta} \right)$$

And

$$(\vec{\nabla} \times \vec{V})_2 = \frac{1}{h_3 h_1} \left( \frac{\partial}{\partial x^3} (h_1 V_1) - \frac{\partial}{\partial x^1} (h_3 V_3) \right)$$

$$\begin{aligned} (\vec{\nabla} \times \vec{V})_r &= \frac{1}{r^2 \sin \theta} \left( \frac{\partial}{\partial \theta} (r \sin \theta V_\phi) - \frac{\partial}{\partial \phi} (r V_\theta) \right) \\ &= \frac{1}{r \sin \theta} \left( \frac{\partial (\sin \theta V_\phi)}{\partial \theta} - \frac{\partial V_\theta}{\partial \phi} \right) \end{aligned}$$

And

$$(\vec{\nabla} \times \vec{V})_3 = \frac{1}{h_1 h_2} \left( \frac{\partial}{\partial x^1} (h_2 V_2) - \frac{\partial}{\partial x^2} (h_1 V_1) \right)$$

$$\begin{aligned} (\vec{\nabla} \times \vec{V})_\theta &= \frac{1}{r \sin \theta} \left( \frac{\partial}{\partial \phi} (V_r) - \frac{\partial}{\partial r} (r \sin \theta V_\phi) \right) \\ &= \frac{1}{r} \left( \frac{1}{\sin \theta} \frac{\partial V_r}{\partial \phi} - \frac{\partial (r V_\phi)}{\partial r} \right) \end{aligned}$$

Therefore given a vector  $\vec{V}$ , its curl is

$$\vec{\nabla} \times \vec{V} = \frac{1}{r} \left( \frac{\partial (r V_\theta)}{\partial r} - \frac{\partial V_r}{\partial \theta} \right) \hat{e}_\phi + \frac{1}{r \sin \theta} \left( \frac{\partial (\sin \theta V_\phi)}{\partial \theta} - \frac{\partial V_\theta}{\partial \phi} \right) \hat{e}_r + \frac{1}{r} \left( \frac{1}{\sin \theta} \frac{\partial V_r}{\partial \phi} - \frac{\partial (r V_\phi)}{\partial r} \right) \hat{e}_\theta$$

### 4.4 Finding Divergence

$$\nabla \cdot V = \nabla_i V^i = \frac{\partial}{\partial x^i} V^i + \Gamma_{ij}^i V^j \quad (1)$$

Where  $\Gamma_{ij}^i = \frac{1}{2}g^{li} \left( \frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right) = \frac{1}{2}g^{li} \left( \frac{\partial g_{il}}{\partial x^j} \right)$  which simplifies to as shown in class notes page 143 to hence above becomes

$$\Gamma_{ij}^i = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j} (\sqrt{g})$$

Hence (1) becomes

$$\begin{aligned} \nabla \cdot V &= \frac{\partial}{\partial x^i} V^i + \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j} (\sqrt{g}) V^j \\ &= \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} (\sqrt{g} V^i) \end{aligned}$$

Using the covariant form the above becomes

$$\nabla \cdot V = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left( \frac{\sqrt{g}}{\sqrt{g_{ii}}} V_i \right)$$

Where in class notes  $h_i$  is used in place of  $\sqrt{g_{ii}}$ , but it is the same.

The sum is over  $i$ . From above, the spherical coordinates are  $x^1 = \phi, x^2 = r, x^3 = \theta$ . And  $g = r^4 \sin^2 \theta$ . Hence the above becomes after expanding

$$\begin{aligned} \nabla \cdot V &= \frac{1}{\sqrt{r^4 \sin^2 \theta}} \left( \frac{\partial}{\partial \phi} \left( \frac{\sqrt{r^4 \sin^2 \theta}}{\sqrt{g_{\phi\phi}}} V_\phi \right) + \frac{\partial}{\partial r} \left( \frac{\sqrt{r^4 \sin^2 \theta}}{\sqrt{g_{rr}}} V_r \right) + \frac{\partial}{\partial \theta} \left( \frac{\sqrt{r^4 \sin^2 \theta}}{\sqrt{g_{\theta\theta}}} V_\theta \right) \right) \\ &= \frac{1}{r^2 \sin \theta} \left( \frac{\partial}{\partial \phi} \left( \frac{r^2 \sin \theta}{r \sin \theta} V_\phi \right) + \frac{\partial}{\partial r} \left( \frac{r^2 \sin \theta}{1} V_r \right) + \frac{\partial}{\partial \theta} \left( \frac{r^2 \sin \theta}{r} V_\theta \right) \right) \\ &= \frac{1}{r^2 \sin \theta} \left( \frac{\partial}{\partial \phi} (r V_\phi) + \frac{\partial}{\partial r} (r^2 \sin \theta V_r) + \frac{\partial}{\partial \theta} (\sin \theta V_\theta) \right) \\ &= \frac{\partial}{\partial \phi} \left( \frac{1}{r \sin \theta} V_\phi \right) + \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 V_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta V_\theta) \end{aligned}$$

## 4.5 Finding Laplacian

The Laplacian is given by

$$\nabla^2 = \frac{1}{\sqrt{\det(g)}} \frac{\partial}{\partial x^i} \left( \frac{\sqrt{\det(g)}}{g_{ii}} \frac{\partial}{\partial x^i} \right)$$

Hence

$$\begin{aligned}
\nabla^2 &= \frac{1}{\sqrt{r^4 \sin^2 \theta}} \frac{\partial}{\partial x_1} \left( \frac{\sqrt{r^4 \sin^2 \theta}}{g_{11}} \frac{\partial}{\partial x_1} \right) + \frac{1}{\sqrt{r^4 \sin^2 \theta}} \frac{\partial}{\partial x_2} \left( \frac{\sqrt{r^4 \sin^2 \theta}}{g_{22}} \frac{\partial}{\partial x_2} \right) + \frac{1}{\sqrt{r^4 \sin^2 \theta}} \frac{\partial}{\partial x_3} \left( \frac{\sqrt{r^4 \sin^2 \theta}}{g_{33}} \frac{\partial}{\partial x_3} \right) \\
&= \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \phi} \left( \frac{r^2 \sin \theta}{r^2 \sin^2 \theta} \frac{\partial}{\partial \phi} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial r} \left( \frac{r^2 \sin \theta}{1} \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \frac{r^2 \sin \theta}{r^2} \frac{\partial}{\partial \theta} \right) \\
&= \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \phi} \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial r} \left( r^2 \sin \theta \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) \\
&= \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} + \frac{1}{r^2} \left( 2r \frac{\partial}{\partial r} + r^2 \frac{\partial^2}{\partial r^2} \right) + \frac{1}{r^2 \sin \theta} \left( \cos \theta \frac{\partial}{\partial \theta} + \sin \theta \frac{\partial^2}{\partial \theta^2} \right) \\
&= \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{\cos \theta}{r^2 \sin \theta} \frac{\partial}{\partial \theta} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \\
&= \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \left( \frac{\cos \theta}{\sin \theta} \frac{\partial}{\partial \theta} + \frac{\partial^2}{\partial \theta^2} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2}
\end{aligned}$$

Therefore

$$\begin{aligned}
\nabla^2 u &= \frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \left( \frac{\cos \theta}{\sin \theta} \frac{\partial u}{\partial \theta} + \frac{\partial^2 u}{\partial \theta^2} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2} \\
&= u_{rr} + \frac{2}{r} u_r + \frac{1}{r^2} \left( \frac{\cos \theta}{\sin \theta} u_\theta + u_{\theta\theta} \right) + \frac{1}{r^2 \sin^2 \theta} u_{\phi\phi}
\end{aligned}$$