(1) $$R = N\int_0^\infty dE\, E\, e^{-\beta E}\, e^{-\alpha E^{-1/2}} = N\int_0^\infty dE\, e^{f(E)}$$

$$f(E) = -\beta E - \alpha E^{-1/2} + \ln E$$

Find the saddle point: $f' = -\beta + \frac{\alpha}{2} E^{-3/2} + \frac{1}{E} = 0$

$1 = \beta E - \frac{1}{2}\frac{\alpha}{\sqrt{E}}$ Solution referred to as E_0.

Define $x = \beta E$ which is dimensionless.

$1 = x - \frac{1}{2}\frac{\alpha\sqrt{\beta}}{\sqrt{x}}$ When $\alpha^2\beta >> 1$ the 2 terms on the right side must balance each other.

$x_0^{3/2} \approx \frac{\alpha\sqrt{\beta}}{2}$ $\quad x_0^3 \approx \frac{\alpha^2\beta}{4}$ $\quad x_0 \approx \left(\frac{\alpha^2\beta}{4}\right)^{1/3} >> 1$

Thus justifies ignoring the 1 on the left side.

$$f(x_0) = -x_0 - \frac{\alpha\sqrt{\beta}}{x_0} + \ln(x_0/\beta)$$

Small compared to other 2 terms

$$f''(E) = -\frac{3}{4}\alpha E^{-5/2} - \frac{1}{E^2}$$

$$f''(x_0) = -\frac{3}{4}\frac{\alpha\beta^{5/2}}{x_0^{5/2}} - \frac{\beta^2}{x_0^2} = -\frac{\beta^2}{x_0^2}\left(\frac{3}{4}\frac{\alpha\sqrt{\beta}}{\sqrt{x_0}} + 1\right) =$$

$$= -\frac{\beta^2}{x_0^2}\left(\underset{\substack{\uparrow \\ \text{This term is much larger}}}{\frac{3}{4}\frac{2x_0^{3/2}}{x_0^{1/2}}} + 1\right) \simeq -\frac{3}{2}\frac{\beta^2}{x_0}$$

Thus $R \simeq N e^{f(x_0)} \underbrace{\int_0^\infty dE\, e^{\frac{1}{2}f''(x_0)(E-E_0)^2}}_{\sqrt{\frac{2\pi}{-f''(x_0)}} = \sqrt{\frac{4\pi}{3}}\frac{\sqrt{x_0}}{\beta}}$

$$= \sqrt{\frac{4\pi}{3\beta}}\, N\left(\frac{x_0}{\beta}\right)^{3/2} e^{-x_0} = \sqrt{\frac{\pi}{3}}\, 2N\,\frac{x_0^{3/2}}{\beta^2}\, e^{-x_0}$$

Now $x_0^{3/2} = \frac{\alpha\sqrt{\beta}}{2}$ so we get

$$\boxed{R \simeq \sqrt{\frac{\pi}{3}}\, N\,(k_B T)^{3/2}\,\alpha\, e^{-(\alpha^2/4k_BT)^{1/3}}}$$

Note that $R \to 0$ rapidly as $T \to 0$.

(2) $$I = \int_a^b f(x)\,\delta(g(x))\,dx$$

$g(x_0) = 0$ and $a < x_0 < b$ Let $y = g(x)$.

$dy = g'(x)\,dx$ Then $I = \int_{y_{min}}^{y_{max}} f(g^{-1}(y))\,\delta(y)\,\dfrac{dy}{|g'(g^{-1}(y))|}$

where $g^{-1}(y) = x$. We use the absolute value of g' in the Jacobian and make the lower limit of the y integration smaller than the upper limit. This takes into account both the possibility that $g'(x) > 0$ and $g'(x) < 0$ in that interval. Then

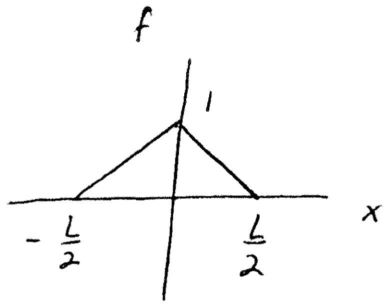
$$I = \frac{f(g^{-1}(0))}{|g'(g^{-1}(0))|} = \frac{f(x_0)}{|g'(x_0)|}$$

$$\boxed{\int_a^b f(x)\,\delta(g(x))\,dx = \frac{f(x_0)}{|g'(x_0)|}}$$

This generalizes straightforwardly if $g(x) = 0$ at multiple points in the interval.

(3)

$$f(x) = \begin{cases} 1 + \dfrac{2x}{L} & -\dfrac{L}{2} \le x \le 0 \\ 1 - \dfrac{2x}{L} & 0 \le x \le \dfrac{L}{2} \end{cases} \qquad \text{periodic}$$



$$f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left(A_n \cos\frac{2\pi n x}{L} + B_n \sin\frac{2\pi n x}{L}\right)$$

$$B_n = \frac{2}{L}\int_{-L/2}^{L/2} f(x) \sin\frac{2\pi n x}{L}\,dx = 0 \quad \text{because } f(x) \text{ is even}$$

$$A_0 = \frac{2}{L}\int_{-L/2}^{L/2} f(x)\,dx = \frac{4}{L}\int_0^{L/2}\left(1 - \frac{2x}{L}\right)dx = \frac{4}{L}\left(\frac{L}{2} - \frac{L}{4}\right) = 1$$

$$A_n = \frac{2}{L}\int_{-L/2}^{L/2} f(x)\cos\frac{2\pi n x}{L}\,dx = \frac{4}{L}\int_0^{L/2}\left(1 - \frac{2x}{L}\right)\cos\frac{2\pi n x}{L}\,dx$$

Use $\displaystyle \frac{d}{dk}\int \sin kx\,dx = \int x \cos kx\,dx \qquad k_n \equiv \frac{2\pi n}{L}$

$$\int_0^{L/2} x \cos k_n x\,dx = \frac{d}{dk_n}\int_0^{L/2} \sin k_n x\,dx = -\frac{d}{dk_n}\left.\frac{\cos k_n x}{k_n}\right|_0^{L/2} =$$

$$= -\frac{d}{dk_n}\left[\frac{\cos\left(\frac{k_n L}{2}\right) - 1}{k_n}\right] = \frac{\left[\cos\left(\frac{k_n L}{2}\right) - 1\right]}{k_n^2} + \frac{L}{2k_n}\sin\left(\frac{k_n L}{2}\right) =$$

$$= \frac{L^2}{(2\pi n)^2}\left[\cos(\pi n) - 1\right] + \frac{L^2}{4\pi n}\sin(\pi n)$$

$$\int_0^{L/2} \cos k_n x \, dx = \frac{\sin k_n x}{k_n}\Bigg|_0^{L/2} = \frac{L}{2\pi n}\sin(\pi n)$$

$$A_n = \frac{4}{L}\left\{\frac{L}{2\pi n}\sin(\pi n) - \frac{L}{2\pi^2 n^2}\left[\cos(\pi n) - 1\right] - \frac{L}{2\pi n}\sin(\pi n)\right\}$$

$$= \frac{2}{\pi^2 n^2}\left[1 - \cos(\pi n)\right] = \frac{2}{\pi^2 n^2}\left[1 - (-1)^n\right]$$

$$= \begin{cases} \frac{4}{\pi^2 n^2} & n \text{ is odd} \\ 0 & n \text{ is even} \end{cases}$$

Let $n = 2m+1, \quad m = 0, 1, 2, \ldots$

$$\boxed{f(x) = \frac{1}{2} + \frac{4}{\pi^2}\sum_{m=0}^{\infty} \frac{1}{(2m+1)^2}\cos\left[\frac{2\pi(2m+1)x}{L}\right]}$$