HW 6
Physics 5041 Mathematical Methods for Physics
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University of Minnesota, Twin Cities

Nasser M. Abbasi

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Contents

1 Problem 1

> **1.** (5 pts) The rate of nuclear reactions in a star is given by the formula
>
> $$R = N \int_0^\infty dE\, E\, \mathrm{e}^{-\beta E}\, \mathrm{e}^{-\alpha E^{-1/2}}$$
>
> where E is energy, $\beta = 1/k_B T$, α is a constant, and N is a normalization. Evaluate this integral using the saddle point approximation when $(\beta\alpha^2)^{1/3} \gg 1$. This is the low temperature limit appropriate for conditions in the star.

Figure 1: Problem statement

Solution

The first step in saddle point method is to write the integral as $\int_0^\infty e^{f(E)} dE$. Hence

$$\begin{aligned} R &= N \int_0^\infty e^{\left(-\beta E - \alpha E^{\frac{-1}{2}} + \ln E\right)} dE \\ &= N \int_0^\infty e^{f(E)} dE \end{aligned} \tag{A}$$

Where

$$f(E) = -\beta E - \alpha E^{\frac{-1}{2}} + \ln E \tag{1}$$

The next step is to determine where $f(E)$ is maximum. Therefore we need to solve $f'(E) = 0$ in order to determine E_0, where $f(E_0)$ is maximum.

$$\begin{aligned} f'(E) &= -\beta + \frac{1}{2}\alpha E^{\frac{-3}{2}} + \frac{1}{E} \\ &= 0 \end{aligned}$$

We need to make this dimensionless. Multiplying both sides of the above by α^2 gives

$$-\alpha^2\beta + \frac{1}{2}\alpha^3 E^{\frac{-3}{2}} + \frac{\alpha^2}{E} = 0$$

Let $E = x\alpha^2$, then the above becomes

$$-\alpha^2\beta + \frac{1}{2}\alpha^3 \left(x\alpha^2\right)^{\frac{-3}{2}} + \frac{\alpha^2}{\left(x\alpha^2\right)} = 0$$

$$-\alpha^2\beta + \frac{1}{2}\frac{1}{x^{\frac{3}{2}}} + \frac{1}{x} = 0 \tag{2}$$

<u>Case 1</u> Ignoring the term $\frac{1}{x^{\frac{3}{2}}}$ in (2) results in

$$\begin{aligned}-\alpha^2\beta + \frac{1}{x} &= 0\\ \frac{1}{x} &= \alpha^2\beta\\ x &= \frac{1}{\alpha^2\beta}\end{aligned}$$

Using this value for x we check if this is larger than or smaller than the term we ignored which is $\frac{1}{x^{\frac{3}{2}}}$.

$$\left[\frac{1}{x^{\frac{3}{2}}}\right]_{x=\frac{1}{\alpha^2\beta}} = \frac{1}{\left(\frac{1}{\alpha^2\beta}\right)^{\frac{3}{2}}} = \frac{1}{\left(\frac{1}{\alpha\beta^2}\right)^3} = \left(\beta^2\alpha\right)^3$$

Since $\left(\alpha^2\beta\right)^{\frac{1}{3}} \gg 1$, then $\alpha^2\beta \gg 1$ and hence $x = \frac{1}{\alpha^2\beta}$ is much smaller than $\left(\beta^2\alpha\right)^3$. So our choice of ignoring $\frac{1}{x^{\frac{3}{2}}}$ was wrong. Hence we need to ignore the term $\frac{1}{x}$ from (2)

<u>Case 2</u> Ignoring the term $\frac{1}{x}$ results in

$$\begin{aligned}-\alpha^2\beta + \frac{1}{2}\frac{1}{x^{\frac{3}{2}}} &= 0\\ \frac{-2x^{\frac{3}{2}}\alpha^2\beta + 1}{2x^{\frac{3}{2}}} &= 0\\ -2x^{\frac{3}{2}}\alpha^2\beta + 1 &= 0\\ x^{\frac{3}{2}} &= \frac{-1}{-2\alpha^2\beta}\end{aligned}$$

Solving gives

$$x = \left(\frac{1}{2\alpha^2\beta}\right)^{\frac{2}{3}}$$

But $E = x\alpha^2$, and from the above we the energy E_0 which makes $f(E)$ maximum as

$$\begin{aligned} E_0 &= \alpha^2 \left(\frac{1}{2\alpha^2\beta}\right)^{\frac{2}{3}} \\ &= \frac{\alpha^{2-\frac{4}{3}}}{2^{\frac{2}{3}}\beta^{\frac{2}{3}}} \\ &= \frac{\alpha^{\frac{2}{3}}}{2^{\frac{2}{3}}\beta^{\frac{2}{3}}} \end{aligned}$$

Hence

$$\boxed{E_0 = \left(\frac{\alpha}{2\beta}\right)^{\frac{2}{3}}}$$

Now that we found which value of E makes $f(E)$ maximum, we can expand $f(E)$ in Taylor series around E_0

$$f(E) = f(E_0) + f'(E_0)(E - E_0) + \frac{f''(E_0)}{2!}(E - E_0)^2 + H.O.T$$

But $f'(E_0) = 0$ then the above becomes, after ignoring H.O.T.

$$f(E) = f(E_0) + \frac{f''(E_0)}{2!}(E - E_0)^2 \tag{3}$$

Since $f'(E) = -\beta + \frac{1}{2}\alpha E^{\frac{-3}{2}} + \frac{1}{E}$ then

$$f''(E_0) = -\frac{3}{4}\alpha E_0^{\frac{-5}{2}} - E_0^{-2}$$

Since $E_0^{\frac{-5}{2}} \gg E_0^{-2}$ the above becomes

$$\begin{aligned} f''(E_0) &= -\frac{3}{4}\alpha E_0^{\frac{-5}{2}} \\ &\simeq -\frac{3}{2}\frac{\beta^2}{E_0} \end{aligned} \tag{4}$$

Equation (A) now becomes

$$\begin{aligned} R &= N\int_0^\infty e^{f(E)} dE \\ &= N\int_0^\infty e^{f(E_0)+\frac{f''(E_0)}{2!}(E-E_0)^2} dE \\ &= Ne^{f(E_0)}\int_0^\infty e^{\frac{f''(E_0)}{2!}(E-E_0)^2} dE \end{aligned}$$

We would like to write the above as $\int_0^\infty e^{-ax^2}dx = \sqrt{\frac{\pi}{a}}$. Therefore, assuming $u = E - E_0$, hence $\frac{du}{dE} = 1$. When $E = 0$ then $u = -E_0$ and when $E = \infty$ then $u = \infty$. Hence the above becomes

$$
\begin{aligned}
R &= Ne^{f(E_0)} \int_{-E_0}^{\infty} e^{\frac{f''(E_0)}{2!}u^2} du \\
&= Ne^{f(E_0)} \int_{-E_0}^{\infty} e^{-\frac{3}{4}\frac{\beta^2}{E_0}u^2} du
\end{aligned}
$$

Since E_0 is positive, then contribution from lower limit $u = -E_0$ to the value of the integral is Negligible. We can then let lower limit go to $-\infty$ without affecting the overall result of the integral. The above becomes

$$
R = Ne^{f(E_0)} \int_{-\infty}^{\infty} e^{-\frac{3}{4}\frac{\beta^2}{E_0}u^2} du
$$

This is now in the form of Gaussian $\int_{-\infty}^\infty e^{-ax^2}dx = \sqrt{\frac{\pi}{a}}$. Hence we can write the above, using $a = \frac{3}{4}\frac{\beta^2}{E_0}$

$$
\begin{aligned}
R &= Ne^{f(E_0)} \sqrt{\frac{\pi}{\frac{3}{4}\frac{\beta^2}{E_0}}} \\
&= Ne^{f(E_0)} \sqrt{\frac{4\pi E_0}{3\beta^2}}
\end{aligned}
$$

But $f(E_0)$ from (1) is $f(E_0) = -\beta E_0 - \alpha E_0^{\frac{-1}{2}} + \ln E_0$, hence the above becomes

$$
\begin{aligned}
R &= NE_0 e^{-\beta E_0 - \alpha E_0^{\frac{-1}{2}}} \sqrt{\frac{4\pi E_0}{3\beta^2}} \\
&= NE_0 e^{-\beta E_0 - \alpha E_0^{\frac{-1}{2}}} \sqrt{\frac{4\pi}{3\alpha E_0^{\frac{-5}{2}}}}
\end{aligned}
$$

But $E_0 = \left(\frac{\alpha}{2\beta}\right)^{2/3}$, therefore the above becomes, after some more simplifications

$$
R = N\left(\frac{\alpha}{2\beta}\right)^{2/3} \exp\left(-\beta\left(\frac{\alpha}{2\beta}\right)^{2/3} - \alpha\left(\frac{\alpha}{2\beta}\right)^{-2/6}\right) \sqrt{\frac{4\pi}{3\alpha\left(\frac{\alpha}{2\beta}\right)^{-10/6}}}
$$

Simplifies to

$$
R = \sqrt{\frac{\pi}{3}} N \left(k_\beta T\right)^{\frac{3}{2}} \alpha e^{-\left(\frac{\alpha^2}{4}k_\beta T\right)^{\frac{1}{3}}}
$$

This was a hard problem. See key solution.

2 Problem 2

2. (5 pts) Assume that $g(x_0) = 0$ for $a < x_0 < b$ and that $g^{-1}(x)$ exists in that range of x. Show that

$$\int_a^b f(x)\delta(g(x))dx = \frac{f(x_0)}{|g'(x_0)|}$$

Figure 2: Problem statement

Solution

Let $u = g(x)$, hence

$$\frac{du}{dx} = g'(x) \tag{1}$$

But

$$\begin{aligned} x &= g^{-1}\left(g(x)\right) \\ &= g^{-1}(u) \end{aligned}$$

Replacing x in (1) by the above results (so everything is in terms of u) gives

$$\frac{du}{dx} = g'\left(g^{-1}(u)\right)$$

Now we take care of the limits of integration. When $x = a$ then $u = g(a)$ and when $x = b$ then $u = g(b)$. Now the integral I becomes in terms of u the following

$$\begin{aligned} I &= \int_{g(a)}^{g(b)} f\left(g^{-1}(u)\right)\delta(u)\frac{du}{g'\left(g^{-1}(u)\right)} \\ &= \int_{g(a)}^{g(b)} \delta(u)\left[\frac{f\left(g^{-1}(u)\right)}{g'\left(g^{-1}(u)\right)}\right]du \end{aligned} \tag{2}$$

Since we do not know the sign of $g'(x_0)$, as it can be positive or negative, so we take its absolute value in the above, so that the limits of integration do not switch. Hence (2) becomes

$$I = \int_{g(a)}^{g(b)} \delta(u)\left[\frac{f\left(g^{-1}(u)\right)}{\left|g'\left(g^{-1}(u)\right)\right|}\right]du \tag{3}$$

We are given that there is one point x_0 between $g(a)$, and $g(b)$ where $g(x_0) = 0$ which is the same as saying $u = 0$ at that point. Hence by applying the standard property of Dirac delta function, which says that $\int_a^b \delta(0)\,\phi(z)\,dz = \phi(0)$ to equation (3) gives

$$I = \frac{f\left(g^{-1}(0)\right)}{\left|g'\left(g^{-1}(0)\right)\right|}$$

But $g^{-1}(0) = x_0$, therefore the above becomes

$$\int_a^b f(x)\,\delta\left(g(x)\right)dx = \frac{f(x_0)}{\left|g'(x_0)\right|}$$

Which is the result required to show.

3 Problem 3

3. (5 pts) Find the Fourier series that represents the periodic function

$$f(x) = 1 + \frac{2x}{L} \quad \text{when} \quad -\frac{L}{2} \leq x \leq 0$$

$$f(x) = 1 - \frac{2x}{L} \quad \text{when} \quad 0 \leq x \leq \frac{L}{2}$$

Figure 3: Problem statement

Solution

A plot of the function to approximate is (using $L = 1$) for illustration

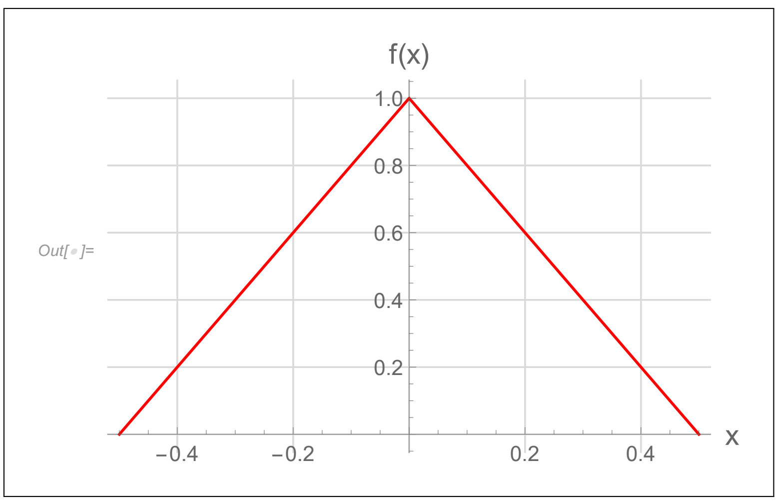

Figure 4: The function $f(x)$ to find its Fourier series

The function period is $T = L$. Hence the Fourier series is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi}{L}nx\right) + b_n \cos\left(\frac{2\pi}{L}nx\right)$$

Since $f(x)$ is an even function, then $b_n = 0$ and the above simplifies to

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi}{L}nx\right)$$

Where

$$a_0 = \frac{2}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)\, dx$$

We can calculate this integral, but it is easier to find a_0 knowing that $\frac{a_0}{2}$ represent the average of the area under the function $f(x)$.

We see right away that the area is $2\left(\frac{1}{2}\frac{L}{2}\right) = \frac{L}{2}$. Hence, solving $\frac{a_0}{2}L = \frac{L}{2}$ for a_0 gives $a_0 = 1$. Now we find a_n

$$a_n = \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)\cos\left(\frac{2\pi}{L}nx\right)dx$$

Since $f(x)$ is even and $\cos\left(\frac{2\pi}{L}nx\right)$ is even, then the above simplifies to

$$\begin{aligned}
a_n &= \frac{4}{L}\int_0^{\frac{L}{2}} f(x)\cos\left(\frac{2\pi}{L}nx\right)dx \\
&= \frac{4}{L}\int_0^{\frac{L}{2}}\left(1-\frac{2x}{L}\right)\cos\left(\frac{2\pi}{L}nx\right)dx \\
&= \frac{4}{L}\left(\int_0^{\frac{L}{2}}\cos\left(\frac{2\pi}{L}nx\right)dx - \frac{2}{L}\int_0^{\frac{L}{2}} x\cos\left(\frac{2\pi}{L}nx\right)dx\right) \qquad (1)
\end{aligned}$$

But

$$\begin{aligned}
\int_0^{\frac{L}{2}}\cos\left(\frac{2\pi}{L}nx\right)dx &= \frac{1}{\frac{2n\pi}{L}}\left[\sin\left(\frac{2\pi}{L}nx\right)\right]_0^{\frac{L}{2}} \\
&= \frac{L}{2n\pi}\left(\sin\left(\frac{2\pi}{L}n\frac{L}{2}\right)\right) \\
&= \frac{L}{2n\pi}\sin(\pi n) \\
&= 0
\end{aligned}$$

And $\int_0^{\frac{L}{2}} x\cos\left(\frac{2\pi}{L}nx\right)dx$ is integrated by parts. Let $u = x, dv = \cos\left(\frac{2\pi}{L}nx\right)$, hence $du = 1$ and

$v=\frac{1}{\frac{2n\pi}{L}}\sin\left(\frac{2\pi}{L}nx\right)$. Therefore

$$\begin{aligned}
\int_0^{\frac{L}{2}} x\cos\left(\frac{2\pi}{L}nx\right)dx &= uv-\int vdu\\
&= \frac{1}{\frac{2n\pi}{L}}\left[x\sin\left(\frac{2\pi}{L}nx\right)\right]_0^{\frac{L}{2}} - \frac{1}{\frac{2n\pi}{L}}\int \sin\left(\frac{2\pi}{L}nx\right)dx\\
&= -\frac{L}{2n\pi}\int \sin\left(\frac{2\pi}{L}nx\right)dx\\
&= \frac{L}{2n\pi}\left[\frac{\cos\left(\frac{2\pi}{L}nx\right)}{\frac{2\pi}{L}n}\right]_0^{\frac{L}{2}}\\
&= \left(\frac{L}{2n\pi}\right)^2\left(\cos\left(\frac{2\pi}{L}n\frac{L}{2}\right)-1\right)\\
&= \left(\frac{L}{2n\pi}\right)^2(\cos(n\pi)-1)\\
&= \left(\frac{L}{2n\pi}\right)^2\left((-1)^n-1\right)
\end{aligned}$$

Substituting these results in (1) gives

$$\begin{aligned}
a_n &= -\frac{4}{L}\left(\frac{2}{L}\left(\frac{L}{2n\pi}\right)^2\left((-1)^n-1\right)\right)\\
&= -\frac{2}{n^2\pi^2}\left((-1)^n-1\right)
\end{aligned}$$

When n is even we see that $a_n=0$ and when n is odd, then $a_n=\frac{4}{n^2\pi^2}$. Therefore

$$\begin{aligned}
f(x) &= \frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\left(\frac{2\pi}{L}nx\right)\\
&= \frac{1}{2}+\sum_{n=1,3,5,\cdots}^{\infty}\left(\frac{4}{n^2\pi^2}\right)\cos\left(\frac{2\pi}{L}nx\right)\\
&= \frac{1}{2}+\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\left(\frac{2\pi}{L}(2n-1)x\right)
\end{aligned}$$

4 Problem 4

4. (10 pts) Consider the Fourier series for the function $f(\theta) = 1$ when $0 < \theta < \pi$ and $f(\theta) = -1$ when $\pi < \theta < 2\pi$. Just to the right of $\theta = 0$ the first n terms in the series exhibit a local maximum of $1+\delta_n$. For large n, $\delta_n \approx 0.2$. Using computer software, make plots of the series for 4 representative values of n of your choosing for $0 < \theta < \pi/2$ for illustration. What is the limit of the overshoot δ_n as $n \to \infty$ to 4 significant figures? Include printouts of the programs you wrote to make the plots and to find the limit. This is called the Gibbs phenomenon.

Figure 5: Problem statement

Solution

A plot of the above function is

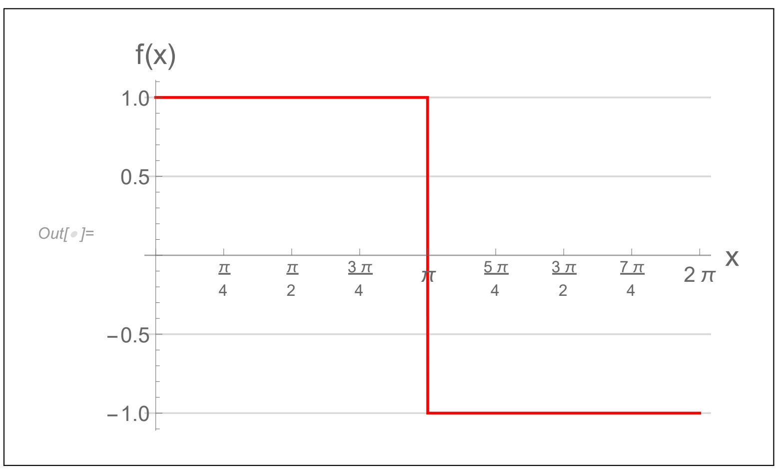

Figure 6: The function $f(x)$ over one period

We first need to find the Fourier series of the function $f(x)$. Since the function is odd, then we only need to determine b_n

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin(nx)$$

Where

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(nx)\, dx$$

Since $f(x)$ is odd, and $\sin$ is odd, then the product is even, and the above simplifies to

$$\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^\pi f(x)\sin(nx)\,dx \\
&= \frac{2}{\pi}\int_0^\pi \sin(nx)\,dx \\
&= \frac{2}{\pi}\left(-\frac{\cos nx}{n}\right)_0^\pi \\
&= \frac{-2}{n\pi}(\cos nx)_0^\pi \\
&= \frac{-2}{n\pi}(\cos n\pi - 1) \\
&= \frac{-2}{n\pi}\left((-1)^n - 1\right) \\
&= \frac{2}{n\pi}\left(1 - (-1)^n\right)
\end{aligned}$$

When n is even, then $b_n = 0$ and when n is odd then $b_n = \frac{4}{n\pi}$, therefore

$$f(x) \sim \frac{4}{\pi}\sum_{n=1,3,5,\cdots}^{\infty}\frac{1}{n}\sin(nx)$$

Which can be written as

$$f(x) \sim \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\sin((2n-1)x) \tag{1}$$

Next, 4 plots were made to see the approximation for $n = 1, 5, 10, 20$.

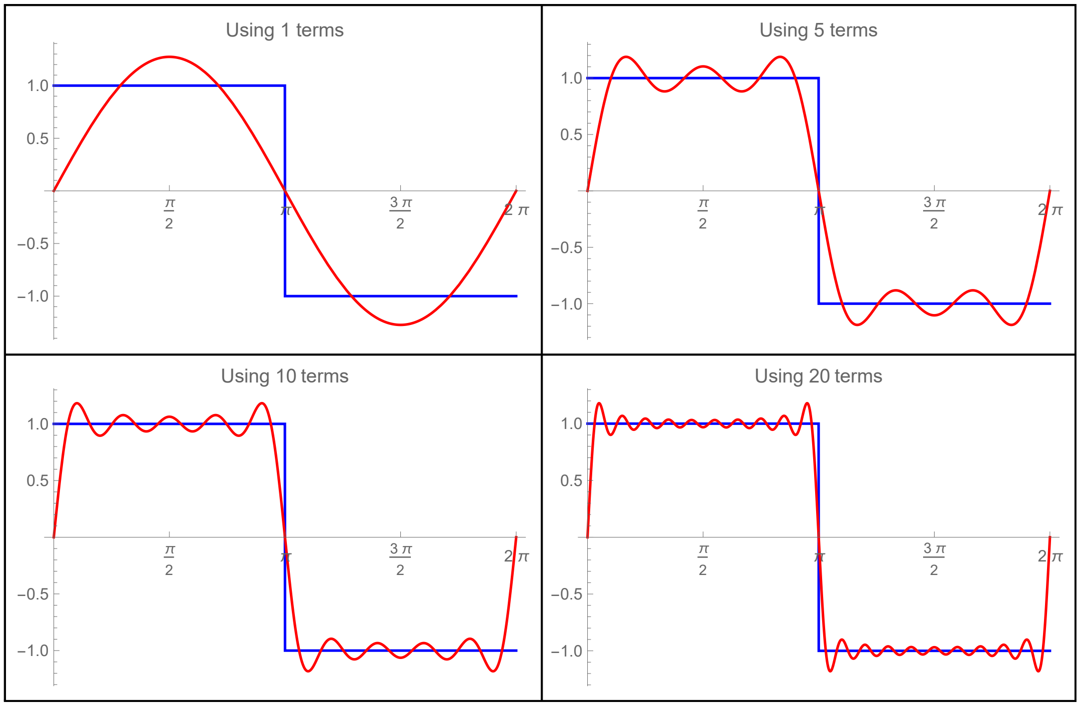

Figure 7: Fourier series approximation for different n values

The source code used is

```
ClearAll[f, x, n];
f[x_ /; 0 ≤ x ≤ 2 Pi] := Piecewise[{{1, 0 ≤ x < Pi}, {-1, Pi ≤ x ≤ 2 Pi}}];
fApprox[x_, nTerms_] := 4/Pi Sum[1/(2 n - 1) Sin[(2 n - 1) x], {n, 1, nTerms}];
Grid[Partition[Table[Plot[{f[x], fApprox[x, n]}, {x, 0, 2 Pi },
    PlotStyle → {Blue, Red}, PlotLabel → Row[{"Using ", n, " terms"}],
    ImageSize → 320, Ticks → {Range[0, 2 Pi, Pi / 2], Automatic}
   ],
   {n, {1, 5, 10, 20}}], 2], Frame → All, Alignment → Center, Spacings → {1,  1}]
```

Figure 8: Source code used to generate the above plot

The partial sum of (1) is

$$f_N(x) = \frac{4}{\pi}\sum_{n=1}^{N}\frac{1}{(2n-1)}\sin((2n-1)x) \tag{2}$$

To determine the overshoot, we need to first find x_0 where the local maximum near $x = 0$ is. This is an illustration, showing the Fourier series approximation to the right of $x = 0$. This plot uses $n = 100$.

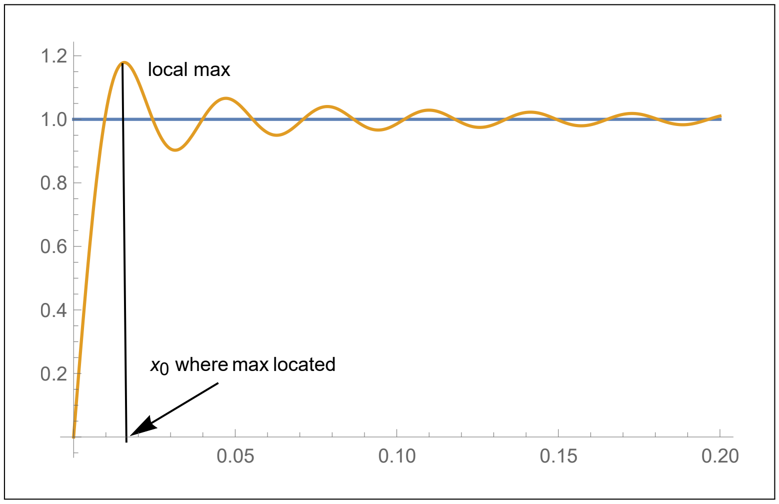

Figure 9: Finding x_0 where maximum overshoot is located

Hence we need to determine $f'(x)$ and then solve for $f'(x) = 0$ in order to find x_0

$$\begin{aligned} f_N'(x) &= \frac{4}{\pi}\sum_{n=1}^{N}\cos((2n-1)x) \\ &= \frac{2}{\pi}\frac{\sin(2Nx)}{\sin x} \end{aligned}$$

Derivation that shows the above is included in the appendix of this problem. Therefore solving $\frac{\sin(2Nx)}{\sin x} = 0$ implies $\sin(2Nx) = 0$ or $2Nx = \pi$ (since we want to be on the right side of $x = 0$, we do not pick 0, but the next zero, this means π is first value). This implies that local maximum to the right of $x = 0$ is located at

$$\boxed{x_0 = \frac{\pi}{2N}}$$

Therefore we need to determine $f_N(x_0)$ to calculate the overshoot due to the Gibbs effect to the right of $x = 0$. From (2) and using x_0 now instead of x gives

$$\begin{aligned} f_N\left(\frac{\pi}{2N}\right) &= \frac{4}{\pi}\sum_{n=1}^{N}\frac{1}{(2n-1)}\sin\left((2n-1)\frac{\pi}{2N}\right) \\ &= \frac{4}{\pi}\left(\frac{\sin\left(\frac{\pi}{2N}\right)}{1} + \frac{\sin\left(3\frac{\pi}{2N}\right)}{3} + \frac{\sin\left(5\frac{\pi}{2N}\right)}{5} + \cdots + \frac{\sin\left((2N-1)\frac{\pi}{2N}\right)}{2N-1}\right) \end{aligned}$$

But $\frac{\sin(\pi z)}{\pi z} = \operatorname{sinc}(z)$, therefore we rewrite the above as

$$
\begin{aligned}
f_N\left(\frac{\pi}{2N}\right) &= 4\left(\frac{\sin\left(\frac{\pi}{2N}\right)}{\pi} + \frac{\sin\left(3\frac{\pi}{2N}\right)}{3\pi} + \frac{\sin\left(5\frac{\pi}{2N}\right)}{5\pi} + \cdots + \frac{\sin\left((2N-1)\frac{\pi}{2N}\right)}{(2N-1)\pi}\right)\\
&= 4\left(\frac{1}{2N}\frac{\sin\left(\pi\frac{1}{2N}\right)}{\pi\frac{1}{2N}} + \frac{1}{2N}\frac{\sin\left(\pi\frac{3}{2N}\right)}{3\pi\frac{1}{2N}} + \frac{1}{2N}\frac{\sin\left(\pi\frac{5}{2N}\right)}{5\pi\frac{1}{2N}} + \cdots + \frac{1}{2N}\frac{\sin\left(\pi\frac{(2N-1)}{2N}\right)}{(2N-1)\pi\frac{1}{2N}}\right)\\
&= 4\left(\frac{1}{2N}\operatorname{sinc}\left(\frac{1}{2N}\right) + \frac{1}{2N}\operatorname{sinc}\left(\frac{3}{2N}\right) + \frac{1}{2N}\operatorname{sinc}\left(\frac{5}{2N}\right) + \cdots + \frac{1}{2N}\operatorname{sinc}\left(\frac{2N-1}{2N}\right)\right)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
f_N\left(\frac{\pi}{2N}\right) &= 2\left(\frac{1}{N}\operatorname{sinc}\left(\frac{1}{2N}\right) + \frac{1}{N}\operatorname{sinc}\left(\frac{3}{2N}\right) + \frac{1}{N}\operatorname{sinc}\left(\frac{5}{2N}\right) + \cdots + \frac{1}{N}\operatorname{sinc}\left(\frac{2N-1}{2N}\right)\right)\\
&= 2\left\{\left[\operatorname{sinc}\left(\frac{1}{2N}\right) + \operatorname{sinc}\left(\frac{3}{2N}\right) + \operatorname{sinc}\left(\frac{5}{2N}\right) + \cdots + \operatorname{sinc}\left(\frac{2N-1}{2N}\right)\right]\frac{1}{N}\right\}
\end{aligned}
$$

Therefore, if we consider a length of 1 and $\frac{1}{N}$ is partition length, then the sum inside {} above is a Riemann sum and the above becomes In the limit, as $N \to \infty$

$$
\lim_{N\to\infty} f_N\left(\frac{\pi}{2N}\right) = 2\int_0^1 \operatorname{sinc}(x)\,dx
$$

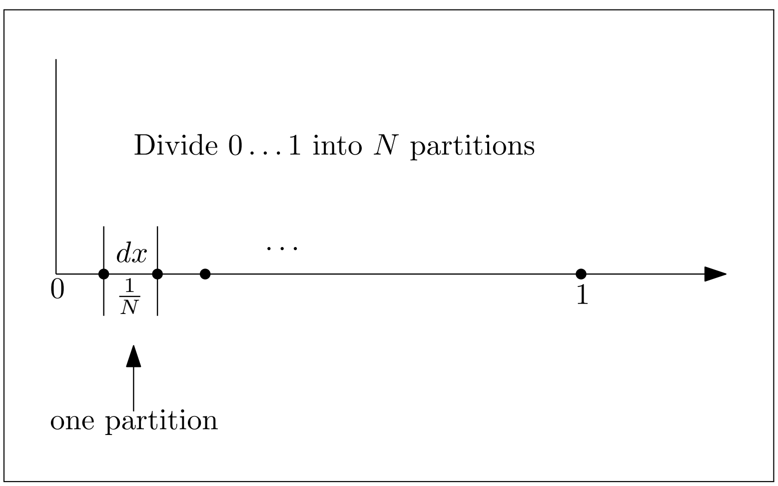

Figure 10: Converting Riemman sum to an integral

Therefore

$$
\lim_{N\to\infty} f_N\left(\frac{\pi}{2N}\right) = 2\int_0^1 \frac{\sin(\pi x)}{\pi x}dx
$$

The $\int_0^1 \frac{\sin(\pi x)}{\pi x}dx$ is known as Si. I could not solve it analytically. It has numerical value of

0.5894898772. Therefore

$$\lim_{N\to\infty} f_N\left(\frac{\pi}{2N}\right) = 2\,(0.5894898772)$$
$$= 1.17897974$$

Since $f(x) = 1$ between 0 and π, then we see that the overshoot is the difference, which is

$$\lim_{N\to\infty} \delta_N = 1.17897974 - 1$$
$$= 0.1789$$

For 4 decimal places. The above result gives good agreement with the plot showing that the overshoot is a little less than 0.2 when viewed on the computer screen. The only use for computation used by the computer for this part of the problem was the evaluation of $\int_0^1 \frac{\sin(\pi x)}{\pi x}dx$. The code is

```
In[ ]:= Integrate[Sin[Pi x] / (Pi x), {x, 0, 1}]
Out[ ]= SinIntegral[π]
        ----------------
               π

In[ ]:= N[%, 16]
Out[ ]= 0.5894898722360836
```

Figure 11: Finding the limit

4.1 Appendix

Here we show the following result used in the above solution.

$$\frac{4}{\pi}\sum_{n=1}^{N}\cos\left((2n-1)x\right) = \frac{2}{\pi}\frac{\sin(2Nx)}{\sin x}$$

Since $\cos z = \operatorname{Re}\left(e^{iz}\right)$, then $\cos\left((2n-1)x\right) = \operatorname{Re}\left(e^{i(2n-1)x}\right)$. Hence the above is the same as

$$\frac{4}{\pi}\sum_{n=1}^{N}\cos\left((2n-1)x\right) = \frac{4}{\pi}\operatorname{Re}\sum_{n=1}^{N}e^{i(2n-1)x} \tag{1}$$

But

$$\sum_{n=1}^{N}e^{i(2n-1)x} = \sum_{n=1}^{N}e^{2ixn-ix}$$
$$= e^{-ix}\sum_{n=1}^{N}e^{2ixn}$$
$$= e^{-ix}\sum_{n=1}^{N}\left(e^{2ix}\right)^{n}$$

Using partial sum property $\sum_{n=1}^{N} r^n = r\frac{1-r^N}{1-r}$, then we can write the above using $r = e^{2ix}$ as

$$\begin{aligned}
\sum_{n=1}^{N} e^{i(2n-1)x} &= e^{-ix}\left(e^{2ix}\frac{1-e^{2iNx}}{1-e^{2ix}}\right)\\
&= e^{ix}\frac{1-e^{2iNx}}{1-e^{2ix}}\\
&= \frac{1-e^{2iNx}}{e^{-ix}-e^{ix}}\\
&= \frac{e^{2iNx}-1}{e^{ix}-e^{-ix}}\\
&= \frac{e^{2iNx}-1}{2i\sin(x)}\\
&= \frac{\cos(2Nx)+i\sin(2Nx)-1}{2i\sin(x)}
\end{aligned}$$

Multiplying numerator and denominator by i gives

$$\begin{aligned}
\sum_{n=1}^{N} e^{i(2n-1)x} &= \frac{i\cos(2Nx)-\sin(2Nx)-i}{-2\sin(x)}\\
&= i\frac{(\cos(2Nx)-1)}{-2\sin x}+\frac{\sin(2Nx)}{2\sin(x)}
\end{aligned}$$

The real part of the above is $\frac{\sin(2Nx)}{2\sin(x)}$, hence (1) becomes

$$\begin{aligned}
\frac{4}{\pi}\sum_{n=1}^{N}\cos((2n-1)x) &= \frac{4}{\pi}\operatorname{Re}\sum_{n=1}^{N} e^{i(2n-1)x}\\
&= \frac{4}{\pi}\left(\frac{\sin(2Nx)}{2\sin(x)}\right)\\
&= \frac{2}{\pi}\frac{\sin(2Nx)}{\sin(x)}
\end{aligned}$$

Which is the result was needed to show.