HW 3
Physics 5041 Mathematical Methods for Physics
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Contents

1 Problem 1

Consider the function $f(z) = z^{\frac{1}{n}}$ where n is a positive integer. The branch point is at $z = 0$ and the branch cut is chosen to be along the positive x axis. How many sheets are there? What is the range of θ corresponding to each sheet?

Solution

Following the example in the class handout, where it showed how to find the number of sheets for $z^{\frac{1}{2}}$, the same method is used here, which is to keep adding a multiple of 2π angles until the same result for the original principal value of the function $g(z)$ evaluated at θ is obtained. This gives the number of sheets.

Let

$$\begin{aligned} g(z) &= z^{\frac{1}{n}} \\ g(r,\theta) &= \left(re^{i\theta}\right)^{\frac{1}{n}} \\ g(r,\theta) &= r^{\frac{1}{n}} e^{i\frac{\theta}{n}} \end{aligned} \tag{1}$$

In the above, θ is called principal argument. And now the idea is to find how many times 2π needs to be added to θ in order to get back the same value of original of $g(r,\theta)$ at the starting θ that one picks. Adding one time 2π to θ, equation (1) becomes

$$\begin{aligned} g(r,\theta+2\pi) &= r^{\frac{1}{n}} e^{i\frac{(\theta+2\pi)}{n}} \\ &= r^{\frac{1}{n}} e^{i\frac{\theta}{n}+i\frac{2\pi}{n}} \\ &= r^{\frac{1}{n}} e^{i\frac{\theta}{n}} e^{i\frac{2\pi}{n}} \end{aligned}$$

And we add another 2π, or now a total of 4π

$$\begin{aligned} g(r,\theta+4\pi) &= r^{\frac{1}{n}} e^{i\frac{(\theta+4\pi)}{n}} \\ &= r^{\frac{1}{n}} e^{i\frac{\theta}{n}+i\frac{4\pi}{n}} \\ &= r^{\frac{1}{n}} e^{i\frac{\theta}{n}} e^{i\frac{4\pi}{n}} \end{aligned}$$

And so on. We keep adding 2π, or a total of $k(2\pi)$ such that the last term above, which in term of k is $e^{\frac{k(2\pi)i}{n}}$ simplifies to 1 which implies getting back original function value at $g(r,\theta)$. Hence for k times we have

$$\begin{aligned} g(r,\theta+k(2\pi)) &= r^{\frac{1}{n}} e^{i\frac{(\theta+k(2\pi))}{n}} \\ &= r^{\frac{1}{n}} e^{i\frac{\theta}{n}+i\frac{k(2\pi)}{n}} \\ &= r^{\frac{1}{n}} e^{i\frac{\theta}{n}} e^{i\frac{k(2\pi)}{n}} \end{aligned}$$

We see from the above, is that only <u>when $k = n$</u>, then $r^{\frac{1}{n}} e^{i\frac{\theta}{n}} e^{i\frac{k(2\pi)}{n}} = r^{\frac{1}{n}} e^{i\frac{\theta}{n}} e^{2\pi i}$. But $e^{2\pi i} = 1$, therefore it reduces to

$$\begin{aligned} g(r,\theta+n(2\pi)) &= r^{\frac{1}{n}}e^{i\frac{\theta}{n}} \\ &= g(r,\theta) \end{aligned}$$

Which is the original value of the function. Therefore there are $\underline{n \text{ sheets}}$.

The formula that can also be used to obtain all values for this multivalued function is

$$g(r,\theta) = r^{\frac{1}{n}}e^{i\left(\frac{\theta}{n}+\frac{2\pi}{n}k\right)} \qquad k = 0,1,\cdots n-1$$

Now to answer the angle θ range question. From the above, we see the range of the angle for each sheet is as follows

$$\begin{aligned} R_1 &: 0 < \theta < 2\pi \\ R_2 &: 2\pi < \theta < 4\pi \\ R_3 &: 4\pi < \theta < 6\pi \\ &\vdots \\ R_n &: (n-1)2\pi < \theta < n(2\pi) \end{aligned}$$

Sheet R_1 is called the principal sheet associated with $k = 0$.

2 Problem 2

Derive the formula

$$\arctan z = \frac{i}{2}\ln\left(\frac{i+z}{i-z}\right)$$

Solution

Let $w = \arctan(z)$ hence

$$z = \tan(w)$$
$$z = \frac{\sin w}{\cos w}$$

But $\sin w = \frac{e^{iw}-e^{-iw}}{2i}$ and $\cos w = \frac{e^{iw}+e^{-iw}}{2}$, hence the above simplifies to

$$z = \frac{\frac{e^{iw}-e^{-iw}}{2i}}{\frac{e^{iw}+e^{-iw}}{2}}$$
$$= \frac{1}{i}\frac{e^{iw}-e^{-iw}}{e^{iw}+e^{-iw}}$$

Or

$$iz = \frac{e^{iw}-e^{-iw}}{e^{iw}+e^{-iw}}$$

Multiplying the numerator and denominator of the right side by e^{iw} gives

$$iz = \frac{e^{2iw}-1}{e^{2iw}+1}$$

Let $e^{iw} = x$ then the above is the same as

$$iz = \frac{x^2-1}{x^2+1}$$
$$iz\left(x^2+1\right) = x^2-1$$
$$x^2 iz + iz = x^2 - 1$$
$$x^2 iz + iz - x^2 + 1 = 0$$
$$x^2(iz-1) + (1+iz) = 0$$
$$x^2 = \frac{-(1+iz)}{(iz-1)}$$
$$= \frac{(1+iz)}{(1-iz)}$$

Simplifying gives

$$\begin{aligned} x^2 &= \frac{i(-i+z)}{i(-i-z)} \\ &= \frac{(z-i)}{(-i-z)} \end{aligned}$$

Hence

$$x = \pm\left(\frac{z-i}{-i-z}\right)^{\frac{1}{2}}$$

But $x = e^{iw}$, and the above becomes

$$e^{iw} = \pm\left(\frac{z-i}{-i-z}\right)^{\frac{1}{2}}$$

We need now to decide which sign to take. Since $z = \tan(w)$, then when $w = 0$, $z = 0$ because $\tan(0) = 0$. Putting $w = 0, z = 0$ in the above gives

$$\begin{aligned} 1 &= \pm\left(\frac{i}{i}\right)^{\frac{1}{2}} \\ &= \pm(1)^{\frac{1}{2}} \\ &= \pm 1 \end{aligned}$$

Hence we need to choose the + sign so both sides is positive. Hence

$$e^{iw} = \left(\frac{z-i}{-i-z}\right)^{\frac{1}{2}}$$

Now, taking the natural log of both sides gives

$$\begin{aligned}
iw &= \ln\left(\frac{z-i}{-i-z}\right)^{\frac{1}{2}} \\
iw &= \frac{1}{2}\ln\left(\frac{z-i}{-i-z}\right) \\
w &= \frac{1}{2i}\ln\left(\frac{z-i}{-i-z}\right) \\
&= \frac{-i}{2}\ln\left(\frac{z-i}{-i-z}\right) \\
&= \frac{i}{2}\ln\left(\left(\frac{z-i}{-i-z}\right)^{-1}\right) \\
&= \frac{i}{2}\ln\left(\frac{-i-z}{z-i}\right) \\
&= \frac{i}{2}\ln\left(\frac{-(z+i)}{-(i-z)}\right) \\
&= \frac{i}{2}\ln\left(\frac{z+i}{i-z}\right)
\end{aligned}$$

But $w = \arctan(z)$, hence the <u>final result</u> is

$$\arctan(z) = \frac{i}{2}\ln\left(\frac{i+z}{i-z}\right)$$

3 Problem 3

Using the formula for $\arctan z$ from the previous problem, find the real functions $u\left(x,y\right)$ and $v\left(x,y\right)$ in the expression $\arctan z = u\left(x,y\right) + iv\left(x,y\right)$

<u>Solution</u>

Let

$$\frac{i}{2}\ln\left(\frac{i+z}{i-z}\right) = u + iv$$

where $u \equiv u\left(x,y\right), v \equiv v\left(x,y\right)$ are the real and imaginary parts of $\arctan(z)$. Therefore

$$\begin{aligned}
\frac{i}{2}\ln\left(\frac{i+z}{i-z}\right) &= \frac{i}{2}\left(\ln\left|\frac{i+z}{i-z}\right| + i\left(\arg\left(\frac{i+z}{i-z}\right) + 2n\pi\right)\right) \qquad n = 0, \pm 1, \pm 2, \cdots \\
&= \frac{i}{2}\ln\left|\frac{i+z}{i-z}\right| - \frac{1}{2}\left(\arg\left(\frac{i+z}{i-z}\right) + 2n\pi\right) \qquad (1)
\end{aligned}$$

Where $\arg\left(\frac{i+z}{i-z}\right)$ is the principal argument. But since $z = x + iy$ then we see that

$$\begin{aligned}
\left|\frac{i+z}{i-z}\right| &= \left|\frac{i+\left(x+iy\right)}{i-\left(x+iy\right)}\right| \\
&= \left|\frac{i+x+iy}{i-x-iy}\right| \\
&= \left|\frac{x+i\left(1+y\right)}{-x+i\left(1-y\right)}\right| \\
&= \frac{\sqrt{x^2+\left(1+y\right)^2}}{\sqrt{x^2+\left(1-y\right)^2}} \\
&= \sqrt{\frac{x^2+\left(1+y\right)^2}{x^2+\left(1-y\right)^2}} \qquad (2)
\end{aligned}$$

And the principal argument is

$$\begin{aligned}
\arg\left(\frac{i+z}{i-z}\right) &= \arg(i+z) - \arg(i-z) \\
&= \arg(i(1-iz)) - \arg(i(1+iz)) \\
&= \arg i + \arg(1-iz) - \arg i + \arg(1+iz) \\
&= \arg(1-iz) + \arg(1+iz)
\end{aligned}$$

Letting $z = x + iy$ in the above results in

$$
\begin{aligned}
\arg\left(\frac{i+z}{i-z}\right) &= \arg\left(1 - i\left(x+iy\right)\right) - \arg\left(1 + i\left(x+iy\right)\right)\\
&= \arg\left(1 - ix + y\right) - \arg\left(1 + ix - y\right)\\
&= \arg\left(\left(1+y\right) - ix\right) - \arg\left(\left(1-y\right) + ix\right)\\
&= \arctan\left(\frac{-x}{1+y}\right) - \arctan\left(\frac{x}{1-y}\right) \qquad (3)
\end{aligned}
$$

Substituting (2,3) into (1) gives

$$
\begin{aligned}
\frac{i}{2}\ln\left(\frac{i+z}{i-z}\right) &= \frac{i}{2}\left(\ln\sqrt{\frac{x^2+\left(1+y\right)^2}{x^2+\left(1-y\right)^2}} + i\left(\arctan\left(\frac{-x}{1+y}\right) - \arctan\left(\frac{x}{1-y}\right) + 2n\pi\right)\right) \qquad n = 0, \pm 1, \pm 2, \cdots\\
&= \frac{i}{4}\ln\left(\frac{x^2+\left(1+y\right)^2}{x^2+\left(1-y\right)^2}\right) - \frac{1}{2}\left(\arctan\left(\frac{-x}{1+y}\right) - \arctan\left(\frac{x}{1-y}\right) + 2n\pi\right)
\end{aligned}
$$

Setting the above equal to $u + iv$ shows that the real part and the imaginary parts are

$$
\begin{aligned}
u &= -\frac{1}{2}\left(\arctan\left(\frac{-x}{1+y}\right) - \arctan\left(\frac{x}{1-y}\right) + 2n\pi\right) \qquad n = 0, \pm 1, \pm 2, \cdots\\
v &= \frac{1}{4}\ln\left(\frac{x^2+\left(y+1\right)^2}{x^2+\left(1-y\right)^2}\right)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\arctan\left(z\right) &= \frac{i}{2}\ln\left(\frac{i+z}{i-z}\right)\\
&= u + iv
\end{aligned}
$$

Where u, v are given above. We see that $\arctan\left(z\right)$ is multivalued as it depends on the value of n.

For illustration of $u\left(x,y\right)$ and $v\left(x,y\right)$, the following is a plot of the above found solution showing the real part $u\left(x,y\right)$ for $n = 0$ (principal sheet)

```
Plot3D[-1/2 (ArcTan[(1 + y), -x] - ArcTan[(1 - y), x]),
 {x, -Pi/2, Pi/2}, {y, -Pi/2, Pi/2},
 AxesLabel → {"x", "y", "u(x,y)"},
 BaseStyle → 14, Ticks → {{-Pi/2, 0, Pi/2}, {-Pi/2, 0, Pi/2},
   Automatic}]
```

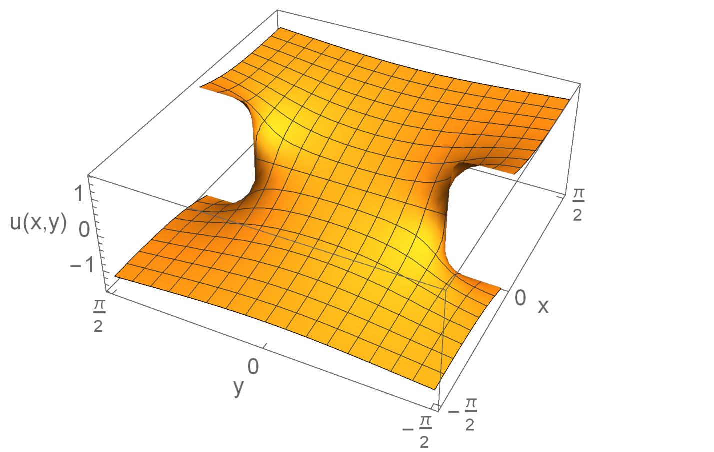

Figure 1: Real part $u(x,y)$ using principal sheet

And the following shows $u\left(x,y\right)$ with both $n=0$ and $n=1$ on the same plot showing two sheets

```
In[•]:= Plot3D[{-1/2 (ArcTan[(1 + y), -x] - ArcTan[(1 - y), x]),
          -1/2 (ArcTan[(1 + y), -x] - ArcTan[(1 - y), x] + 2 Pi)},
         {x, -Pi/2, Pi/2}, {y, -Pi/2, Pi/2},
         AxesLabel → {"x", "y", "u(x,y)"},
         BaseStyle → 14, Ticks → {{-Pi/2, 0, Pi/2}, {-Pi/2, 0, Pi/2},
           Automatic}]
```

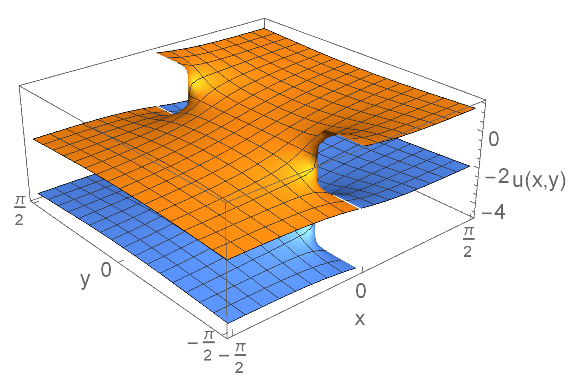

Figure 2: Real part $u(x,y)$ showing $n=0, n=1$ on same plot

And the following plot shows the imaginary part $v\left(x, y\right)$

```
In[ ]:= Plot3D[1 / 4 Log[(x^2 + (y + 1)^2)/(x^2 + (1 - y)^2)], {x, -Pi / 2, Pi / 2}, {y, -1.5, 1.5},
  AxesLabel → {"x", "y", "u(x,y)"}, BaseStyle → 14,
  Ticks → {{-Pi / 2, 0, Pi / 2}, {-Pi / 2, 0, Pi / 2}, Automatic}]
```

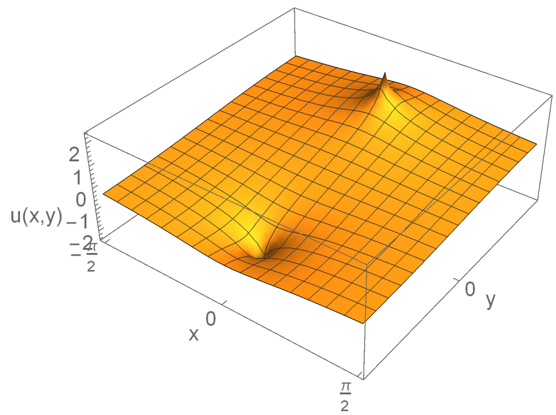

Figure 3: Imaginary part $v(x, y)$

4 Problem 4

In the domain $r > 0, 0 < \theta < 2\pi$. show that the function $u = \ln r$ is harmonic and find its conjugate. Do this in both Cartesian and polar coordinates.

4.1 Part (a) Using Cartesian

A function $u\left(x, y\right)$ is harmonic if it satisfies the Laplace PDE $u_{xx} + u_{yy} = 0$. Since

$$r = \sqrt{x^2 + y^2}$$

Then

$$\begin{aligned} u &= \ln r \\ &= \ln \sqrt{x^2 + y^2} \\ &= \frac{1}{2} \ln \left(x^2 + y^2\right) \end{aligned}$$

We now need to calculate u_{xx} and u_{yy}.

$$\begin{aligned} u_x &= \frac{1}{2} \frac{\partial}{\partial x} \ln \left(x^2 + y^2\right) \\ &= \frac{1}{2} \frac{2x}{x^2 + y^2} \\ &= \frac{x}{x^2 + y^2} \end{aligned}$$

And

$$u_{xx} = \frac{\partial}{\partial x} \frac{x}{x^2 + y^2}$$

Applying the integration rule $\frac{\partial}{\partial x} \frac{f(x)}{g(x)} = \frac{f'g - fg}{g^2}$ to the above, where $f = x$ and $g = x^2 + y^2$ results in

$$\begin{aligned} u_{xx} &= \frac{x^2 + y^2 - x\left(2x\right)}{\left(x^2 + y^2\right)^2} \\ &= \frac{x^2 + y^2 - 2x^2}{\left(x^2 + y^2\right)^2} \\ &= \frac{y^2 - x^2}{\left(x^2 + y^2\right)^2} \qquad (1) \end{aligned}$$

Similarly

$$\begin{aligned}
u_y &= \frac{1}{2}\frac{\partial}{\partial y}\ln\left(x^2+y^2\right)\\
&= \frac{1}{2}\frac{2y}{x^2+y^2}\\
&= \frac{y}{x^2+y^2}
\end{aligned}$$

Applying the integration rule $\frac{\partial}{\partial y}\frac{f(y)}{g(y)} = \frac{f'g-fg}{g^2}$ to the above, where $f = y$ and $g = x^2+y^2$ results in

$$\begin{aligned}
u_{yy} &= \frac{x^2+y^2-y\left(2y\right)}{\left(x^2+y^2\right)^2}\\
&= \frac{x^2+y^2-2y^2}{\left(x^2+y^2\right)^2}\\
&= \frac{x^2-y^2}{\left(x^2+y^2\right)^2} \qquad (2)
\end{aligned}$$

Now that we found u_{xx} and u_{yy}, we need to verify that $u_{xx}+u_{yy}=0$. Adding (1,2) gives

$$\begin{aligned}
u_{xx}+u_{yy} &= \frac{y^2-x^2}{\left(x^2+y^2\right)^2}+\frac{x^2-y^2}{\left(x^2+y^2\right)^2}\\
&= \frac{y^2-x^2+x^2-y^2}{\left(x^2+y^2\right)^2}\\
&= 0
\end{aligned}$$

Hence $u = \ln r$ is <u>harmonic</u>.

To find its conjugate. Let the conjugate be $v\left(x,y\right)$. Let u be the real part of analytic function

$$f = u + iv$$

Applying Cauchy Riemann equations to f results in

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad (3)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad (4)$$

From (3) and using the earlier result found for u_x gives

$$\frac{\partial v}{\partial y} = \frac{x}{x^2+y^2}$$

Integrating the above w.r.t. y gives

$$
\begin{aligned}
v &= \int \frac{x}{x^2+y^2} dy + \Phi(x) \\
&= x \int \frac{1}{x^2+y^2} dy + \Phi(x) \\
&= \frac{1}{x} \int \frac{1}{1+\left(\frac{y}{x}\right)^2} dy + \Phi(x)
\end{aligned}
$$

The above is integrated using substitution. Let $u = \frac{y}{x}$, then $\frac{du}{dy} = \frac{1}{x}$ and the integral becomes

$$
\begin{aligned}
v &= \frac{1}{x}\left(\int \frac{1}{1+u^2}(x du)\right) + \Phi(x) \\
&= \int \frac{1}{1+u^2} du + \Phi(x)
\end{aligned}
$$

But $\int \frac{1}{1+u^2} du = \arctan(u) = \arctan\left(\frac{y}{x}\right)$, therefore the above becomes

$$
v = \arctan\left(\frac{y}{x}\right) + \Phi(x) \tag{5}
$$

Taking derivative of (5) w.r.t. x gives an ODE to solve for $\Phi(x)$

$$
\frac{\partial v}{\partial x} = \frac{d}{dx}\left(\arctan\left(\frac{y}{x}\right)\right) + \Phi'(x) \tag{5A}
$$

To find $\frac{d}{dx}\arctan\left(\frac{y}{x}\right)$, let

$$
w = \arctan\left(\frac{y}{x}\right)
$$

Now the goal is to find $\frac{dw}{dx}$. The above is the same as

$$
\tan(w) = \frac{y}{x} \tag{6}
$$

Taking derivative of both sides of the above w.r.t. x gives

$$
\frac{d}{dx}\tan(w) = -\frac{y}{x^2}
$$

But $\frac{d}{dx}\tan(w) = \sec^2(w)\frac{dw}{dx}$, and the above can be written as

$$
\begin{aligned}
\sec^2(w)\frac{dw}{dx} &= -\frac{y}{x^2} \\
\frac{dw}{dx} &= -\frac{y}{x^2}\frac{1}{\sec^2(w)}
\end{aligned} \tag{7}
$$

But $\sec^2(w) = \frac{1}{\cos^2 w}$ and $\cos^2 w + \sin^2 w = 1$. Therefore dividing by $\cos^2 w$ gives $1 + \frac{\sin^2 w}{\cos^2 w} = \sec^2(w)$ or $1 + \tan^2 w = \sec^2(w)$. But from (6) we know that $\tan(w) = \frac{y}{x}$, therefore $1 + \left(\frac{y}{x}\right)^2 =$

$\sec^2(w)$. Replacing this expression for $\sec^2(w)$ in (7) gives

$$\begin{aligned}\frac{dw}{dx} &= -\frac{y}{x^2}\frac{1}{1+\left(\frac{y}{x}\right)^2}\\ &= -\frac{y}{x^2}\frac{x^2}{x^2+y^2}\\ &= \frac{-y}{x^2+y^2}\end{aligned}$$

Now that we found $\frac{dw}{dx}$ which is $\frac{d}{dx}\arctan\left(\frac{y}{x}\right)$, then 5A becomes

$$\frac{\partial v}{\partial x} = \frac{-y}{x^2+y^2} + \Phi'(x)$$

But from Cauchy Riemann equation (4) above, we know that $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, therefore the above is the same as

$$\frac{\partial u}{\partial y} = -\left(\frac{-y}{x^2+y^2} + \Phi'(x)\right)$$

We know what $\frac{\partial u}{\partial y}$ is. We found this earlier which is $\frac{\partial u}{\partial y} = \frac{y}{x^2+y^2}$. Hence the above equation becomes

$$\begin{aligned}\frac{y}{x^2+y^2} &= \frac{y}{x^2+y^2} - \Phi'(x)\\ \Phi'(x) &= 0\end{aligned}$$

Therefore <u>Φ is constant</u>, say C_1. Equation (5) becomes

$$\boxed{v(x,y) = \arctan\left(\frac{y}{x}\right) + C_1} \tag{8}$$

Which is the conjugate of $u = \frac{1}{2}\ln\left(x^2+y^2\right)$. To verify the result in (8), we now check that $v(x,y)$ is indeed harmonic by checking that it satisfies the Laplace PDE.

$$\begin{aligned}v_x &= \frac{-y}{x^2+y^2}\\ v_{xx} &= \frac{y(2x)}{\left(x^2+y^2\right)^2}\end{aligned}$$

And

$$\begin{aligned}v_y &= \frac{x}{x^2+y^2}\\ v_{yy} &= \frac{-x(2y)}{\left(x^2+y^2\right)^2}\end{aligned}$$

Using the above we see that

$$\begin{aligned} v_{xx} + v_{yy} &= \frac{y(2x)}{\left(x^2+y^2\right)^2} - \frac{x\left(2y\right)}{\left(x^2+y^2\right)^2} \\ &= 0 \end{aligned}$$

This shows that $v\left(x,y\right)$ obtained above is harmonic. It is the conjugate of $u\left(x,y\right)$.

$v\left(x,y\right)$ is not a unique conjugate of $u\left(x,y\right)$, since the constant C_1 is arbitrary.

4.2 Part (b) Using Polar coordinates

Here $z = re^{i\theta}$ and we are told that $u(r,\theta) = \ln r$. To show this is harmonic in polar coordinates, we need to show it satisfies Laplacian in polar coordinates, which is

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

But $u_r = \frac{d}{dr}\ln r = \frac{1}{r}$ and $u_{rr} = -\frac{1}{r^2}$ and $u_{\theta\theta} = 0$. Substituting these into the above gives

$$\begin{aligned} -\frac{1}{r^2} + \frac{1}{r}\frac{1}{r} &= 0 \\ 0 &= 0 \end{aligned}$$

Therefore $u = \ln r$ is harmonic since it satisfies the Laplacian in polar coordinates. To find its conjugate, we use C-R in polar coordinates, and these are given by

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \tag{1}$$

$$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} \tag{2}$$

From (1), and since we know that $\frac{\partial u}{\partial r} = \frac{1}{r}$, then this gives

$$\begin{aligned} \frac{1}{r} &= \frac{1}{r}\frac{\partial v}{\partial \theta} \\ \frac{\partial v}{\partial \theta} &= 1 \end{aligned}$$

Or by integration w.r.t. θ

$$v = \theta + \Phi(r)$$

Where $\Phi(r)$ is the constant of integration (a function). Taking derivative of the above w.r.t. r gives

$$\frac{\partial v}{\partial r} = \Phi'(r)$$

But from (2) $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} = 0$. (Because u does not depend on θ). Hence the above results in $\Phi'(r) = 0$ or $\Phi = C_1$ a constant. Therefore the conjugate harmonic function is

$$\boxed{v(r,\theta) = \theta + C_1}$$

Now we verify this satisfies Laplacian in Polar. From

$$v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\theta\theta} = 0$$

We see since $v_r = 0$ and $v_{rr} = 0$ and $v_\theta = 1$ and $v_{\theta\theta} = 0$, therefore we obtain $0 = 0$ also. Hence $v = \theta + C_1$ satisfies the Laplacian.

5 Problem 5

Find the value of $\int_C f(z)\,dz$ where $f(z) = e^z$ for two different contours. C_1 is straight line from the origin to the point $(2,1)$. C_2 is a straight line from the origin to the point $(2,0)$ followed by another straight line from $(2,0)$ to $(2,1)$

Solution

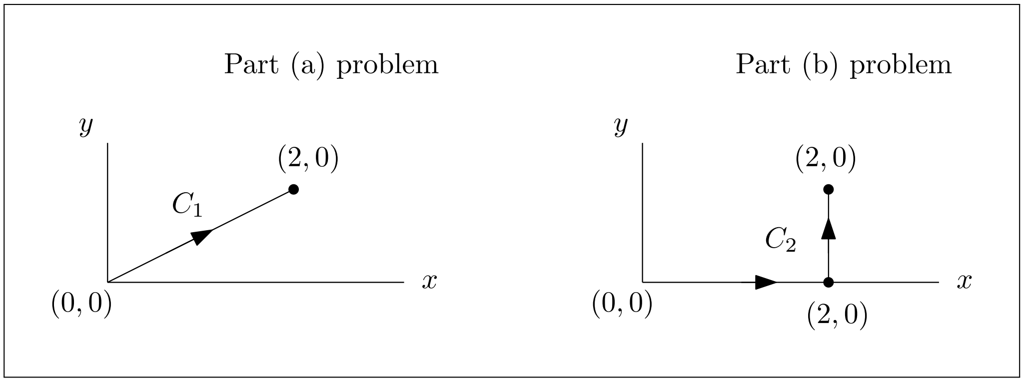

Figure 4: Showing contours for part(a) and pat (b)

5.1 Part a

Using contour C_1. The line starts from $(x_0, y_0) = (0,0)$ and ends at $(x_1, y_1) = (2,1)$. Hence the parametrization for this line is given by

$$
\begin{aligned}
x(t) &= (1-t)\,x_0 + t x_1 \\
&= 2t
\end{aligned}
$$

And

$$
\begin{aligned}
y(t) &= (1-t)\,y_0 + t y_1 \\
&= t
\end{aligned}
$$

Now $f(z) = e^z = e^{x+iy}$, Therefore in terms of t this becomes

$$
\begin{aligned}
f(t) &= e^{2t+it} \\
&= e^{t(2+i)}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\int_{C_1} f(z)\,dz &= \int_{t=0}^{t=1} f(t)\,z'(t)\,dt \\
&= \int_0^1 e^{t(2+i)} z'(t)\,dt
\end{aligned}
$$

But $z(t) = x(t) + iy(t) = 2t + it$, hence $z'(t) = 2 + i$ and the above becomes

$$\begin{aligned}\int_{C_1} f(z)\,dz &= \int_0^1 e^{t(2+i)}(2+i)\,dt\\ &= (2+i)\int_0^1 e^{t(2+i)}dt\\ &= (2+i)\left(\frac{e^{t(2+i)}}{(2+i)}\right)_0^1\\ &= \left(e^{t(2+i)}\right)_0^1\end{aligned}$$

Hence the final result is

$$\int_{C_1} f(z)\,dz = e^{2+i} - 1$$

5.2 Part b

Using C_2. The first line starts from $(x_0, y_0) = (0,0)$ and ends at $(x_1, y_1) = (2,0)$. Hence the parametrization for this line is given by

$$\begin{aligned}x(t) &= (1-t)\,x_0 + tx_1\\ &= 2t\end{aligned}$$

And

$$\begin{aligned}y(t) &= (1-t)\,y_0 + ty_1\\ &= 0\end{aligned}$$

Now $f(z) = e^z = e^{x+iy}$, Therefore in terms of t the function $f(z)$ becomes

$$f(t) = e^{2t}$$

Hence, for the line from $(0,0)$ to $(2,0)$ we have

$$\begin{aligned}\int_{C_{2_1}} f(z)\,dz &= \int_{t=0}^{t=1} f(t)\,z'(t)\,dt\\ &= \int_0^1 e^{2t}z'(t)\,dt\end{aligned}$$

But $z = x + iy = 2t$ since $y(t) = 0$. hence $z'(t) = 2$ and the above becomes

$$\begin{aligned}\int_{C_{2_1}} f(z)\,dz &= 2\int_0^1 e^{2t}dt\\ &= 2\left(\frac{e^{2t}}{2}\right)_0^1\\ &= e^2 - 1 \qquad (1)\end{aligned}$$

The <u>second line</u> starts from $(x_0, y_0) = (2, 0)$ and ends at $(x_1, y_1) = (2, 1)$. Hence the parametrization for this line is given by

$$\begin{aligned} x(t) &= (1-t)\, x_0 + t x_1 \\ &= (1-t)\, 2 + 2t \\ &= 2 \end{aligned}$$

And

$$\begin{aligned} y(t) &= (1-t)\, y_0 + t y_1 \\ &= t \end{aligned}$$

Now $f(z) = e^z = e^{x+iy}$, Therefore in terms of t this becomes

$$f(t) = e^{2+it}$$

Hence, for the line from $(2, 0)$ to $(2, 1)$ we have

$$\begin{aligned} \int_{C_{2_2}} f(z)\, dz &= \int_{t=0}^{t=1} f(t)\, z'(t)\, dt \\ &= \int_0^1 e^{2+it} z'(t)\, dt \end{aligned}$$

But $z = x + iy = 2 + it$. hence $z'(t) = i$ and the above becomes

$$\begin{aligned} \int_{C_{2_2}} f(z)\, dz &= \int_0^1 i e^{2+it} dt \\ &= i \left(\frac{e^{2+it}}{i} \right)_0^1 \\ &= \left(e^{2+it} \right)_0^1 \\ &= e^{2+i} - e^2 \end{aligned} \tag{2}$$

Therefore the total is the sum of (1) and (2)

$$\int_{C_2} f(z)\, dz = e^2 - 1 + e^{2+i} - e^2$$

Hence <u>the final result</u> is

$$\int_{C_2} f(z)\, dz = e^{2+i} - 1 \tag{3}$$

To verify this, since e^z is analytic then $\int_{C_2} f(z)\, dz - \int_{C_1} f(z)\, dz$ should come out to be zero (By Cauchy theorem). This is because $\oint f(z)\, dz = 0$ around the closed contour, going clockwise.

Let us see if this is true:

$$\begin{aligned}\int_{C_2} f(z)\,dz - \int_{C_1} f(z)\,dz &= \left[e^{2+i}-1\right]-\left[e^{2+i}-1\right]\\ &= 0\\ &= \oint f(z)\,dz\end{aligned}$$

Verified. A small note: $\oint_C f(z)\,dz = 0$ does not necessarily mean that $f(z)$ is analytic on and inside C as some non analytic function can give zero, depending on C. But if $f(z)$ happened to be analytic, then $\oint_C f(z)\,dz$ is always zero. But here we now that e^{az} is analytic.