① $S = 1 + \frac{1}{4} - \frac{1}{16} - \frac{1}{64} + \frac{1}{256} + \frac{1}{1024} - \cdots$

Series converges absolutely since $\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{4}$

$$S = \left(1 - \frac{1}{16} + \frac{1}{256} - \cdots\right) + \underbrace{\left(\frac{1}{4} - \frac{1}{64} + \frac{1}{1024} - \cdots\right)}_{\frac{1}{4}\left(1 - \frac{1}{16} + \frac{1}{256} - \cdots\right)}$$

$$= \frac{5}{4} \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{16}\right)^n = \frac{5}{4} \frac{1}{1 + \frac{1}{16}} = \boxed{\frac{20}{17} = S}$$

② $S = \frac{1}{0!} + \frac{2}{1!} + \frac{3}{2!} + \cdots$ Looks related to e^x.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \frac{d}{dx}\left(x e^x\right) = \frac{d}{dx} \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} = \sum_{n=0}^{\infty} \frac{(n+1)x^n}{n!} =$$

$$= \frac{1}{0!} + \frac{2}{1!} x + \frac{3}{2!} x^2 + \cdots$$ S has $x = 1$

$$\frac{d}{dx}(x e^x) = (1+x) e^x \qquad \boxed{S = 2e}$$

③ $f(\theta) = \sin(\theta) + \frac{1}{3}\sin(2\theta) + \frac{1}{5}\sin(3\theta) + \frac{1}{7}\sin(4\theta) + \cdots$

$$= \mathrm{Im}\left\{ e^{i\theta} + \frac{1}{3}e^{i2\theta} + \frac{1}{5}e^{i3\theta} + \frac{1}{7}e^{i4\theta} + \cdots \right\}$$

$$= \mathrm{Im}\left\{ e^{\frac{i\theta}{2}}\left[e^{\frac{i\theta}{2}} + \frac{1}{3}e^{\frac{i3\theta}{2}} + \frac{1}{5}e^{\frac{i5\theta}{2}} + \frac{1}{7}e^{\frac{i7\theta}{2}} + \cdots \right] \right\}$$

$$= \mathrm{Im}\, g \quad \text{where} \quad g = z\left(z + \frac{1}{3}z^3 + \frac{1}{5}z^5 + \frac{1}{7}z^7 + \cdots\right) = zh$$

and $z = e^{\frac{i\theta}{2}}$

$$h' = 1 + z^2 + z^4 + z^6 + \cdots = \frac{1}{1-z^2}$$

Now $h = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right) + \underbrace{a + ib}_{\text{real constants}}$

$$\ln\left(\frac{1+z}{1-z}\right) = \ln\left(\frac{1+e^{i\frac{\theta}{2}}}{1-e^{i\frac{\theta}{2}}}\right) = \ln\left(\frac{e^{\frac{i\theta}{4}} + e^{-i\frac{\theta}{4}}}{e^{-i\frac{\theta}{4}} - e^{i\frac{\theta}{4}}}\right) =$$

$$= \ln\left(\frac{\cos\frac{\theta}{4}}{-i\sin\frac{\theta}{4}}\right) = \ln\left(\frac{e^{\frac{i\pi}{2}}}{\tan\frac{\theta}{4}}\right) = i\frac{\pi}{2} - \ln\left(\tan\frac{\theta}{4}\right)$$

$$g = \frac{1}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)\left[i\frac{\pi}{2} - \ln\left(\tan\frac{\theta}{4}\right) + 2a + 2ib\right]$$

$$f = \mathrm{Im}\, g = \cos\frac{\theta}{2}\left[b + \frac{\pi}{4}\right] - \frac{1}{2}\sin\frac{\theta}{2}\ln\left(\tan\frac{\theta}{4}\right) + a\sin\frac{\theta}{2}$$

Now $f(0) = f(\pi) = 0 \Rightarrow a = 0,\ b = -\frac{\pi}{4}$

$$\boxed{f(\theta) = -\frac{1}{2}\sin\frac{\theta}{2}\ln\left(\tan\frac{\theta}{4}\right)}$$

④ $$f = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2 x^{2n-1}}{(2n-1)!}$$

Looks like 2 derivatives to get n^2 factor.

$$g = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n}}{(2n-1)!} \qquad x\frac{dg}{dx} = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n x^{2n}}{(2n-1)!}$$

$$\frac{d}{dx}\left(x\frac{dg}{dx}\right) = 4\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2 x^{2n-1}}{(2n-1)!} = 4f$$

$$g = x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \cdots = x\sin x$$

$$\frac{dg}{dx} = x\cos x + \sin x \qquad \frac{d}{dx}\left(x\frac{dg}{dx}\right) = \frac{d}{dx}\left(x^2\cos x + x\sin x\right) =$$

$$= 2x\cos x - x^2\sin x + \sin x + x\cos x$$

$$= 3x\cos x + (1-x^2)\sin x$$

$$\boxed{f(x) = \frac{3}{4}x\cos x + \frac{(1-x^2)}{4}\sin x}$$

(5) $\sec(x) = \sum_{n=0}^{\infty} (-1)^n \frac{E_{2n}}{(2n)!} x^{2n}$

(a) $x = 0$ $\quad \sec(0) = \boxed{1 = E_0}$

(b) $1 = \cos(x) \sum_{n=0}^{\infty} (-1)^n \frac{E_{2n}}{(2n)!} x^{2n} = \left[\sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} \right] \left[\sum_{n=0}^{\infty} \frac{(-1)^n E_{2n} x^{2n}}{(2n)!} \right]$

The $m=n=0$ term gives 1. The rest must vanish order by order. For $n \geq 1$

$$\frac{(-1)^n E_{2n}}{(2n)!} - \frac{1}{2!} \frac{(-1)^{n-1} E_{2n-2}}{(2n-2)!} + \frac{1}{4!} \frac{(-1)^n E_{2n-4}}{(2n-4)!} - \cdots \frac{(-1)^n E_0}{(2n)!} = 0$$

$$\boxed{E_{2n} + \frac{(2n)!}{2!(2n-2)!} E_{2n-2} + \frac{(2n)!}{4!(2n-4)!} E_{2n-4} + \cdots + E_0 = 0}$$

Note that the coefficients are binomial coefficients.

$$\boxed{E_2 = -1 \quad E_4 = 5 \quad E_6 = -61 \quad E_8 = 1385}$$

(c) For small k we can expand

$$\frac{1}{(2m+1)^2 - 4k^2} = \frac{1}{(2m+1)^2} \left[1 - \left(\frac{2k}{2m+1} \right)^2 \right]^{-1} =$$

$$= \frac{1}{(2m+1)^2} \sum_{n=0}^{\infty} \frac{(4k^2)^n}{(2m+1)^{2n}}$$

$$\sec(k\pi) = \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} \sum_{n=0}^{\infty} \frac{(2k)^{2n}}{(2m+1)^{2n}}$$

$$= \frac{4}{\pi} \sum_{n=0}^{\infty} (2k)^{2n} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^{2n+1}}$$

Compare to $\sec(k\pi) = \sum_{n=0}^{\infty} (-1)^n \frac{E_{2n}}{(2n)!} (k\pi)^{2n}$

order by order in k.

$$\frac{4}{\pi} \cdot 2^{2n} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^{2n+1}} = \frac{(-1)^n \pi^{2n} E_{2n}}{(2n)!}$$

$$\Rightarrow \boxed{\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^{2n+1}} = \frac{(-1)^n}{2} \left(\frac{\pi}{2}\right)^{2n+1} \frac{E_{2n}}{(2n)!}}$$