HW 2
Physics 5041 Mathematical Methods for Physics
Spring 2019
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November 2, 2019 Compiled on November 2, 2019 at 10:24pm [public]

Contents

1 Problem 1

Find the sum of $1+\frac{1}{4}-\frac{1}{16}-\frac{1}{64}+\frac{1}{256}+\frac{1}{1024}-\cdots$

Solution

We would like to combine each two consecutive negative terms and combine each two consecutive positive terms in the series in order to obtain an alternating series which is easier to work with. But to do that, we first need to check that the series is absolutely convergent. The $|a_n|$ term is $\frac{1}{4^n}$, therefore

$$\begin{aligned}
L &= \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \\
&= \lim_{n\to\infty}\left|\frac{\frac{1}{4^{n+1}}}{\frac{1}{4^n}}\right| \\
&= \lim_{n\to\infty}\left|\frac{4^n}{4^{n+1}}\right| \\
&= \left|\frac{1}{4}\right|
\end{aligned}$$

Since $|L|<1$ then the series is absolutely convergent so we are allowed now to group (or rearrange) terms as follows

$$\begin{aligned}
S &= \left(1+\frac{1}{4}\right)-\left(\frac{1}{16}+\frac{1}{64}\right)+\left(\frac{1}{256}+\frac{1}{1024}\right)-\left(\frac{1}{4096}+\frac{1}{16384}\right)+\cdots \\
&= \frac{5}{4}-\frac{5}{64}+\frac{5}{1024}-\frac{5}{16\,384}+\cdots \\
&= \frac{5}{4}\left(1-\frac{1}{16}+\frac{1}{256}-\frac{1}{4096}+\cdots\right) \\
&= \frac{5}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{4^{2n}} \\
&= \frac{5}{4}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{16}\right)^n \qquad (1)
\end{aligned}$$

But $\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{16}\right)^n$ has the form $\sum_{n=0}^{\infty}(-1)^n r^n$ where $r=\frac{1}{16}$ and since $|r|<1$ then by the binomial series

$$\begin{aligned}
\sum_{n=0}^{\infty}(-1)^n r^n &= 1-r+r^2-r^3+\cdots \\
&= \frac{1}{1+r}
\end{aligned}$$

Therefore the sum in (1) becomes, when using $r=\frac{1}{16}$ the following

$$\begin{aligned}
S &= \frac{5}{4}\left(\frac{1}{1+\frac{1}{16}}\right) \\
&= \frac{5}{4}\left(\frac{16}{17}\right)
\end{aligned}$$

Hence

$$\boxed{S=\frac{20}{17}}$$

Or

$$S\approx 1.176$$

2 Problem 2

Find the sum of $\frac{1}{0!} + \frac{2}{1!} + \frac{3}{2!} + \cdots$

Solution

$$
\begin{aligned}
S &= \sum_{n=0} \frac{n+1}{n!} \\
&= \sum_{n=0} \frac{n}{n!} + \sum_{n=0} \frac{1}{n!} \\
&= \sum_{n=0} \frac{n}{(n)(n-1)!} + e \\
&= \sum_{n=0} \frac{1}{(n-1)!} + e \\
&= \sum_{n=-1} \frac{1}{n!} + e \\
&= \frac{1}{(-1)!} + \sum_{n=0} \frac{1}{n!} + e \\
&= \frac{1}{(-1)!} + e + e \\
&= \frac{1}{(-1)!} + 2e
\end{aligned}
$$

Now to handle $\frac{1}{(-1)!}$, we use Gamma function definition for factorials $\Gamma(n) = (n-1)!$ for positive integers, and the generalized $\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx$ for non positive integers. By definition $\Gamma(-k)$ where k is negative integer is ∞. (Gamma function is defined only for negative values other than the negative integers).

Hence $\frac{1}{(-1)!} = \frac{1}{\infty} = 0$. So the above result now simplifies to

$$S = 2e$$

3 Problem 3

Sum the following series assuming that $0 < \theta < \pi$ for definiteness.

$$f(\theta) = \sin(\theta) + \frac{1}{3}\sin(2\theta) + \frac{1}{5}\sin(3\theta) + \frac{1}{7}\sin(4\theta) + \cdots$$

<u>Solution</u>

Since $e^{in\theta} = \cos(n\theta) + i\sin(n\theta)$ then the above is the same as writing

$$f(\theta) = \operatorname{Im}(e^{i\theta} + \frac{1}{3}e^{2i\theta} + \frac{1}{5}e^{3i\theta} + \frac{1}{7}e^{4i\theta} + \cdots) \tag{1}$$

Let $e^{i\frac{\theta}{2}} = x$, then the above becomes

$$\begin{aligned} f(\theta) &= \operatorname{Im}(x^2 + \frac{1}{3}x^4 + \frac{1}{5}x^6 + \frac{1}{7}x^8 + \cdots) \\ &= \operatorname{Im}\left(x\left(x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 + \cdots\right)\right) \end{aligned}$$

Let $g(x) = x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 + \cdots$, hence the above becomes

$$\begin{aligned} f(\theta) &= \operatorname{Im}\left(xg(x)\right) \\ &= \operatorname{Im}\left(x\int g'(x)\,dx\right) \end{aligned} \tag{2}$$

But $g'(x) = 1 + \frac{3x^2}{3} + \frac{5}{5}x^4 + \cdots = 1 + x^2 + x^4 + x^6 + \cdots$. Now for $|x| < 1$ and using Binomial series this has the sum

$$g'(x) = \frac{1}{1-x^2}$$

Substituting the above into (2) gives

$$f(\theta) = \operatorname{Im}\left(x\int \frac{1}{1-x^2}dx\right) \tag{3}$$

But

$$\int \frac{1}{1-x^2}dx = \int \frac{1}{(1-x)(1+x)}dx$$

Let $\frac{1}{(1-x)(1+x)} = \frac{A}{1-x} + \frac{B}{1+x}$. Hence $A(1+x) + B(1-x) = 1$ or $A + Ax + B - Bx = 1$ or $x(A-B) + (A+B) = 1$. Therefore $A = 1 - B$ and $A = B$. Hence $2B = 1$ or $B = \frac{1}{2}$ and also $A = \frac{1}{2}$. It follows that the above integral becomes

$$\begin{aligned} \int \frac{1}{1-x^2}dx &= \int \frac{A}{1-x} + \frac{B}{1+x}dx \\ &= \frac{1}{2}\int \frac{1}{1-x} + \frac{1}{1+x}dx \\ &= \frac{1}{2}\left(\ln(1+x) - \ln(1-x)\right) \\ &= \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \end{aligned}$$

Substituting the above into (3) gives

$$f(\theta) = \operatorname{Im}\left(\frac{x}{2}\ln\left(\frac{1+x}{1-x}\right)\right)$$

Now, replacing x back by $e^{i\frac{\theta}{2}}$ gives

$$f(\theta) = \frac{1}{2}\operatorname{Im}\left(e^{i\frac{\theta}{2}}\ln\left(\frac{1+e^{i\frac{\theta}{2}}}{1-e^{i\frac{\theta}{2}}}\right)\right)$$

Multiplying the numerator and denominator inside the $\ln$ by $e^{\frac{-i\theta}{4}}$ gives

$$f(\theta) = \frac{1}{2}\operatorname{Im}\left(e^{i\frac{\theta}{2}}\ln\left(\frac{e^{\frac{-i\theta}{4}} + e^{i\frac{\theta}{4}}}{e^{\frac{-i\theta}{4}} - e^{i\frac{\theta}{4}}}\right)\right)$$

$$= \frac{1}{2}\operatorname{Im}\left(e^{i\frac{\theta}{2}}\ln\left(\frac{e^{i\frac{\theta}{4}} + e^{\frac{-i\theta}{4}}}{-\left(e^{i\frac{\theta}{4}} - e^{\frac{-i\theta}{4}}\right)}\right)\right) \tag{4}$$

But $\cos\left(\frac{\theta}{4}\right) = \frac{e^{i\frac{\theta}{4}}+e^{\frac{-i\theta}{4}}}{2}$ and $\sin\left(\frac{\theta}{4}\right) = \frac{e^{\frac{-i\theta}{4}}-e^{i\frac{\theta}{4}}}{2i}$, therefore

$$e^{i\frac{\theta}{4}} + e^{\frac{-i\theta}{4}} = 2\cos\left(\frac{\theta}{4}\right)$$

$$e^{i\frac{\theta}{4}} - e^{\frac{-i\theta}{4}} = 2i\sin\left(\frac{\theta}{4}\right)$$

Using these in (4) gives

$$f(\theta) = \frac{1}{2}\operatorname{Im}\left(e^{i\frac{\theta}{2}}\ln\left(\frac{2\cos\left(\frac{\theta}{4}\right)}{-2i\sin\left(\frac{\theta}{4}\right)}\right)\right)$$

$$= \frac{1}{2}\operatorname{Im}\left(e^{i\frac{\theta}{2}}\ln\left(i\frac{\cos\left(\frac{\theta}{4}\right)}{\sin\left(\frac{\theta}{4}\right)}\right)\right) \tag{5}$$

Using $\ln z = \ln|z| + i\arg(z)$, where the principal argument is used. Here $z = i\frac{\cos\left(\frac{\theta}{4}\right)}{\sin\left(\frac{\theta}{4}\right)}$. This gives $|z| = \frac{\cos\left(\frac{\theta}{4}\right)}{\sin\left(\frac{\theta}{4}\right)}$ and

$$\arg(z) = \arg\left(i\frac{\cos\left(\frac{\theta}{4}\right)}{\sin\left(\frac{\theta}{4}\right)}\right)$$

Since since $\frac{\cos\left(\frac{\theta}{4}\right)}{\sin\left(\frac{\theta}{4}\right)} > 0$ for all θ in the range $0 < \theta < \pi$ then $i\frac{\cos\left(\frac{\theta}{4}\right)}{\sin\left(\frac{\theta}{4}\right)}$ is complex in the positive i direction. Hence

$$\arg(z) = \frac{\pi}{2}$$

Therefore we can write that

$$\ln\left(i\frac{\cos\left(\frac{\theta}{4}\right)}{\sin\left(\frac{\theta}{4}\right)}\right) = \ln\left(\frac{\cos\left(\frac{\theta}{4}\right)}{\sin\left(\frac{\theta}{4}\right)}\right) + i\frac{\pi}{2}$$

But we can simplify the above more using

$$\ln\left(\frac{\cos\left(\frac{\theta}{4}\right)}{\sin\left(\frac{\theta}{4}\right)}\right) = \ln\left(\frac{1}{\tan\left(\frac{\theta}{4}\right)}\right)$$

$$= \ln 1 - \ln\tan\left(\frac{\theta}{4}\right)$$

$$= -\ln\tan\left(\frac{\theta}{4}\right)$$

Substituting all the above back into (5) gives

$$\begin{aligned}
f(\theta) &= \frac{1}{2}\operatorname{Im}\left(e^{i\frac{\theta}{2}}\left[-\ln\tan\frac{\theta}{4}+i\frac{\pi}{2}\right]\right) \\
&= \frac{1}{2}\operatorname{Im}\left(\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)\left(-\ln\tan\frac{\theta}{4}+i\frac{\pi}{2}\right)\right) \\
&= \frac{1}{2}\operatorname{Im}\left(-\cos\frac{\theta}{2}\ln\tan\frac{\theta}{4}-i\sin\frac{\theta}{2}\ln\tan\frac{\theta}{4}+i\frac{\pi}{2}\cos\frac{\theta}{2}-\frac{\pi}{2}\sin\frac{\theta}{2}\right) \\
&= \frac{1}{2}\operatorname{Im}\left(i\left[-\sin\frac{\theta}{2}\ln\tan\frac{\theta}{4}+\frac{\pi}{2}\cos\frac{\theta}{2}\right]+\left[-\cos\frac{\theta}{2}\ln\tan\frac{\theta}{4}+\frac{\pi}{2}\sin\frac{\theta}{2}\right]\right)
\end{aligned}$$

Now we can take the imaginary part, giving the final answer as

$$f(\theta) = \frac{1}{2}\left(\frac{\pi}{2}\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\ln\tan\left(\frac{\theta}{4}\right)\right)$$

4 Problem 4

Evaluate the series $f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2 x^{2n-1}}{(2n-1)!} = x - \frac{4x^3}{3!} + \frac{9x^5}{5!} + \cdots$ by comparing it with $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

Solution

Since

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2 x^{2n-1}}{(2n-1)!}$$

And since

$$\sin(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n-1}}{(2n-1)!}$$

Then we start by taking derivative of $\sin(x)$ twice, which gives

$$\frac{d}{dx}\sin(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2n-1)x^{2n-2}}{(2n-1)!} \tag{1}$$

And differentiating one more time

$$\begin{aligned}
\frac{d^2}{dx^2}\sin(x) &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2n-1)(2n-2)x^{2n-3}}{(2n-1)!} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\left(4n^2-6n+2\right)x^{2n-3}}{(2n-1)!} \\
&= 4\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2 x^{2n-3}}{(2n-1)!} - 6\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n x^{2n-3}}{(2n-1)!} + 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n-3}}{(2n-1)!}
\end{aligned}$$

Multiplying both sides by x^2 gives

$$\begin{aligned}
x^2\frac{d^2}{dx^2}\sin(x) &= 4\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2 x^{2n-1}}{(2n-1)!} - 6\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n x^{2n-1}}{(2n-1)!} + 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n-1}}{(2n-1)!} \\
&= 4f(x) - 6\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n x^{2n-1}}{(2n-1)!} + 2\sin(x) \qquad (2)
\end{aligned}$$

Let

$$g(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n x^{2n-1}}{(2n-1)!}$$

Then (2) becomes

$$\begin{aligned}
x^2\frac{d^2}{dx^2}\sin(x) &= 4f(x) - 6g(x) + 2\sin(x) \\
-x^2\cos x &= 4f(x) - 6g(x) + 2\sin(x) \qquad (3)
\end{aligned}$$

So we just need to find $g(x)$. For this we can use (1). Writing (1) as

$$\begin{aligned}
\frac{d}{dx}\sin(x) &= 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n x^{2n-2}}{(2n-1)!} - \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n-2}}{(2n-1)!} \\
x\frac{d}{dx}\sin(x) &= 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n x^{2n-1}}{(2n-1)!} - \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n-1}}{(2n-1)!} \\
x\frac{d}{dx}\sin(x) &= 2g(x) - \sin(x)
\end{aligned}$$

Hence

$$g(x) = \frac{x\frac{d}{dx}\sin(x) + \sin(x)}{2}$$

Using the above in (3) gives

$$
\begin{aligned}
x^2 \frac{d^2}{dx^2} \sin(x) &= 4f(x) - 6\left(\frac{x \frac{d}{dx} \sin(x) + \sin(x)}{2}\right) + 2\sin(x) \\
-x^2 \sin(x) &= 4f(x) - 3(x\cos x + \sin x) + 2\sin x
\end{aligned}
$$

Solving for $f(x)$

$$
\begin{aligned}
f(x) &= \frac{-x^2 \sin + 3x\cos x + \sin x}{4} \\
&= \frac{\left(1 - x^2\right)\sin x + 3x\cos x}{4}
\end{aligned}
$$

Or

$$
x - \frac{4x^3}{3!} + \frac{9x^5}{5!} + \cdots = \frac{1}{4}\left(1 - x^2\right)\sin x + \frac{3}{4} x\cos x
$$

5 Problem 5

The Euler numbers are defined by

$$\sec(x) = \sum_{n=0}^{\infty} (-1)^n \frac{E_{2n}}{(2n)!} x^{2n}$$

(a) What is E_0?

(b) Find recursion expansion for E_{2n} when $n \geq 1$. Determine E_2, E_4, E_6, E_8 explicitly.

(c) The partial fraction expansion of secant is

$$\sec(k\pi) = \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m (2m+1)}{(2m+1)^2 - 4k^2}$$

Expand the right side in a power series in k and use it to evaluate the sum

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^{2n+1}}$$

In terms of one or more Euler numbers.

Solution

5.1 Part (a)

Using the formula given, we see that

$$\sec(x) = E_0 - \frac{E_2}{2!} x^2 + \frac{E_4}{4!} x^4 - \frac{E_6}{6!} x^6 + \cdots$$

When $x = 0$ the above gives

$$\sec(0) = E_0$$

Hence

$$E_0 = 1$$

5.2 Part (b)

Since $\cos(x)\sec(x) = 1$ then

$$1 = \cos(x) \left(\sum_{n=0}^{\infty} (-1)^n \frac{E_{2n}}{(2n)!} x^{2n} \right)$$

Using power series expansion for $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$, then the above becomes

$$1 = \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} \right) \left(\sum_{n=0}^{\infty} (-1)^n \frac{E_{2n}}{(2n)!} x^{2n} \right)$$

To see the pattern, so that we can combine the product above, let us multiply few terms, and collect on powers of x

$$\begin{aligned}
1 &= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)\left(E_0 - \frac{E_2}{2!}x^2 + \frac{E_4}{4!}x^4 - \frac{E_6}{6!}x^6 \cdots\right) \\
&= x^0 (E_0) + x^2 \left(-\frac{E_2}{2!} - \frac{E_0}{2!}\right) + x^4 \left(\frac{E_4}{4!} + \frac{E_2}{2!2!} + \frac{E_0}{4!}\right) + x^6 \left(-\frac{E_6}{6!} - \frac{E_4}{2!4!} - \frac{E_2}{4!2!} - \frac{E_0}{6!}\right) + \cdots \\
&= x^0 (E_0) - x^2 \left(\frac{E_0}{2!} + \frac{E_2}{2!}\right) + x^4 \left(\frac{E_0}{4!} + \frac{E_2}{2!2!} + \frac{E_4}{4!}\right) - x^6 \left(\frac{E_0}{6!} + \frac{E_2}{4!2!} + \frac{E_4}{2!4!} + \frac{E_6}{6!}\right) + \cdots
\end{aligned}$$

Therefore the above can be written as

$$1 = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{1}{(2n-2k)!\,(2k)!} E_{2k} \right) (-1)^n x^{2n}$$

When $n=0$ then the RHS $\sum_{k=0}^{n}\frac{1}{(2n-2k)!(2k)!}E_{2k}=E_0=1$. Hence we can rewrite the above by starting sum from $n=1$ as follows

$$\begin{aligned}1&=1+\sum_{n=1}^{\infty}\left(\sum_{k=0}^{n}\frac{1}{(2n-2k)!\,(2k)!}E_{2k}\right)(-1)^n x^{2n}\\ 0&=\sum_{n=1}^{\infty}\left(\sum_{k=0}^{n}\frac{1}{(2n-2k)!\,(2k)!}E_{2k}\right)(-1)^n x^{2n}\end{aligned}$$

Equating terms of powers of x on both sides: since left side has no x, then this implies the coefficient of x in the RHS must be zero. This implies

$$\begin{aligned}\sum_{k=0}^{n}\frac{(-1)^n}{(2n-2k)!\,(2k)!}E_{2k}&=0\\ \sum_{k=0}^{n}\frac{1}{(2n-2k)!\,(2k)!}E_{2k}&=0\end{aligned}$$

Since we want E_{2n}, then we make the sum stop at $n-1$ to isolate that term. Hence the above becomes

$$\begin{aligned}\left(\sum_{k=0}^{n-1}\frac{1}{(2n-2k)!\,(2k)!}E_{2k}\right)+\frac{1}{(2n-2n)!\,(2n)!}E_{2n}&=0\\ \left(\sum_{k=0}^{n-1}\frac{1}{(2n-2k)!\,(2k)!}E_{2k}\right)+\frac{1}{(2n)!}E_{2n}&=0\\ \left(\sum_{k=0}^{n-1}\frac{(2n)!}{(2n-2k)!\,(2k)!}E_{2k}\right)+E_{2n}&=0\\ \left(\sum_{k=0}^{n-1}\binom{2n}{2k}E_{2k}\right)+E_{2n}&=0\end{aligned}$$

Therefore the recursion formula is finally found as

$$E_{2n}=-\sum_{k=0}^{n-1}\binom{2n}{2k}E_{2k}\tag{4}$$

Using (4), we now calculate E_2,E_4,E_6,E_8.

<u>For $n=1$</u> then (4) becomes

$$\begin{aligned}E_2&=-\sum_{k=0}^{0}\frac{(2)!}{(2-2k)!\,(2k)!}E_{2k}\\ &=\frac{(2)!}{(2)!}E_0\\ &=-E_0\\ &=-1\end{aligned}$$

<u>For $n=2$</u> then (5) becomes

$$\begin{aligned}E_4&=-\sum_{k=0}^{1}\frac{(4)!}{(4-2k)!\,(2k)!}E_{2k}\\ &=-\left(\frac{(4)!}{(4)!}E_0+\frac{(4)!}{(4-2)!\,(2)!}E_2\right)\\ &=-\left(E_0+\frac{(4)\,(3)\,(2)}{(2)\,(2)}E_2\right)\\ &=-\left(1+(2)\,(3)\,(-1)\right)\\ &=-\left(1-6\right)\\ &=5\end{aligned}$$

For $n = 3$ then (5) becomes

$$
\begin{aligned}
E_6 &= -\sum_{k=0}^{2} \frac{(6)!}{(6-2k)!\,(2k)!} E_{2k} \\
&= -\left(\frac{(6)!}{(6)!} E_0 + \frac{(6)!}{(6-2)!\,(2)!} E_2 + \frac{(6)!}{(2)!\,(4)!} E_4 \right) \\
&= -\left(E_0 + \frac{(6)\,(5)}{2} E_2 + \frac{(6)\,(5)}{2} E_4 \right) \\
&= -\left(E_0 + 15E_2 + 15E_4 \right) \\
&= -\left(1 + 15\,(-1) + 15\,(5) \right) \\
&= -61
\end{aligned}
$$

For $n = 4$ then (5) becomes

$$
\begin{aligned}
E_6 &= -\sum_{k=0}^{3} \frac{(8)!}{(8-2k)!\,(2k)!} E_{2k} \\
&= -\left(\frac{(8)!}{(8)!} E_0 + \frac{(8)!}{(8-2)!\,(2)!} E_2 + \frac{(8)!}{(4)!\,(4)!} E_4 + \frac{(8)!}{(8-6)!\,(6)!} E_6 \right) \\
&= -\left(E_0 + \frac{(8)!}{(6)!\,(2)!} E_2 + \frac{(8)\,(7)\,(6)\,(5)}{(4)!} E_4 + \frac{(8)!}{(2)!\,(6)!} E_6 \right) \\
&= -\left(E_0 + \frac{(8)\,(7)}{(2)!} E_2 + \frac{(8)\,(7)\,(6)\,(5)}{(4)\,(3)\,(2)} E_4 + \frac{(8)\,(7)}{(2)} E_6 \right) \\
&= -\left(E_0 + 28E_2 + 70E_4 + 28E_6 \right) \\
&= -\left(1 + 28\,(-1) + 70\,(5) + 28\,(-61) \right) \\
&= 1385
\end{aligned}
$$

Summary

n	E_{2n}
0	$E_0 = 1$
1	$E_2 = -1$
2	$E_4 = 5$
3	$E_6 = -61$
4	$E_8 = 1385$

5.3 Part c

$$
\begin{aligned}
\sec(k\pi) &= \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m\,(2m+1)}{(2m+1)^2 - (2k)^2} \\
&= \frac{4}{\pi} \sum_{m=0}^{\infty} (-1)^m \frac{(2m+1)}{(2m+1)^2 - (2k)^2} \\
&= \frac{4}{\pi} \sum_{m=0}^{\infty} (-1)^m \frac{1}{(2m+1) - \frac{(2k)^2}{(2m+1)}} \\
&= \frac{4}{\pi} \sum_{m=0}^{\infty} (-1)^m \frac{1}{(2m+1)\left(1 - \frac{(2k)^2}{(2m+1)^2}\right)} \\
&= \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)} \frac{1}{\left(1 - \frac{(2k)^2}{(2m+1)^2}\right)} \\
&= \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)} \frac{1}{\left(1 - \left(\frac{2k}{2m+1}\right)^2\right)}
\end{aligned}
$$

Assuming $\left|\frac{2k}{2m+1}\right| < 1$ then $\frac{1}{\left(1-\left(\frac{2k}{2m+1}\right)^2\right)} = \sum_{n=0}^{\infty}\left(\frac{2k}{2m+1}\right)^{2n}$. From Binomial series. Then the above can be written as

$$\sec(k\pi) = \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)}\left(\sum_{n=0}^{\infty}\left(\frac{2k}{2m+1}\right)^{2n}\right)$$

Interchanging the order of summation in order to combine m terms

$$\begin{aligned}\sec(k\pi) &= \frac{4}{\pi}\sum_{n=0}^{\infty}k^{2n}\left(\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)}\left(\frac{2}{2m+1}\right)^{2n}\right)\\ &= \frac{4}{\pi}\sum_{n=0}^{\infty}k^{2n}\left(\sum_{m=0}^{\infty}(-1)^m\frac{2^{2n}}{(2m+1)^{2n+1}}\right)\end{aligned}\tag{1}$$

But since $\sec(x) = \sum_{n=0}^{\infty}(-1)^n\frac{E_{2n}}{(2n)!}x^{2n}$, then when $x = k\pi$, this becomes

$$\begin{aligned}\sec(k\pi) &= \sum_{n=0}^{\infty}(-1)^n\frac{E_{2n}}{(2n)!}(k\pi)^{2n}\\ &= \sum_{n=0}^{\infty}k^{2n}(-1)^n\frac{E_{2n}}{(2n)!}(\pi)^{2n}\end{aligned}\tag{2}$$

Comparing (1) and (2), we see this correspondence

$$\frac{4}{\pi}\sum_{n=0}^{\infty}k^{2n}\left(\sum_{m=0}^{\infty}(-1)^m\frac{2^{2n}}{(2m+1)^{2n+1}}\right) = \sum_{n=0}^{\infty}k^{2n}(-1)^n\frac{E_{2n}}{(2n)!}(\pi)^{2n}$$

Hence

$$\begin{aligned}\frac{4}{\pi}\sum_{m=0}^{\infty}(-1)^m\frac{2^{2n}}{(2m+1)^{2n+1}} &= (-1)^n\frac{E_{2n}}{(2n)!}(\pi)^{2n}\\ \sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^{2n+1}} &= (-1)^n\left(\frac{\pi}{4}\right)\left(\frac{1}{2^{2n}}\right)\frac{E_{2n}}{(2n)!}(\pi)^{2n}\\ &= (-1)^n\left(\frac{1}{2^2}\right)\left(\frac{1}{2^{2n}}\right)\frac{E_{2n}}{(2n)!}(\pi)^{2n+1}\end{aligned}$$

Therefore

$$\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^{2n+1}} = (-1)^n\left(\frac{1}{2^{2(n+1)}}\right)\frac{E_{2n}}{(2n)!}(\pi)^{2n+1}$$

Where E_{2n} are the Euler numbers found above.