(1) $g_1 = [5\,4\,1\,2\,3] \quad g_2 = [2\,1\,5\,3\,4]$

What g will give $g^{-1} g_1 g = g_2$ or $g_1 g = g g_2$?

Answer: $\boxed{g = [3\,1\,4\,2\,5]}$

check:

$g_1 g = [5\,4\,1\,2\,3][3\,1\,4\,2\,5] = [5\,2\,3\,1\,4]$

$g g_2 = [3\,1\,4\,2\,5][2\,1\,5\,3\,4] = [5\,2\,3\,1\,4]$

② $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $A_2 = \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix}$

Using the fact that $z^3 = 1$ we have

$$A_2 A_2 = \boxed{\begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix} \equiv A_3} \qquad A_1 A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$A_1 A_2 = \boxed{\begin{pmatrix} 0 & z^2 \\ z & 0 \end{pmatrix} \equiv A_4} \qquad A_2 A_1 = \boxed{\begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix} \equiv A_5}$$

Let's make a table.

$A_1 A_1 = I$	$A_2 A_1 = A_5$	$A_3 A_1 = A_4$	$A_4 A_1 = A_3$	$A_5 A_1 = A_2$
$A_1 A_2 = A_4$	$A_2 A_2 = A_3$	$A_3 A_2 = I$	$A_4 A_2 = A_5$	$A_5 A_2 = A_1$
$A_1 A_3 = A_5$	$A_2 A_3 = I$	$A_3 A_3 = A_2$	$A_4 A_3 = A_1$	$A_5 A_3 = A_4$
$A_1 A_4 = A_1$	$A_2 A_4 = A_1$	$A_3 A_4 = A_5$	$A_4 A_4 = I$	$A_5 A_4 = A_3$
$A_1 A_5 = A_3$	$A_2 A_5 = A_4$	$A_3 A_5 = A_1$	$A_4 A_5 = A_2$	$A_5 A_5 = I$

The inverses are: $A_2^{-1} = A_3$, $A_1^{-1} = A_1$, $A_4^{-1} = A_4$, $A_5^{-1} = A_5$

Hence this is a group.

The only <u>proper</u> subgroups are

$\{I, A_1\}$ $\{I, A_4\}$ $\{I, A_5\}$ $\{I, A_2, A_3\}$

3 $$M(v_2)M(v_1) = \frac{1}{\sqrt{(1-v_1^2)(1-v_2^2)}}\begin{pmatrix} 1 & v_2 \\ v_2 & 1 \end{pmatrix}\begin{pmatrix} 1 & v_1 \\ v_1 & 1 \end{pmatrix} =$$

$$= \frac{1}{\sqrt{(1-v_1^2)(1-v_2^2)}}\begin{pmatrix} 1+v_1v_2 & v_1+v_2 \\ v_1+v_2 & 1+v_1v_2 \end{pmatrix}$$

$$= \sqrt{\frac{(1+v_1v_2)^2}{(1-v_1^2)(1-v_2^2)}}\begin{pmatrix} 1 & \frac{v_1+v_2}{1+v_1v_2} \\ \frac{v_1+v_2}{1+v_1v_2} & 1 \end{pmatrix}$$

Define $\boxed{v_{12} = \frac{v_1+v_2}{1+v_1v_2}}$

$$1 - v_{12}^2 = 1 - \frac{(v_1+v_2)^2}{(1+v_1v_2)^2} = \frac{(1+2v_1v_2+v_1^2v_2^2) - (v_1^2+2v_1v_2+v_2^2)}{(1+v_1v_2)^2}$$

$$= \frac{1+v_1^2v_2^2-v_1^2v_2^2}{(1+v_1v_2)^2} = \frac{(1-v_1^2)(1-v_2^2)}{(1+v_1v_2)^2}$$

or $$\frac{1}{1-v_{12}^2} = \frac{(1+v_1v_2)^2}{(1-v_1^2)(1-v_2^2)}$$

Thus $\boxed{M(v_2)M(v_1) = M(v_{12})}$ with v_{12} as above.

This is closure, Associativity follows from matrix multiplication. $I = M(0)$ is the identity.

The inverse for each element is $M^{-1}(v) = M(-v)$.

4

$$[X_i, X_j] = c_{ij}^{k} X_k$$

$$\parallel$$

$$X_i X_j - X_j X_i$$

(a) $$[X_i, X_j] = -[X_j, X_i] = -c_{ji}^{k}$$

$$\Rightarrow \boxed{C_{ji}^{k} = -C_{ij}^{k}}$$

(b) $$[[X_i, X_j], X_k] + [[X_j, X_k], X_i] + [[X_k, X_i], X_j] =$$

$$= [X_i X_j - X_j X_i, X_k] + [X_j X_k - X_k X_j, X_i] + [X_k X_i - X_i X_k, X_j]$$

$$= X_i X_j X_k - X_j X_i X_k - X_k X_i X_j + X_k X_j X_i$$

$$+ X_j X_k X_i - X_k X_j X_i - X_i X_j X_k + X_i X_k X_j$$

$$+ X_k X_i X_j - X_i X_k X_j - X_j X_k X_i + X_j X_i X_k$$

These all cancel pairwise to give 0.

(c) $$\underbrace{[[X_i, X_j]}_{c_{ij}^{l} X_l}, X_k] + \underbrace{[[X_j, X_k]}_{C_{jk}^{l} X_l}, X_i] + \underbrace{[[X_k, X_i]}_{C_{ki}^{l} X_l}, X_j] = 0$$

$$0 = \underbrace{C_{ij}^{l}\,[X_l, X_k]}_{C_{lk}^{m} X_m} + \underbrace{C_{jk}^{l}\,[X_l, X_i]}_{C_{li}^{m} X_m} + \underbrace{C_{ki}^{l}\,[X_l, X_j]}_{C_{lj}^{m} X_m}$$

$$= \left(C_{ij}^{l} C_{lk}^{m} + C_{jk}^{l} C_{li}^{m} + C_{ki}^{l} C_{lj}^{m} \right) X_m$$

Since none of the X_m are zero we must have

$$\boxed{C_{ij}^{l} C_{lk}^{m} + C_{jk}^{l} C_{li}^{m} + C_{ki}^{l} C_{lj}^{m} = 0}$$