HW 12
Physics 5041 Mathematical Methods for Physics
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Contents

1 Problem 1

<u>Problem</u> Consider the following two elements of S_5

$$g_1 = [54123]$$
$$g_2 = [21534]$$

Find a third element g of this group such that $g^{-1}g_1g = g_2$

<u>Solution</u>

When $g^{-1}xg = y$, we say that y is conjugate to x using g.

$$gg^{-1}g_1g = gg_2$$
$$g_1g = gg_2 \tag{1}$$

But the class of conjugate pairs is symmetric. This means that

$$g^{-1}g_2g = g_1$$
$$gg^{-1}g_2g = gg_1$$
$$g_2g = gg_1 \tag{2}$$

We have two equations (1,2). Let us now apply g_1, g_2 on them. Let $g = [abcde]$ and the goal is to determine the unknowns a, b, c, d, e. Equation (1) becomes

$$[54123]\,[abcde] = [abcde]\,[21534]$$
$$[edabc] = [abcde]\,[21534] \tag{1A}$$

Similarly for (2)

$$[21534]\,[abcde] = [abcde]\,[54123]$$
$$[baecd] = [abcde]\,[54123] \tag{2A}$$

OK, this is some progress. But how are we going to find a, b, c, d, e?. Let use try <u>$a = 1$</u> and see what we get. If $a = 1$ then (1A) implies $e = 2$ and (2A) implies $b = 5$. Now, if $b = 5$ then (1A) gives $d = 4$ and (2A) gives $a = 3$. Which is conflict with our assumption that $a = 1$ we started with.

Let us next assume that <u>$a = 2$</u> and see if we get a conflict or not. If $a = 2$ then (1A) gives $e = 1$ and (2A) gives $b = 4$. Now, if $b = 4$ then (1A) gives $d = 3$ and (2A) gives $a = 2$. Good no conflict so far. Now taking $d = 3$ then (1A) gives $b = 5$, which is a conflict of what we found so far. So our starting guess of $a = 2$ is not correct.

Let us next assume that <u>$a = 3$</u> and see if we get a conflict or not. If $a = 3$ then (1A) gives $e = 5$ and (2A) gives $b = 1$. Now using $b = 1$ then (1A) gives $d = 2$ and (2A) gives $a = 5$, which is conflict with our assumption that $a = 3$.

Let us next assume that <u>$a = 4$</u> and see if we get a conflict or not. If $a = 4$ then (1A) gives $e = 3$ and (2A) gives $b = 2$. Now using $b = 2$ then (1A) gives $d = 1$ and (2A) gives $a = 4$. Good. No conflict so far. So far we found $a, b, e, d = 4, 2, 3, 1$. It must mean this case that $c = 5$ since it it only entry left. Let us check if this works or not.

From above we have a candidate element to check which is

$$g = [42513]$$

Trying it on (1,2). From (1)

$$\begin{aligned} g_1 g &= g g_2 \\ [54123]\,[42513] &= [42513]\,[21534] \\ [31425] &= [31425] \end{aligned}$$

OK. Let us check (2)

$$\begin{aligned} g_2 g &= g g_1 \\ [21534]\,[42513] &= [42513]\,[54123] \\ [24351] &= [24351] \end{aligned}$$

Verified. Hence one element is $\underline{g = [42513]}$.

This means that

$$[42513]^{-1}\,[54123]\,[42513] = [21534]$$

2 Problem 2

Do the following matrices for a group?

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Here $z = e^{i\frac{2\pi}{3}}$. If not, add the minimum number of 2×2 matrices to form a group. Then make a list of all possible subgroups.

Solution

The group G with elements g_i must have the following properties (using matrix multiplication as the binary operation $\circ$)

1. $g_i \circ g_j$ is also an element in the group G
2. Binary operation is associative: $(g_i \circ g_j) \circ g_k = g_i \circ (g_j \circ g_k)$
3. There is element I called the identity element such that $I \circ g_i = g_i \circ I = g_i$ for all $g_i \in G$
4. Each group element g_i has inverse g_i^{-1} such that $g_i \circ g_i^{-1} = g_i^{-1} \circ g_i = I$

Checking the first property. Let $g_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, g_2 = \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix}, g_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then since g_1 is the identity element, all products with it will also be in G. Looking at products with g_2

$$\begin{aligned} g_2 g_3 &= \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix} \end{aligned}$$

But $\begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix}$ is not in G. Hence it is not a group since not closed under the matrix multiplication.

Adding this as new element and calling it g_4

$$g_4 = \begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix}$$

But now we see that

$$g_2 g_4 = \begin{pmatrix} 0 & z^2 \\ z & 0 \end{pmatrix}$$

Is not in G. Calling the g_5.

$$g_5 = \begin{pmatrix} 0 & z^2 \\ z & 0 \end{pmatrix}$$

Check again if closed

$$g_2g_5 = \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix}\begin{pmatrix} 0 & z^2 \\ z & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & z^3 \\ z^3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & e^{i2\pi} \\ e^{i2\pi} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = g_3$$

Which is in G. Now checking all products with g_3 to see if they are in G.

$$g_3g_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix} = \begin{pmatrix} 0 & z^2 \\ z & 0 \end{pmatrix} = g_5$$

Which is in G. And

$$g_3g_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix} = \begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix}$$

But this is not in G. Adding the above as new element g_6

$$g_6 = \begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix}$$

Checking again from the start that the group we have now is closed, which now contains $g_1, g_2, g_3, g_4, g_5, g_6$.

<u>Checking all products with g_2</u>

$$g_2g_2 = \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix} = \begin{pmatrix} z^2 & 0 \\ 0 & z^4 \end{pmatrix} = \begin{pmatrix} z^2 & 0 \\ 0 & e^{i\frac{2\pi}{3}(4)} \end{pmatrix} = \begin{pmatrix} z^2 & 0 \\ 0 & e^{i\frac{\pi}{3}} \end{pmatrix} = \begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix} = g_6$$

$$g_2g_3 = \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix} = g_4$$

$$g_2g_4 = \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix}\begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & z^2 \\ z^4 & 0 \end{pmatrix} = \begin{pmatrix} 0 & z^2 \\ z & 0 \end{pmatrix} = g_5$$

$$g_2g_5 = \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix}\begin{pmatrix} 0 & z^2 \\ z & 0 \end{pmatrix} = \begin{pmatrix} 0 & z^3 \\ z^3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = g_3$$

$$g_2g_6 = \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix}\begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix} = \begin{pmatrix} z^3 & 0 \\ 0 & z^3 \end{pmatrix} = \begin{pmatrix} e^{i2\pi} & 0 \\ 0 & e^{i2\pi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = g_1$$

Checking all products with g_3

$$g_3g_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix} = \begin{pmatrix} 0 & z^2 \\ z & 0 \end{pmatrix} = g_5$$

$$g_3g_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = g_1$$

$$g_3g_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix} = \begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix} = g_6$$

$$g_3g_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & z^2 \\ z & 0 \end{pmatrix} = \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix} = g_2$$

$$g_3g_6 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix} = \begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix} = g_4$$

Checking all products with g_4

$$g_4g_2 = \begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix} = \begin{pmatrix} 0 & z^3 \\ z^3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = g_3$$

$$g_4g_3 = \begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix} = g_2$$

$$g_4g_4 = \begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix}\begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix} = \begin{pmatrix} z^3 & 0 \\ 0 & z^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = g_1$$

$$g_4g_5 = \begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix}\begin{pmatrix} 0 & z^2 \\ z & 0 \end{pmatrix} = \begin{pmatrix} z^2 & 0 \\ 0 & z^4 \end{pmatrix} = \begin{pmatrix} z^2 & 0 \\ 0 & e^{i\frac{2\pi}{3}(4)} \end{pmatrix} = \begin{pmatrix} z^2 & 0 \\ 0 & e^{i\frac{\pi}{3}} \end{pmatrix} = \begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix} = g_6$$

$$g_4g_6 = \begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix}\begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix} = \begin{pmatrix} 0 & z^2 \\ z^4 & 0 \end{pmatrix} = \begin{pmatrix} 0 & z^2 \\ z & 0 \end{pmatrix} = g_5$$

Checking all products with g_5

$$g_5 g_2 = \begin{pmatrix} 0 & z^2 \\ z & 0 \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix} = \begin{pmatrix} 0 & z^4 \\ z^2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix} = g_4$$

$$g_5 g_3 = \begin{pmatrix} 0 & z^2 \\ z & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix} = g_6$$

$$g_5 g_4 = \begin{pmatrix} 0 & z^2 \\ z & 0 \end{pmatrix}\begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix} = \begin{pmatrix} z^4 & 0 \\ 0 & z^2 \end{pmatrix} = \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix} = g_2$$

$$g_5 g_5 = \begin{pmatrix} 0 & z^2 \\ z & 0 \end{pmatrix}\begin{pmatrix} 0 & z^2 \\ z & 0 \end{pmatrix} = \begin{pmatrix} z^3 & 0 \\ 0 & z^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = g_1$$

$$g_5 g_6 = \begin{pmatrix} 0 & z^2 \\ z & 0 \end{pmatrix}\begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix} = \begin{pmatrix} 0 & z^3 \\ z^3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = g_3$$

Checking all products with g_6

$$g_6 g_2 = \begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix} = \begin{pmatrix} z^3 & 0 \\ 0 & z^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = g_1$$

$$g_6 g_3 = \begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & z^2 \\ z & 0 \end{pmatrix} = g_5$$

$$g_6 g_4 = \begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix}\begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & z^3 \\ z^3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = g_3$$

$$g_6 g_5 = \begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix}\begin{pmatrix} 0 & z^2 \\ z & 0 \end{pmatrix} = \begin{pmatrix} 0 & z^4 \\ z^2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix} = g_4$$

$$g_6 g_6 = \begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix}\begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix} = \begin{pmatrix} z^4 & 0 \\ 0 & z^2 \end{pmatrix} = \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix} = g_2$$

Therefore the group

$$G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & z^2 \\ z & 0 \end{pmatrix} \begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix}$$

Is closed under matrix multiplication. To check the associative property, which says that $g_i \circ \left(g_j \circ g_k\right) = \left(g_i \circ g_j\right) \circ g_k$ for all i, j, k in G. But from the property of matrix multiplication, we know this property is already satisfied since the matrices are all of same order which is 2×2. Checking that There is element I called the identity element such that $I \circ g_i = g_i \circ I = g_i$, then we see that $g_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is clearly I in this case. Checking the last property: Each group element g_i has inverse g_i^{-1} such that $g_i \circ g_i^{-1} = g_i^{-1} \circ g_i = I$. In this case g_i^{-1} is the inverse.

For g_1 then g_1^{-1} is itself.

Checking g_2

$$g_2^{-1} = \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix}^{-1} = \begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix} = g_6$$

$$g_2^{-1} \circ g_2 = \begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix} = \begin{pmatrix} z^3 & 0 \\ 0 & z^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Checking g_3

$$g_3^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = g_3$$

$$g_3^{-1} \circ g_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Checking for g_4

$$g_4^{-1} = \begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix} = g_4$$

$$g_4^{-1} \circ g_4 = \begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix}\begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix} = \begin{pmatrix} z^3 & 0 \\ 0 & z^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Checking g_5

$$g_5^{-1} = \begin{pmatrix} 0 & z^2 \\ z & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & z^2 \\ z & 0 \end{pmatrix} = g_5$$

$$g_5^{-1} \circ g_5 = \begin{pmatrix} 0 & z^2 \\ z & 0 \end{pmatrix}\begin{pmatrix} 0 & z^2 \\ z & 0 \end{pmatrix} = \begin{pmatrix} z^3 & 0 \\ 0 & z^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Checking g_6

$$g_6^{-1} = \begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix}^{-1} = \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix} = g_2$$

$$g_6^{-1} \circ g_6 = \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix}\begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix} = \begin{pmatrix} z^3 & 0 \\ 0 & z^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

OK. All elements checked. Hence G is indeed a group.

$$G = \overbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}^{g_1}, \overbrace{\begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix}}^{g_2}, \overbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}^{g_3}, \overbrace{\begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix}}^{g_4}, \overbrace{\begin{pmatrix} 0 & z^2 \\ z & 0 \end{pmatrix}}^{g_5}, \overbrace{\begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix}}^{g_6}$$

Setting up the Group table. In this table $g_1 = I$ the identity element.

$\circ$	I	g_2	g_3	g_4	g_5	g_6
I	I	g_2	g_3	g_4	g_5	g_6
g_2	g_2	g_6	g_4	g_5	g_3	I
g_3	g_3	g_5	I	g_6	g_2	g_4
g_4	g_4	g_3	g_2	I	g_6	g_5
g_5	g_5	g_4	g_6	g_2	I	g_3
g_6	g_6	I	g_5	g_3	g_4	g_2

Now we need to find all subgroups. By Lagrange theorem, we know for finite group such as G above, all subgroups are of order that divides the order of G. This means the order of the subgroups (if they exist) must be 2 or 3. (not counting order 1 which is just I and order 6 which is the group G itself).

Let us consider possible subgroups of order 2 first. Since subgroup must include the identity element $g_1 = I$, then all possible subgroups of order 2 are the following

$$[I, g_2], [I, g_3], [I, g_4], [I, g_5], [I, g_6]$$

Clearly each one of these is closed under $\circ$. Since $I \circ g_i = g_i \circ I = g_i \in G_{sub}$. But when checking for the property that each group element g_i has inverse g_i^{-1} such that $g_i \circ g_i^{-1} = g_i^{-1} \circ g_i = I$, then this fails unless each element is the same as its inverse. From earlier we found that

$$g_3^{-1} = g_3$$
$$g_4^{-1} = g_4$$
$$g_5^{-1} = g_5$$

Only. This implies that out of the above 6 candidate subgroups of order 2 only the following are subgroups

$$[I, g_3], [I, g_4], [I, g_5]$$

We found 3 subgroups so far. Now we need to consider all possible subgroups of order 3. Candidates are

$$[I, g_2, g_3], [I, g_2, g_4], [I, g_2, g_5], [I, g_2, g_6], [I, g_3, g_4], [I, g_3, g_5], [I, g_3, g_6], [I, g_4, g_5], [I, g_4, g_6], [I, g_5, g_6]$$

There are 10 candidates subgroups of order 3 above that we need to check. Easiest check is if the subgroup is closed. We know they satisfy the associative property.

Checking $[I, g_2, g_3]$

$$g_2 \circ g_3 = \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = g_4$$

Not closed.

Checking $\left[I, g_2, g_4\right]$

$$g_2 \circ g_4 = \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix}\begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix} = g_5$$

Not closed.

Checking $\left[I, g_2, g_5\right]$

$$g_2 \circ g_5 = \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix}\begin{pmatrix} 0 & z^2 \\ z & 0 \end{pmatrix} = g_3$$

Not closed.

Checking $\left[I, g_2, g_6\right]$

$$g_2 \circ g_6 = \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix}\begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix} = I$$

$$g_6 \circ g_2 = \begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix} = \begin{pmatrix} z^3 & 0 \\ 0 & z^3 \end{pmatrix} = I$$

Closed. Associativity is met since these are matrices of same order. Let check inverse property: Each subgroup element g_i has inverse g_i^{-1} such that $g_i \circ g_i^{-1} = g_i^{-1} \circ g_i = I$. In this case g_i^{-1} is the inverse matrix.

For g_2

$$g_2^{-1} = \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix}^{-1} = \begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix} = g_6$$

And $g_2^{-1} \circ g_2 = I$. OK. And

$$g_6^{-1} = \begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix}^{-1} = \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix} = g_2$$

And $g_6^{-1} \circ g_6 = I$. OK. Therefore $\left[I, g_2, g_6\right]$ is indeed a subgroup.

Checking $\left[I, g_3, g_4\right]$

$$g_3 \circ g_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix} = \begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix} = g_6$$

But g_6 is not in this subgroup. Hence not closed.

Checking $\left[I, g_3, g_5\right]$

$$g_3 \circ g_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & z^2 \\ z & 0 \end{pmatrix} = \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix} = g_2$$

But g_2 is not in this subgroup. Hence not closed.

<u>Checking $[I, g_3, g_6]$</u>

$$g_3 \circ g_6 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix} = \begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix} = g_4$$

But g_4 is not in this subgroup. Hence not closed.

<u>Checking $[I, g_4, g_5]$</u>

$$g_4 \circ g_5 = \begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix} \begin{pmatrix} 0 & z^2 \\ z & 0 \end{pmatrix} = \begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix} = g_6$$

But g_5 is not in this subgroup. Hence not closed.

<u>Checking $[I, g_4, g_6]$</u>

$$g_4 \circ g_6 = \begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix} \begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix} = \begin{pmatrix} 0 & z^2 \\ z^4 & 0 \end{pmatrix} = \begin{pmatrix} 0 & z^2 \\ z & 0 \end{pmatrix} = g_5$$

But g_5 is not in this subgroup. Hence not closed.

<u>Checking $[I, g_5, g_6]$</u>

$$g_5 \circ g_6 = \begin{pmatrix} 0 & z^2 \\ z & 0 \end{pmatrix} \begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix} = \begin{pmatrix} 0 & z^3 \\ z^3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = g_3$$

But g_3 is not in this subgroup. Hence not closed.

All subgroups of order 3 are checked. Therefore the following are the subgroups found. There are 4 in total

$$[I, g_3], [I, g_4], [I, g_5], [I, g_2, g_6]$$

Or

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & z^2 \\ z & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix}, \begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix}$$

3 Problem 3

The Lorentz transformation with velocity v along the x axis is described by

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = M(v) \begin{pmatrix} x \\ t \end{pmatrix}$$

Where $M(v) = \frac{1}{\sqrt{1-v^2}} \begin{pmatrix} 1 & v \\ v & 1 \end{pmatrix}$. Show that the product of two such transformations is again a Lorentz transformation. i.e. $M(v_2) M(v_1) = M(v_{12})$ and find v_{12}. Using this result, show that these transformations form a group.

solution

The following diagram is used to help in understanding what we are trying to show.

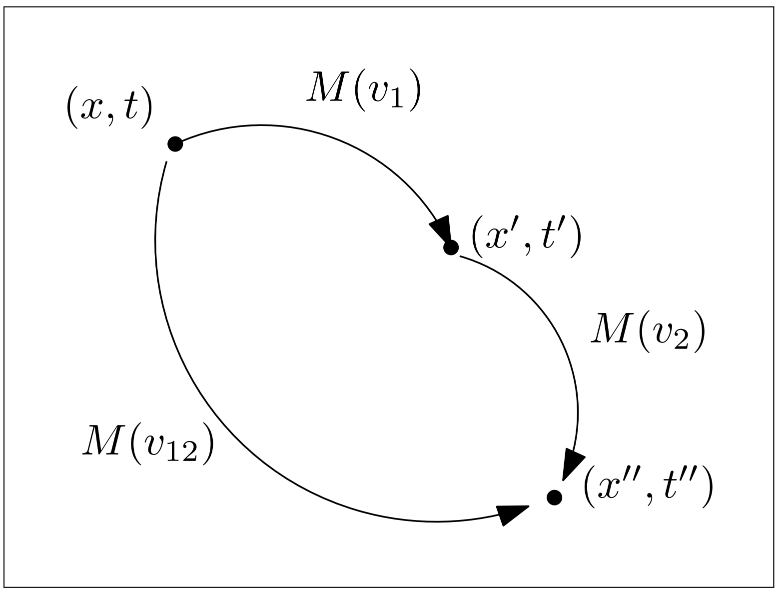

Figure 1: Lorentz transformations involved

Given

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = M(v_1) \begin{pmatrix} x \\ t \end{pmatrix}$$

And

$$\begin{pmatrix} x'' \\ t'' \end{pmatrix} = M(v_2) \begin{pmatrix} x' \\ t' \end{pmatrix}$$

We need to show that, with the help of the diagram above, that

$$\begin{pmatrix} x'' \\ t'' \end{pmatrix} = M(v_2) \begin{pmatrix} x' \\ t' \end{pmatrix} = M(v_2) M(v_1) \begin{pmatrix} x \\ t \end{pmatrix} = M(v_{12}) \begin{pmatrix} x \\ t \end{pmatrix}$$

So we need to find $M(v_{12})$ and see if it is a Lorentz transformation also. In other words, to see if $M(v_{12})$ has the form of $\frac{1}{\sqrt{1-v_{12}^2}}\begin{pmatrix}1 & v_{12}\\ v_{12} & 1\end{pmatrix}$ and need to find what v_{12} is. Starting by finding $M(v_2)$. Given that

$$\begin{aligned}\begin{pmatrix}x'\\ t'\end{pmatrix} &= M(v_1)\begin{pmatrix}x\\ t\end{pmatrix}\\ &= \frac{1}{\sqrt{1-v_1^2}}\begin{pmatrix}1 & v_1\\ v_1 & 1\end{pmatrix}\begin{pmatrix}x\\ t\end{pmatrix}\\ &= \frac{1}{\sqrt{1-v_1^2}}\begin{pmatrix}x+v_1 t\\ v_1 x + t\end{pmatrix}\end{aligned}$$

The above gives

$$\begin{aligned}x' &= \frac{1}{\sqrt{1-v_1^2}}(x+v_1 t)\\ t' &= \frac{1}{\sqrt{1-v_1^2}}(v_1 x + t)\end{aligned}$$

Applying the transformation again on the above result gives

$$\begin{aligned}
\begin{pmatrix} x'' \\ t'' \end{pmatrix} &= M(v_2)\begin{pmatrix} x' \\ t' \end{pmatrix} \\
&= \frac{1}{\sqrt{1-v_2^2}}\begin{pmatrix} 1 & v_2 \\ v_2 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{1-v_1^2}}(x+v_1t) \\ \frac{1}{\sqrt{1-v_1^2}}(v_1x+t) \end{pmatrix} \\
&= \frac{1}{\sqrt{1-v_2^2}}\begin{pmatrix} \frac{1}{\sqrt{1-v_1^2}}(x+v_1t)+\frac{v_2}{\sqrt{1-v_1^2}}(v_1x+t) \\ \frac{v_2}{\sqrt{1-v_1^2}}(x+v_1t)+\frac{1}{\sqrt{1-v_1^2}}(v_1x+t) \end{pmatrix} \\
&= \frac{1}{\sqrt{1-v_2^2}}\frac{1}{\sqrt{1-v_1^2}}\begin{pmatrix} (x+v_1t)+v_2(v_1x+t) \\ v_2(x+v_1t)+(v_1x+t) \end{pmatrix} \\
&= \frac{1}{\sqrt{\left(1-v_2^2\right)\left(1-v_1^2\right)}}\begin{pmatrix} x+v_1t+v_2v_1x+v_2t \\ v_2x+v_2v_1t+v_1x+t \end{pmatrix} \\
&= \frac{1}{\sqrt{1-v_1^2-v_2^2+v_2^2v_1^2}}\begin{pmatrix} x(1+v_2v_1)+t(v_1+v_2) \\ x(v_2+v_1)+t(1+v_2v_1) \end{pmatrix} \\
&= \frac{(1+v_2v_1)}{\sqrt{1-v_1^2-v_2^2+v_2^2v_1^2}}\begin{pmatrix} x+t\frac{(v_1+v_2)}{(1+v_2v_1)} \\ x\frac{(v_2+v_1)}{(1+v_2v_1)}+t \end{pmatrix} \\
&= \frac{1}{\sqrt{\frac{1-v_1^2-v_2^2+v_2^2v_1^2}{(1+v_2v_1)^2}}}\begin{pmatrix} 1 & \frac{(v_1+v_2)}{(1+v_2v_1)} \\ \frac{(1+v_2v_1)}{(1+v_2v_1)} & 1 \end{pmatrix}\begin{pmatrix} x \\ t \end{pmatrix}
\end{aligned}$$

But

$$\frac{1-v_1^2-v_2^2+v_2^2v_1^2}{(1+v_2v_1)^2} = \frac{(1+v_1v_2)^2-(v_1+v_2)^2}{(1+v_2v_1)^2}$$

Therefore

$$\begin{pmatrix} x'' \\ t'' \end{pmatrix} = \frac{1}{\sqrt{1-\frac{(v_1+v_2)^2}{(1+v_2v_1)^2}}}\begin{pmatrix} 1 & \frac{(v_1+v_2)}{(1+v_2v_1)} \\ \frac{(1+v_2v_1)}{(1+v_2v_1)} & 1 \end{pmatrix}\begin{pmatrix} x \\ t \end{pmatrix} \tag{1}$$

Now it is in the form of Lorentz transformation. $\begin{pmatrix} x'' \\ t'' \end{pmatrix} = M(v_{12})\begin{pmatrix} x \\ t \end{pmatrix}$. Comparing this (1) shows that

$$M(v_{12}) = \frac{1}{\sqrt{1-\frac{(v_1+v_2)^2}{(1+v_2v_1)^2}}}\begin{pmatrix} 1 & \frac{(v_1+v_2)}{(1+v_2v_1)} \\ \frac{(1+v_2v_1)}{(1+v_2v_1)} & 1 \end{pmatrix}$$

But $M(v) = \frac{1}{\sqrt{1-v^2}}\begin{pmatrix} 1 & v \\ v & 1 \end{pmatrix}$. By comparing to the above shows that

$$v_{12} = \frac{v_1+v_2}{1+v_2v_1}$$

Therefore what we did above is apply Lorentz transformation again $M(v_2)$ on result we obtained from $M(v_1)$ and we obtained a result which also a valid Lorentz transformation. This means the group is closed under this transformation. We need to show associativity. Which means

$$\begin{aligned} M(v_3)[M(v_2)M(v_1)] &= [M(v_3)M(v_2)]M(v_1) \\ M(v_3)M(v_{12}) &= M(v_{23})M(v_1) \end{aligned} \tag{3}$$

But we found from the above that $M(v_2)M(v_1) = M(v_{12})$ results in $v_{12} = \frac{v_1+v_2}{1+v_2v_1}$. Therefore we can conclude that left side of (2) which is $M(v_3)M(v_{12})$ will also result in

$$v_{321} = \frac{v_{12}+v_3}{1+v_3v_{12}}$$

But $v_{12} = \frac{v_1+v_2}{1+v_2v_1}$, therefore the above simplifies to

$$\begin{aligned} v_{321} &= \frac{\frac{v_1+v_2}{1+v_2v_1}+v_3}{1+v_3\frac{v_1+v_2}{1+v_2v_1}} \\ &= \frac{v_1+v_2+v_3(1+v_2v_1)}{1+v_2v_1+v_3v_1+v_2} \\ &= \frac{v_1+v_2+v_3+v_3v_2v_1}{1+v_2v_1+v_3v_1+v_3v_2} \end{aligned} \tag{3A}$$

And the right side of (3) which is $M(v_{23})M(v_1)$ also gives

$$v_{123} = \frac{v_1+v_{23}}{1+v_1v_{23}}$$

But again, $v_{23} = \frac{v_2+v_3}{1+v_3v_2}$ and the above simplifies to

$$\begin{aligned} v_{123} &= \frac{v_1+\frac{v_2+v_3}{1+v_3v_2}}{1+v_1\frac{v_2+v_3}{1+v_3v_2}} \\ &= \frac{v_1(1+v_3v_2)+v_2+v_3}{1+v_3v_2+v_1v_2+v_1v_3} \\ &= \frac{v_1+v_3v_2v_1+v_2+v_3}{1+v_3v_2+v_1v_2+v_1v_3} \end{aligned} \tag{3B}$$

By comparing (3A) and (3B) we see they are the same. Hence associativity is satisfied. Next we need to check the inverse property. What this means that for each $M(v_i)$ there exist $M^{-1}(v_i)$ such that $M(v_i)M^{-1}(v_i) = I$. where the identity in this case is $M(0) = I$ since

$$\begin{aligned} M(0) &= \frac{1}{\sqrt{1-0}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Since $M(v) = \frac{1}{\sqrt{1-v^2}}\begin{pmatrix} 1 & v \\ v & 1 \end{pmatrix}$ then $M(-v) = \frac{1}{\sqrt{1-v^2}}\begin{pmatrix} 1 & -v \\ -v & 1 \end{pmatrix}$ and

$$\begin{aligned} M(v)M(-v) &= \frac{1}{\sqrt{1-v^2}}\begin{pmatrix} 1 & v \\ v & 1 \end{pmatrix}\frac{1}{\sqrt{1-v^2}}\begin{pmatrix} 1 & -v \\ -v & 1 \end{pmatrix} \\ &= \frac{1}{1-v^2}\begin{pmatrix} 1 & v \\ v & 1 \end{pmatrix}\begin{pmatrix} 1 & -v \\ -v & 1 \end{pmatrix} \\ &= \frac{1}{1-v^2}\begin{pmatrix} 1-v^2 & 0 \\ 0 & 1-v^2 \end{pmatrix} \\ &= \frac{1-v^2}{1-v^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= M(0) \end{aligned}$$

Which is the identity. Hence we showed that for each $M(v_i)$ there exists an inverse $M(-v_i)$. All properties of group have been satisfied. Hence the given Lorentz transformation forms a group.

4 Problem 4

Using $\left[X_i, X_j\right] = c_{ij}^k X_k$ where c_{ij}^k are the structure constants and a summation over k is implied.

1. Show that $c_{ji}^k = -c_{ij}^k$
2. Prove the Jacobi identity $\left[\left[X_i, X_j\right], X_k\right] + \left[\left[X_j, X_k\right], X_i\right] + \left[\left[X_k, X_i\right], X_j\right] = 0$
3. Show that the Jacobi identity implies $c_{ij}^l c_{lk}^m + c_{jk}^l c_{li}^m + c_{ki}^l c_{lj}^m = 0$

Conditions (1,3) are the only conditions on the structure constants. Any set of real numbers c_{ij}^k obeying these two conditions defines a Lie algebra.

solution

4.1 Part (1)

The commutator of 2 generators (X_i, X_j) is linear combination of the generators. Hence

$$\left[X_i, X_j\right] = X_i X_j - X_j X_i = c_{ij}^k X_k \tag{1}$$

Therefore, we also have

$$\left[X_j, X_i\right] = X_j X_i - X_i X_j = c_{ji}^k X_k \tag{2}$$

Adding (1) and (2) gives

$$\begin{aligned} \left(X_i X_j - X_j X_i\right) + \left(X_j X_i - X_i X_j\right) &= c_{ij}^k X_k + c_{ji}^k X_k \\ 0 &= X_k \left(c_{ij}^k + c_{ji}^k\right) \\ 0 &= c_{ij}^k + c_{ji}^k \\ c_{ij}^k &= -c_{ji}^k \end{aligned}$$

4.2 Part (2)

Applying the commutator relation

$$\left[X_i, X_j\right] = X_i X_j - X_j X_i$$

Let LHS of the Jacobi identity be Δ. Applying the above to each term in Δ gives

$$\Delta = \left[\left(X_i X_j - X_j X_i\right), X_k\right] + \left[\left(X_j X_k - X_k X_j\right), X_i\right] + \left[\left(X_k X_i - X_i X_k\right), X_j\right] \tag{1}$$

We want to show that $\Delta = 0$. Now, applying commutator relation again each term of the above gives for the first term

$$\begin{aligned} \left[\left(X_i X_j - X_j X_i\right), X_k\right] &= \left(X_i X_j - X_j X_i\right) X_k - X_k \left(X_i X_j - X_j X_i\right) \\ &= X_i X_j X_k - X_j X_i X_k - X_k X_i X_j + X_k X_j X_i \end{aligned} \tag{2}$$

And for the second term in (1)

$$\begin{aligned}\left[\left(X_jX_k - X_kX_j\right), X_i\right] &= \left(X_jX_k - X_kX_j\right)X_i - X_i\left(X_jX_k - X_kX_j\right)\\ &= X_jX_kX_i - X_kX_jX_i - X_iX_jX_k + X_iX_kX_j \end{aligned} \tag{3}$$

And for the third term in (1)

$$\begin{aligned}\left[\left(X_kX_i - X_iX_k\right), X_j\right] &= \left(X_kX_i - X_iX_k\right)X_j - X_j\left(X_kX_i - X_iX_k\right)\\ &= X_kX_iX_j - X_iX_kX_j - X_jX_kX_i + X_jX_iX_k \end{aligned} \tag{4}$$

Substituting (2,3,4) back into (1) gives

$$\begin{aligned}\Delta &= \left(X_iX_jX_k - X_jX_iX_k - X_kX_iX_j + X_kX_jX_i\right)\\ &+ \left(X_jX_kX_i - X_kX_jX_i - X_iX_jX_k + X_iX_kX_j\right)\\ &+ \left(X_kX_iX_j - X_iX_kX_j - X_jX_kX_i + X_jX_iX_k\right)\end{aligned}$$

We see that all terms cancel each other. Hence $\Delta = 0$ which is what we wanted to show.

4.3 Part (3)

The Jacobi identity is

$$\left[\left[X_i, X_j\right], X_k\right] + \left[\left[X_j, X_k\right], X_i\right] + \left[\left[X_k, X_i\right], X_j\right] = 0$$

Applying $\left[X_i, X_j\right] = c_{ij}^l X_l$ on each term in the LHS above gives, where the summation index l is used in each term, which is OK to do since the terms are separated from each others

$$\begin{aligned}0 &= \left[c_{ij}^l X_l, X_k\right] + \left[c_{jk}^l X_l, X_i\right] + \left[c_{ki}^l X_l, X_j\right]\\ &= c_{ij}^l\left[X_l, X_k\right] + c_{jk}^l\left[X_l, X_i\right] + c_{ki}^l\left[X_l, X_j\right]\end{aligned}$$

Now, applying $\left[X_i, X_j\right] = c_{ij}^m X_m$ again on each term above and now using m as the summation index gives

$$\begin{aligned}0 &= c_{ij}^l c_{lk}^m X_m + c_{jk}^l c_{li}^m X_m + c_{ki}^l c_{lj}^m\\ &= \left(c_{ij}^l c_{lk}^m + c_{jk}^l c_{li}^m + c_{ki}^l c_{lj}^m\right) X_m\\ &= c_{ij}^l c_{lk}^m + c_{jk}^l c_{li}^m + c_{ki}^l c_{lj}^m\end{aligned}$$

Which is what the problem asked to show.