1. $$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0 \qquad 0 \le x \le L_x$$
$$0 \le y \le L_y$$

$\psi(x,y,t) = f(x)\,g(y)\,T(t)$ separation of variables

$$\frac{1}{fgT}\left[f''gT + fg''T - \frac{1}{c^2} fgT'' \right] = 0$$

$$\frac{f''}{f} + \frac{g''}{g} = \frac{1}{c^2}\frac{T''}{T} = \text{constant} = -k^2$$

$$T'' = -k^2c^2T \Rightarrow T(t) = A_1\cos\omega t + B_1\sin\omega t \qquad \omega = kc$$

$$\frac{f''}{f} = -\frac{g''}{g} + \text{constant} = -p^2$$

$$\Rightarrow f(x) = A_2\cos px + B_2\sin px$$

and $g(y) = A_3\cos qy + B_3\sin qy$

Now k, p, and q are related by $p^2+q^2 = k^2$

Boundary conditions ; $f(0) = f(L_x) = 0 \Rightarrow A_2 = 0$

$g(0) = g(L_y) = 0 \Rightarrow A_3 = 0$

$$f(L_x) = B_2\sin(pL_x) = 0 \Rightarrow pL_x = m\pi \qquad m = 1,2,3,\ldots$$

$$g(L_y) = B_3\sin(qL_y) = 0 \Rightarrow qL_y = n\pi \qquad n = 1,2,3,\ldots$$

Thus $k^2 = \left(\frac{m^2}{L_x^2} + \frac{n^2}{L_y^2}\right)\pi^2$

$$\boxed{\omega_{mn} = \sqrt{\frac{m^2}{L_x^2} + \frac{n^2}{L_y^2}}\ \pi c}$$

$$\boxed{\psi_{mn}(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \sin\left(\frac{m\pi x}{L_x}\right) \sin\left(\frac{n\pi y}{L_y}\right) \cdot \left[A_{mn}\cos(\omega_{mn} t) + B_{mn}\sin(\omega_{mn} t)\right]}$$

(2) $$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)$$

$$+ \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0$$

We know that the solution which is zero or finite at the origin is

$$\psi(r,\theta,\phi,t) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} j_l(kr)\, Y_{lm}(\theta,\phi)\left[A_{lm}\cos(\omega t) + B_{lm}\sin(\omega t)\right]$$ where $\omega = kc$.

We only need to apply the boundary conditions.

$$j_l(x) = \sqrt{\frac{\pi}{2x}}\, J_{l+\frac{1}{2}}(x) = \sqrt{\frac{\pi}{4}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,\Gamma(n+l+\frac{3}{2})}\left(\frac{x}{2}\right)^{l+2n}$$

$$\frac{d}{dx} j_l(x) = \frac{\sqrt{\pi}}{4}\sum_{n=0}^{\infty}\frac{(-1)^n(l+2n)}{n!\,\Gamma(n+l+\frac{3}{2})}\left(\frac{x}{2}\right)^{l+2n-1} \quad \text{for } l \geq 1$$

$$\frac{d}{dx} j_0 = \frac{d}{dx}\frac{\sin x}{x} = \frac{\cos x}{x} - \frac{\sin x}{x^2}$$

Thus $\left.\frac{d}{dx} j_0(x)\right|_{x=0} = 0$

$$\left.\frac{d}{dx} j_1(x)\right|_{x=0} = \frac{1}{3} \qquad \left.\frac{d}{dx} j_l(x)\right|_{x=0} = 0 \quad \text{for } l \geq 2$$

Hence $l = 1$ is excluded.

For the rest we require that

$\frac{d}{dx} j_l(x) \Big|_{x = kR} = 0$. There are an infinite number of x's which satisfy this condition. Label them $x_{l,n}$ where n is the n'th one.

$$\boxed{\text{Finally } \psi(r,\theta,\phi,t) = \sum_{\substack{l=0 \\ l \neq 1}}^{\infty} \sum_{m=-l}^{l} \sum_{n=1}^{\infty} j_l(k_{l,n} r)\, Y_{lm}(\theta,\phi) \cdot \left[A_{lmn} \cos(k_{l,n} ct) + B_{lmn} \sin(k_{l,n} ct) \right]}$$

Lowest frequency corresponds to $l = 0$, $n = 1$.

$\frac{d}{dx} j_0(x) = \frac{\cos x}{x} - \frac{\sin x}{x^2}$ so $\tan x_{0,1} = x_{0,1}$

This is the same as encountered in lecture.

$x_{0,1} \approx 4.49$ so $\boxed{\omega_{0,1} = 4.49 \frac{c}{R}}$

③ In class we showed that

$$T(\vec{x},t) = \int d^3x' \, T(\vec{x}',0)\, G(\vec{x},t;\vec{x}') \quad \text{with}$$

$$G(\vec{x},t;\vec{x}') = \frac{e^{-(\vec{x}-\vec{x}')^2/4\kappa t}}{(4\pi\kappa t)^{3/2}}$$

$$\text{Now} \quad T(\vec{x}',0) = \begin{cases} 0 & \text{if } |\vec{x}'| < R \\ T_0 & \text{if } |\vec{x}'| > R \end{cases} = T_0\,\theta(r'-R)$$

$$T(\vec{x},t) = T_0 \int d^3x' \, \theta(r'-R) \, \frac{e^{-(\vec{x}-\vec{x}')^2/4\kappa t}}{(4\pi\kappa t)^{3/2}} =$$

$$= \frac{2\pi T_0}{(4\pi\kappa t)^{3/2}} \int_R^\infty dr'\, r'^2 \underbrace{\int_{-1}^{1} d(\cos\theta)\, e^{-(r^2+r'^2)/4\kappa t}\, e^{\frac{rr'}{2\kappa t}\cos\theta}}_{\frac{2\kappa t}{rr'}\left[e^{-(r-r')^2/4\kappa t} - e^{-(r+r')^2/4\kappa t}\right]}$$

$$= \frac{T_0}{r\sqrt{4\pi\kappa t}} \int_R^\infty dr'\, r' \left[e^{-(r-r')^2/4\kappa t} - e^{-(r+r')^2/4\kappa t} \right]$$

$$\text{Now} \int_R^\infty dr' r' e^{-(r'-r)^2/4\kappa t} = \sqrt{4\kappa t} \int_{\frac{R-r}{\sqrt{4\kappa t}}}^\infty du \left(\sqrt{4\kappa t}\, u + r\right) e^{-u^2} =$$

$$\uparrow$$

$$u = \frac{r'-r}{\sqrt{4\kappa t}} \qquad r' = \sqrt{4\kappa t}\, u + r$$

$$= 4\kappa t \int_{\frac{R-r}{\sqrt{4\kappa t}}}^\infty du\, u\, e^{-u^2} + \sqrt{4\kappa t}\, r \int_{\frac{R-r}{\sqrt{4\kappa t}}}^\infty du\, e^{-u^2} =$$

$$= 2\kappa t\, e^{-(R-r)^2/4\kappa t} + \sqrt{4\kappa t}\, r \frac{\sqrt{\pi}}{2}\left[1 - \operatorname{erf}\left(\frac{R-r}{\sqrt{4\kappa t}}\right)\right]$$

This is for $r \leq R$. The other integral is the same after replacing r with $-r$.

$$\boxed{\frac{T(r,t)}{T_0} = \frac{1}{r}\sqrt{\frac{\kappa t}{\pi}}\left[e^{-(R-r)^2/4\kappa t} - e^{-(R+r)^2/4\kappa t}\right] + 1 - \frac{1}{2}\operatorname{erf}\left(\frac{R-r}{\sqrt{4\kappa t}}\right) - \frac{1}{2}\operatorname{erf}\left(\frac{R+r}{\sqrt{4\kappa t}}\right)}$$

④ $\nabla^2 u + k^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + k^2 u = 0$

This is the same as the drum head problem worked out in lecture except that now the boundary condition is $u(R,\theta) = f(\theta)$ instead of $u(R,\theta) = 0$.

Look for a solution of the form

$$u(r,\theta) = \int_0^{2\pi} f(\theta')\, G(r,\theta;\theta')\, d\theta'$$

(i) $G(r,\theta;\theta')$ must satisfy the original differential equation

(ii) $G(R,\theta;\theta') = \delta(\theta-\theta')$ so that $u(R,\theta) = f(\theta)$

The general solution - without applying boundary conditions and which is finite at $r=0$ - is

$$G(r,\theta;\theta') = \sum_{n=0}^{\infty} J_n(kr)\left[A_n(\theta')e^{in\theta} + B_n(\theta')e^{-in\theta}\right]$$

$$G(R,\theta;\theta') = \sum_{n=0}^{\infty} J_n(kR)\left[A_n(\theta')e^{in\theta} + B_n(\theta')e^{-in\theta}\right]$$

$$= \delta(\theta-\theta')$$

Let's choose the representation

$$\delta(\theta-\theta') = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(\theta-\theta')} = \lim_{M\to\infty} \frac{1}{2\pi}\sum_{m=-M}^{M} e^{im(\theta-\theta')}$$

But to use this we need to sum over both positive and negative n so we write

$$G(r,\theta;\theta') = \sum_{n=-\infty}^{\infty} C_n(\theta')\, J_n(kr)\, e^{in\theta}$$

where $J_{-n}(x) = (-1)^n J_n(x)$.

$$G(R,\theta;\theta') = \sum_{n=-\infty}^{\infty} J_n(kR)\, C_n(\theta')\, e^{in\theta}$$

$$= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{in(\theta-\theta')}$$

$$\Rightarrow \quad C_n(\theta') = \frac{e^{-in\theta'}}{2\pi\, J_n(kR)}$$

$$\boxed{G(r,\theta;\theta') = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \frac{J_n(kr)}{J_n(kR)}\, e^{in(\theta-\theta')}}$$

This can also be written as

$$G(r,\theta;\theta') = \frac{1}{2\pi}\frac{J_0(kr)}{J_0(kR)} + \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{J_n(kr)}{J_n(kR)} \cos[n(\theta-\theta')]$$