HW 11
Physics 5041 Mathematical Methods for Physics
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Contents

1 Problem 1

Find the normal modes of a rectangular drum with sides of length L_x and L_y

solution

The geometry of the problem is

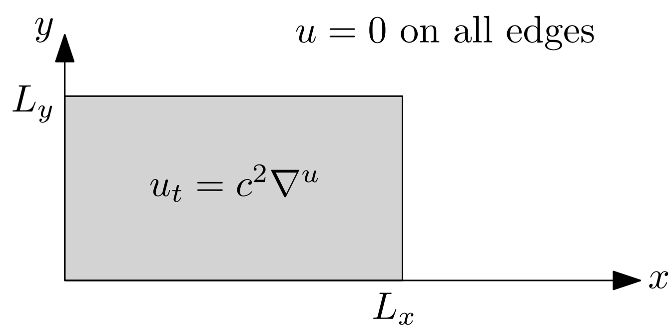

Figure 1: Problem to solve

Using Cartesian coordinates. Wave displacement is $u \equiv u(x, y, t)$ (out of page).

$$\frac{\partial^2 u(x,y,t)}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$
$$0 < x < L_x$$
$$0 < y < L_y$$

Boundary conditions on x

$$u(0, y, t) = 0$$
$$u(L_x, y, t) = 0$$

And boundary conditions on y

$$u(x, 0, t) = 0$$
$$u(x, L_y, t) = 0$$

Solution

Let $u = X(x)\,Y(y)\,T(t)$. Substituting into the PDE gives

$$\frac{1}{c^2}T''XY = X''YT + Y''XT$$
$$\frac{1}{c^2}\frac{T''}{T} = \frac{X''}{X} + \frac{Y''}{Y}$$

Hence, using λ as first separation constant we obtain

$$\frac{1}{c^2}\frac{T''}{T} = -\lambda$$
$$\frac{X''}{X} + \frac{Y''}{Y} = -\lambda$$

The time ODE becomes

$$T'' + c^2\lambda T = 0$$

And the space ODE becomes

$$\frac{X''}{X} + \frac{Y''}{Y} = -\lambda$$

Separating the space ODE again

$$\frac{X''}{X} = -\lambda - \frac{Y''}{Y} = -\mu$$

Where μ is the new separation variable. This gives two new separate ODE's

$$\frac{X''}{X} = -\mu$$
$$-\lambda - \frac{Y''}{Y} = -\mu$$

Or

$$X'' + \mu X = 0$$
$$Y'' + Y\left(\lambda - \mu\right) = 0$$

Solving for X ODE first, and knowing that $\mu > 0$ from nature of boundary conditions, we obtain

$$X(x) = A\cos\left(\sqrt{\mu}x\right) + B\sin\left(\sqrt{\mu}x\right)$$

Applying B.C. at $x = 0$

$$0 = A$$

Hence $X(x) = B\sin\left(\sqrt{\mu}x\right)$. Applying B.C. at $x = L_x$

$$0 = B\sin\left(\sqrt{\mu}L_x\right)$$

Hence

$$\sqrt{\mu}L_x = n\pi$$
$$\mu_n = \left(\frac{n\pi}{L_x}\right)^2 \qquad n = 1, 2, 3, \cdots \tag{1}$$

Therefore the $X_n(x)$ solution is

$$X_n(x) = B_n \sin\left(\frac{n\pi}{L_x}x\right) \qquad n = 1, 2, 3, \cdots \tag{2}$$

Solving the $Y\left(y\right)$ ODE using the same eigenvalues found above

$$Y'' + Y\left(\lambda - \left(\frac{n\pi}{L_x}\right)^2\right) = 0$$

The solution is

$$Y\left(y\right) = C\cos\left(\sqrt{\lambda - \left(\frac{n\pi}{L_x}\right)^2}\, y\right) + D\sin\left(\sqrt{\lambda - \left(\frac{n\pi}{L_x}\right)^2}\, y\right)$$

Applying first B.C. $Y(0) = 0$ gives

$$0 = C$$

Hence

$$Y\left(y\right) = D\sin\left(\sqrt{\lambda - \left(\frac{n\pi}{L_x}\right)^2}\, y\right)$$

Applying second B.C. $Y\left(L_y\right) = 0$

$$0 = D\sin\left(\sqrt{\lambda - \left(\frac{n\pi}{L_x}\right)^2}\, L_y\right)$$

Hence

$$\begin{aligned}
\sqrt{\lambda - \left(\frac{n\pi}{L_x}\right)^2}\, L_y &= m\pi \qquad m = 1,2,3,\cdots \\
\lambda_{nm} - \left(\frac{n\pi}{L_x}\right)^2 &= \left(\frac{m\pi}{L_y}\right)^2 \\
\lambda_{nm} &= \left(\frac{m\pi}{L_y}\right)^2 + \left(\frac{n\pi}{L_x}\right)^2 \qquad n = 1,2,3,\cdots, m = 1,2,3,\cdots
\end{aligned}$$

Hence the Y_{nm} solution is

$$Y_{nm} = D_{nm}\sin\left(\frac{m\pi}{L_y} y\right) \qquad n = 1,2,3,\cdots, m = 1,2,3,\cdots$$

We notice that $X_n\left(x\right)$ solution depends on n only, while $Y_{nm}\left(y\right)$ solution depends on n and m. Now that we found λ we can we solve the time $T\left(t\right)$ ode

$$\begin{aligned}
T''_{nm} + c^2\lambda_{nm}T_{nm} &= 0 \\
T_{nm}\left(t\right) &= E_{nm}\cos\left(c\sqrt{\lambda_{nm}}t\right) + F_{nm}\sin\left(c\sqrt{\lambda_{nm}}t\right)
\end{aligned}$$

Combining all solution , and merging all constants into two, we find

$$
\begin{aligned}
u_{nm}\left(x,y,t\right) &= X_n\left(x\right) Y_{nm}\left(y\right) T_{nm}\left(t\right)\\
&= \left(B_n X_n\right)\left(D_{nm}\sin\left(\frac{m\pi}{L_y}y\right)\right)\left(E_{nm}\cos\left(c\sqrt{\lambda_{nm}}t\right)+F_{nm}\sin\left(c\sqrt{\lambda_{nm}}t\right)\right)\\
&= B_n X_n \sin\left(\frac{m\pi}{L_y}y\right)\left(E'_{nm}\cos\left(c\sqrt{\lambda_{nm}}t\right)+F'_{nm}\sin\left(c\sqrt{\lambda_{nm}}t\right)\right)\\
&= X_n \sin\left(\frac{m\pi}{L_y}y\right)\left(E''_{nm}\cos\left(c\sqrt{\lambda_{nm}}t\right)+F''_{nm}\sin\left(c\sqrt{\lambda_{nm}}t\right)\right)
\end{aligned}
$$

Where E''_{nm}, F''_{nm} are the new constants after merging them with the other constants. Renaming $E''_{nm} = A_{nm}, F''_{nm} = B_{nm}$ the above solution can be written as

$$
\begin{aligned}
u\left(x,y,t\right) &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} X_n\left(x\right) Y_{mn}\left(y\right) T_{mn}\left(t\right)\\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm}\sin\left(\frac{n\pi}{L_x}x\right)\sin\left(\frac{m\pi}{L_y}y\right)\cos\left(c\sqrt{\lambda_{nm}}t\right)\\
&+ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{nm}\sin\left(\frac{n\pi}{L_x}x\right)\sin\left(\frac{m\pi}{L_y}y\right)\sin\left(c\sqrt{\lambda_{nm}}t\right) \qquad (3)
\end{aligned}
$$

To solve this completely, we apply initial conditions to find A_{nm}, B_{nm}. But the problem is just asking for the normal modes. These are given by $X_n\left(x\right) Y_{mn}\left(y\right)$. Therefore for $n=1$, we have the modes $\sin\left(\frac{\pi}{L_x}x\right)\sin\left(\frac{\pi}{L_y}y\right), \sin\left(\frac{\pi}{L_x}x\right)\sin\left(\frac{2\pi}{L_y}y\right), \sin\left(\frac{\pi}{L_x}x\right)\sin\left(\frac{3\pi}{L_y}y\right), \cdots$ and for $n=2$ we have $\sin\left(\frac{2\pi}{L_x}x\right)\sin\left(\frac{\pi}{L_y}y\right), \sin\left(\frac{2\pi}{L_x}x\right)\sin\left(\frac{2\pi}{L_y}y\right), \sin\left(\frac{2\pi}{L_x}x\right)\sin\left(\frac{3\pi}{L_y}y\right), \cdots$ and so on.

n	$m=1$	2	3	4
1	$\sin\left(\frac{\pi}{L_x}x\right)\sin\left(\frac{\pi}{L_y}y\right)$	$\sin\left(\frac{\pi}{L_x}x\right)\sin\left(\frac{2\pi}{L_y}y\right)$	$\sin\left(\frac{\pi}{L_x}x\right)\sin\left(\frac{3\pi}{L_y}y\right)$	$\cdots$
2	$\sin\left(\frac{2\pi}{L_x}x\right)\sin\left(\frac{\pi}{L_y}y\right)$	$\sin\left(\frac{2\pi}{L_x}x\right)\sin\left(\frac{2\pi}{L_y}y\right)$	$\sin\left(\frac{2\pi}{L_x}x\right)\sin\left(\frac{3\pi}{L_y}y\right)$	$\cdots$
3	$\sin\left(\frac{3\pi}{L_x}x\right)\sin\left(\frac{\pi}{L_y}y\right)$	$\sin\left(\frac{3\pi}{L_x}x\right)\sin\left(\frac{2\pi}{L_y}y\right)$	$\sin\left(\frac{3\pi}{L_x}x\right)\sin\left(\frac{3\pi}{L_y}y\right)$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

To draw these modes, let us assume that $L_x = 1, L_y = 1$. This gives

n	$m=1$	2	3	4
1	$\sin\left(\pi x\right)\sin\left(\pi y\right)$	$\sin\left(\pi x\right)\sin\left(2\pi y\right)$	$\sin\left(\pi x\right)\sin\left(3\pi y\right)$	$\cdots$
2	$\sin\left(2\pi x\right)\sin\left(\pi y\right)$	$\sin\left(2\pi x\right)\sin\left(2\pi y\right)$	$\sin\left(2\pi x\right)\sin\left(3\pi y\right)$	$\cdots$
3	$\sin\left(3\pi x\right)\sin\left(\pi y\right)$	$\sin\left(3\pi x\right)\sin\left(2\pi y\right)$	$\sin\left(3\pi x\right)\sin\left(3\pi y\right)$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

The following is a plot of the above modes for illustrations with the code used to generate these plots.

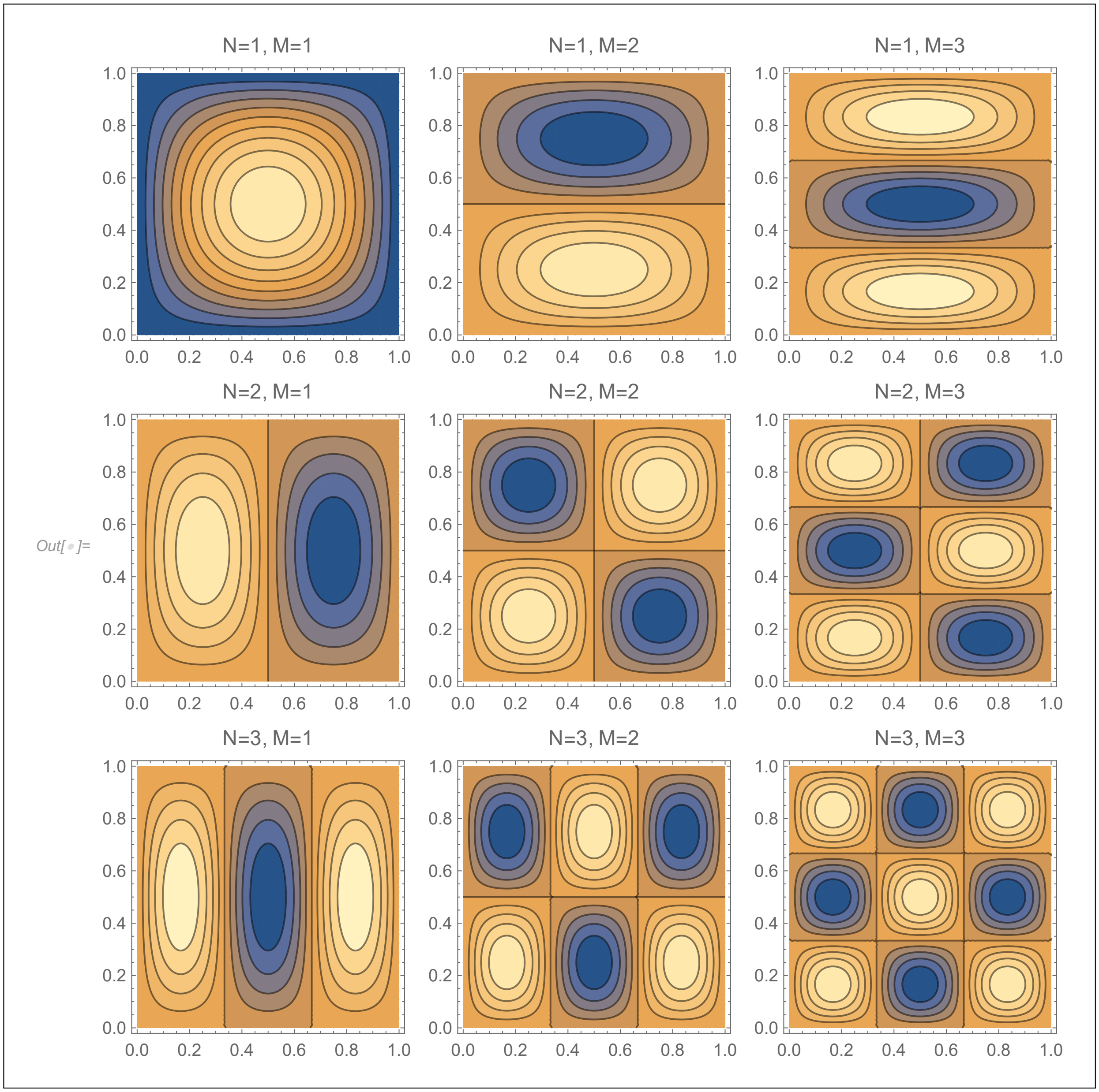

Figure 2: Modes using $L_x = 1, L_y = 1$

```
makePlot[n_, m_] :=
    ContourPlot[Sin[n Pi x] * Sin[m Pi y], {x, 0, 1}, {y, 0, 1},
      PlotLegends → None,
      Frame → True, FrameLabel → {{None, None}, {None, Style[Row[{"N=", n, ", M=", m}], 12]}}];
Grid@Table[makePlot[n, m], {n, 1, 3}, {m, 1, 3}]
```

Figure 3: Code used to draw above plot

The following is 3D view of the above modes.

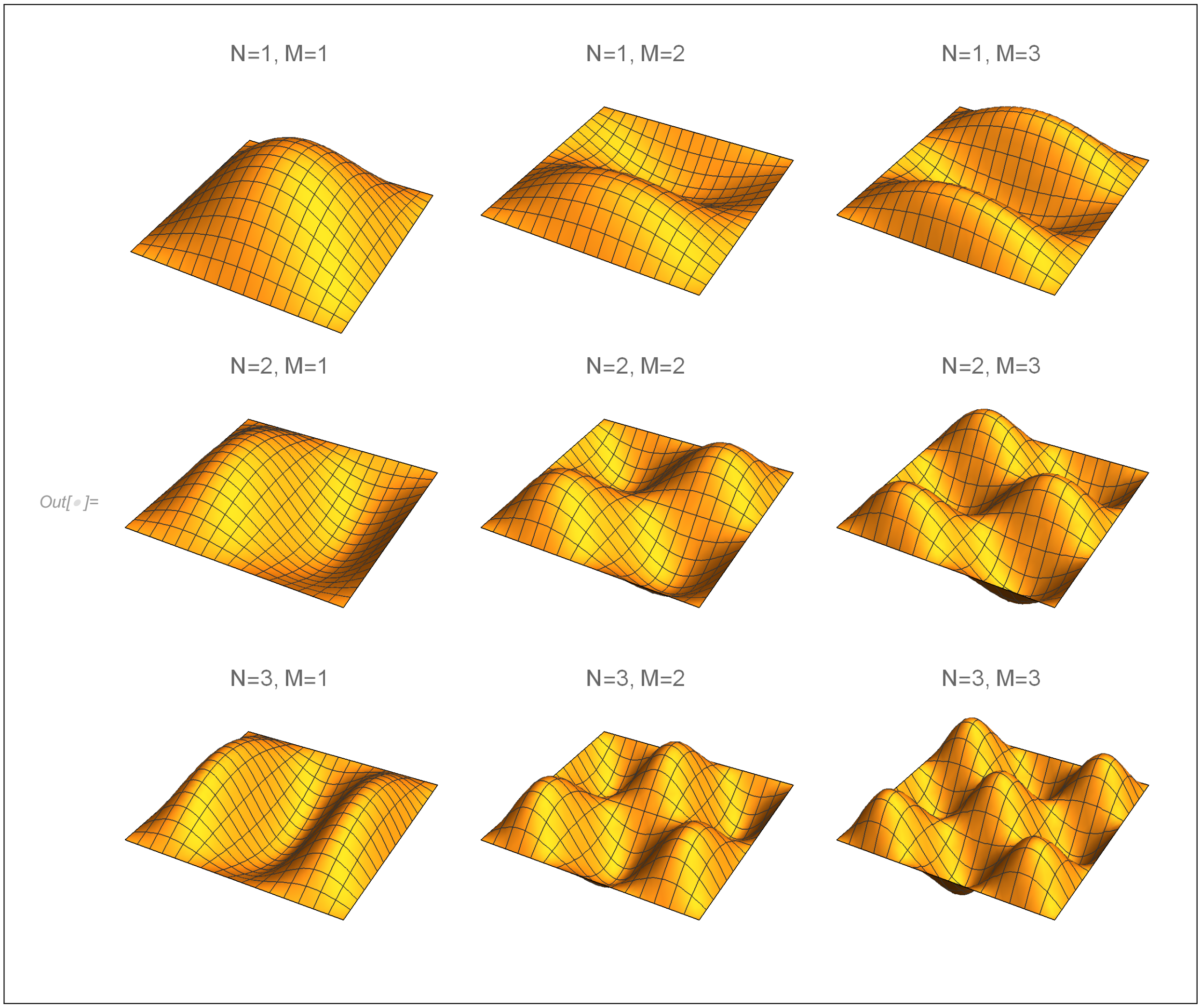

Figure 4: 3D view of the modes using $L_x = 1, L_y = 1$

```
In[ ]:= makePlot[n_, m_] :=
          Plot3D[Sin[n Pi x] * Sin[m Pi y], {x, 0, 1}, {y, 0, 1},
            PlotLabel → Style[Row[{"N=", n, ", M=", m}], 12],
            Boxed -> False, Axes → False
          ];
        Grid@Table[makePlot[n, m], {n, 1, 3}, {m, 1, 3}]
```

Figure 5: Code used to draw above plot

2 Problem 2

Find the normal modes of an acoustic waves in a hollow sphere of radius R. The wave equation is

$$\nabla^2\psi\left(r,\theta,\phi,t\right)=\frac{1}{c^2}\psi_{tt}$$

With boundary conditions $\psi_r=0$ at $r=0$ and at $r=r_0$. (I used r_0 in place of R because wanted to use $R(r)$ for separation of variables).

What is the lowest frequency?

solution

Let

$$\psi\left(r,\theta,\phi,t\right)=u\left(r,\theta,\phi\right)e^{-i\omega t}$$

Substituting this back in the original PDE gives

$$\nabla^2u\left(r,\theta,\phi\right)+\frac{\omega^2}{c^2}u\left(r,\theta,\phi\right)=0$$

Let $k=\frac{\omega}{c}$ (wave number) and the above becomes

$$\nabla^2u+k^2u=0 \tag{1}$$

The above is called the Helmholtz PDE. In spherical coordinates it becomes

$$\overbrace{u_{rr}+\frac{2}{r}u_r}^{\text{Radial part}}+\overbrace{\frac{1}{r^2}\left(\frac{\cos\theta}{\sin\theta}u_\theta+u_{\theta\theta}\right)+\frac{1}{r^2\sin^2\theta}u_{\phi\phi}}^{\text{Angular part}}+k^2u=0$$

Let $u\left(r,\theta,\phi\right)=R(r)\Theta(\theta)\Phi\left(\phi\right)$ and the above becomes

$$R''T\Theta\Phi+\frac{2}{r}R'T\Theta\Phi+\frac{1}{r^2}\left(\frac{\cos\theta}{\sin\theta}\Theta'RT\Phi+\Theta''RT\Phi\right)+\frac{1}{r^2\sin^2\theta}\Phi''R\Theta T+k^2R\Theta T=0$$

Dividing by $R\Theta\Phi\neq0$ gives

$$\frac{R''}{R}+\frac{2}{r}\frac{R'}{R}+\frac{1}{r^2}\left(\frac{\cos\theta}{\sin\theta}\frac{\Theta'}{\Theta}+\frac{\Theta''}{\Theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\Phi''}{\Phi}+k^2=0$$

$$r^2\sin^2\theta\frac{R''}{R}+r^2\sin^2\theta\frac{2}{r}\frac{R'}{R}+\sin^2\theta\left(\frac{\cos\theta}{\sin\theta}\frac{\Theta'}{\Theta}+\frac{\Theta''}{\Theta}\right)+k^2r^2\sin^2\theta=-\frac{\Phi''}{\Phi}$$

The left side depends only on r,θ and the right side depends only on ϕ. Let the second separation constant be m^2 and the above becomes

$$r^2\sin^2\theta\frac{R''}{R}+r^2\sin^2\theta\frac{2}{r}\frac{R'}{R}+\sin^2\theta\left(\frac{\cos\theta}{\sin\theta}\frac{\Theta'}{\Theta}+\frac{\Theta''}{\Theta}\right)+k^2r^2\sin^2\theta=-\frac{\Phi''}{\Phi}=m^2 \tag{2}$$

Which gives the first angular ODE as

$$\Phi''+m^2\Phi=0 \tag{2A}$$

We now go back to (2) to obtain the rest of the solutions. We now have

$$r^2\sin^2\theta\frac{R''}{R}+r^2\sin^2\theta\frac{2}{r}\frac{R'}{R}+\sin^2\theta\left(\frac{\cos\theta}{\sin\theta}\frac{\Theta'}{\Theta}+\frac{\Theta''}{\Theta}\right)+k^2r^2\sin^2\theta=m^2$$

$$k^2r^2+r^2\left(\frac{R''}{R}+\frac{2}{r}\frac{R'}{R}\right)+\left(\frac{\cos\theta}{\sin\theta}\frac{\Theta'}{\Theta}+\frac{\Theta''}{\Theta}\right)=\frac{m^2}{\sin^2\theta}$$

$$k^2r^2+r^2\left(\frac{R''}{R}+\frac{2}{r}\frac{R'}{R}\right)=-\left(\frac{\cos\theta}{\sin\theta}\frac{\Theta'}{\Theta}+\frac{\Theta''}{\Theta}\right)+\frac{m^2}{\sin^2\theta}$$

The left side depends on r and the right side depends on θ only. Let the separation constant be $l(l+1)$ where l is integer which results in

$$k^2r^2+r^2\left(\frac{R''}{R}+\frac{2}{r}\frac{R'}{R}\right)=-\left(\frac{\cos\theta}{\sin\theta}\frac{\Theta'}{\Theta}+\frac{\Theta''}{\Theta}\right)+\frac{m^2}{\sin^2\theta}=l(l+1) \tag{3}$$

Therefore the next angular ODE is

$$-\left(\frac{\cos\theta}{\sin\theta}\frac{\Theta'}{\Theta}+\frac{\Theta''}{\Theta}\right)+\frac{m^2}{\sin^2\theta}=l(l+1)$$

$$-\left(\frac{\cos\theta}{\sin\theta}\frac{\Theta'}{\Theta}+\frac{\Theta''}{\Theta}\right)+\frac{m^2}{\sin^2\theta}-l(l+1)=0$$

$$\left(\frac{\cos\theta}{\sin\theta}\frac{\Theta'}{\Theta}+\frac{\Theta''}{\Theta}\right)-\frac{m^2}{\sin^2\theta}+l(l+1)=0$$

$$\Theta''+\frac{\cos\theta}{\sin\theta}\Theta'+\left(l(l+1)-\frac{m^2}{\sin^2\theta}\right)\Theta=0 \tag{4}$$

Let $z=\cos\theta$, then $\frac{d\Theta}{d\theta}=\frac{d\Theta}{dz}\frac{dz}{d\theta}=-\frac{d\Theta}{dz}\sin\theta$ and

$$\begin{aligned}\frac{d^2\Theta}{d\theta^2}&=\frac{d}{d\theta}\left(-\frac{d\Theta}{dz}\sin\theta\right)\\&=-\frac{d^2\Theta}{dz^2}\frac{dz}{d\theta}\sin\theta-\frac{d\Theta}{dz}\cos\theta\\&=\frac{d^2\Theta}{dz^2}\sin^2\theta-\frac{d\Theta}{dz}\cos\theta\end{aligned}$$

But $\sin^2\theta=1-\cos^2\theta=1-z^2$ and the above becomes

$$\frac{d^2\Theta}{d\theta^2}=\frac{d^2\Theta}{dz^2}\left(1-z^2\right)-\frac{d\Theta}{dz}z$$

Using these in (4) gives

$$\frac{d^2\Theta}{dz^2}\left(1-z^2\right)-\frac{d\Theta}{dz}z+\frac{z}{\sin\theta}\left(-\frac{d\Theta}{dz}\sin\theta\right)+\left(l(l+1)-\frac{m^2}{1-z^2}\right)\Theta(z)=0$$

$$\left(1-z^2\right)\Theta''-2z\Theta'+\left(l(l+1)-\frac{m^2}{1-z^2}\right)\Theta(z)=0 \tag{3A}$$

And finally, we obtain the final ODE, which is the radial ODE from (3)

$$
\begin{aligned}
k^2r^2 + r^2\left(\frac{R''}{R} + \frac{2}{r}\frac{R'}{R}\right) &= l\,(l+1)\\
k^2r^2R + r^2\left(R'' + \frac{2}{r}R'\right) - l\,(l+1)\,R &= 0\\
r^2R'' + 2rR' + \left(k^2r^2 - l\,(l+1)\right)R &= 0\\
R'' + \frac{2}{r}R' + \left(k^2 - \frac{l\,(l+1)}{r^2}\right)R &= 0 \qquad (4A)
\end{aligned}
$$

In summary we have obtained the following 4 ODE's to solve (1A,2A,3A,4A)

$$
\begin{aligned}
\Phi'' + m^2\Phi &= 0 \qquad (2A)\\
\left(1-z^2\right)\Theta'' - 2z\Theta' + \left(l\,(l+1) - \frac{m^2}{1-z^2}\right)\Theta\,(z) &= 0 \qquad (3A)\\
R'' + \frac{2}{r}R' + \left(k^2 - \frac{l\,(l+1)}{r^2}\right)R &= 0 \qquad (4A)
\end{aligned}
$$

Solution to (2A) requires m to be integer due to periodicity requirements of solution. The solution is $\Phi\left(\phi\right) = e^{\pm im\phi}$. Equation (3A) is the associated Legendre ODE. Since we are taking l as integer then the solution is known to be $\Theta\,(z) = P_l^m\,(z) + Q_l^m\,(z)$ where $P_l^m\,(z)$ is called the associated Legendre polynomial and Q_l^m is the Legendre function of the second kind. Finally (4A) can be converted to Bessel ODE as shown in class notes using the transformation $R\,(r) = \frac{u(r)}{\sqrt{r}}$ which results in

$$
u'' + \frac{1}{r}u' + \left(k^2 - \frac{\left(l+\frac{1}{2}\right)^2}{r^2}\right)u = 0
$$

Which has solution $J_{l+\frac{1}{2}}\,(kr)$. The second solution $J_{-\left(l+\frac{1}{2}\right)}\,(kr)$ is rejected since it is not finite at zero and hence makes the solution blow up at center of sphere. Therefore solution to (4A) is

$$
\begin{aligned}
R\,(r) &= C\sqrt{\frac{\pi}{2kr}}J_{l+\frac{1}{2}}\,(kr)\\
&= Cj_l\,(kr)
\end{aligned}
$$

Where C is arbitrary constant. Putting all the above together, then the final solution is

$$
\psi\left(r,\theta,\phi,t\right) = \left\{\ e^{-i\omega t}\ \left\{\begin{array}{c} e^{im\phi}\\ e^{-im\phi}\end{array}\right.\ \left\{\begin{array}{c} P_l^m\,(\cos\theta)\\ Q_l^m\,(\cos\theta)\end{array}\right.\ \left\{\ j_l\,(kr)\right.\right.
$$

Where $j_l\,(kr)$ are the spherical Bessel functions. Now we need to satisfy the boundary conditions. Since only $j_l\,(kr)$ depends on r, then $\psi_r = 0$ at $r = 0$ and at $r = r_0$ are equivalent to looking at $R'\,(r) = 0$ at $r = 0$ and $r = r_0$. Therefore we need to find the smallest l, k which satisfy both conditions. This will give the lowest frequency.

I found from DLMF that the series expansion of $j_l(kr)$ is

$$j_l(kr) = \frac{(kr)^l}{(2l+1)!!}\left(1 - \frac{(kr)^2}{2(2l+3)} + \frac{(kr)^4}{8(2l+5)(2l+3)} + \cdots\right) \tag{5}$$

Hence for $r \to 0$, we can approximate the above as the following by ignoring all higher order terms

$$\lim_{r\to 0} j_l(kr) = \frac{(kr)^l}{(2l+1)!!}$$

Which means for small r, the derivative is

$$\frac{d}{dr} j_l(kr) = \frac{l(kr)^{l-1}}{(2l+1)!!}$$

At $r = 0$ then setting $\left[\frac{d}{dr} j_l(kr)\right]_{r\to 0} = 0$ is satisfied for all l. Now taking derivative of (5) gives

$$\frac{d}{dr} j_l(kr) = \frac{l(kr)^{l-1}}{(2l+1)!!}\left(1 - \frac{(kr)^2}{2(2l+3)} + \frac{(kr)^4}{8(2l+5)(2l+3)} + \cdots\right) + \frac{(kr)^l}{(2l+1)!!}\left(1 - \frac{2(kr)}{2(2l+3)} + \frac{4(kr)^3}{8(2l+5)(2l+3)} + \cdots\right)$$

At $r = r_0$ the above becomes

$$\left[\frac{d}{dr} j_l(kr)\right]_{r\to r_0} = \frac{l(kr_0)^{l-1}}{(2l+1)!!}\left(1 - \frac{(kr_0)^2}{2(2l+3)} + \frac{(kr_0)^4}{8(2l+5)(2l+3)} + \cdots\right) + \frac{(kr_0)^l}{(2l+1)!!}\left(1 - \frac{2(kr_0)}{2(2l+3)} + \frac{4(kr_0)^3}{8(2l+5)(2l+3)} + \cdots\right)$$

Now we ask, for which values of l is the above zero? If we let $l \to \infty$ then we obtain

$$\begin{aligned}\left[\frac{d}{dr} j_l(kr)\right]_{\substack{r\to r_0\\ l\to\infty}} &= \lim_{l\to\infty} \frac{l(kr_0)^{l-1}}{(2l+1)!!} + \frac{(kr_0)^l}{(2l+1)!!}\\ &= 0\end{aligned}$$

Therefore, to satisfy both $\left[\frac{d}{dr} j_l(kr)\right]_{r\to 0} = 0$ and $\left[\frac{d}{dr} j_l(kr)\right]_{r\to r_0} = 0$ we need $l \to \infty$. In other words, a very large integer. The larger l is, the lower the radial frequency. In addition, increasing k while keeping l fixed will increase the frequency. And decreasing k while keeping l fixed decreases the frequency. And for fixed k, increasing l decreases the frequency.

3 Problem 3

A sphere of radius R is at temperature $u = 0$. At time $t = 0$ it is immersed in a heat bath of temperature u_0. What is the temperature distribution $u(r,t)$ as function of time?

solution

Note: I Used $u(r,t)$ instead of $T(r,t)$ as the dependent variable to allow using $T(t)$ for separation of variables without confusing it with the original $T(r,t)$.

The PDE specification is, solve for $u(r,t)$

$$u_t = k\nabla^2 u \qquad t > 0, 0 < r < R$$

With initial conditions

$$u(r,0) = 0$$

And boundary conditions

$$\begin{aligned} u(R,t) &= u_0 \\ |u(0,t)| &< \infty \end{aligned}$$

Where the second B.C. above means the temperature u is bounded at origin (center of sphere). In spherical coordinates, the PDE becomes (There are no dependency on θ, ϕ due to symmetry), and only radial dependency.

$$\frac{1}{k}u_t = \frac{1}{r}(ru)_{rr} \tag{1}$$

To simplify the solution, let

$$U(r,t) = ru(r,t)$$

And we obtain a new PDE

$$\frac{1}{k}U_t = U_{rr} \tag{2}$$

And the boundary conditions $u(R,t) = u_0$ becomes $U(R,t) = Ru_0$ and the initial conditions becomes $U(r,0) = 0$. So we will solve (2) and not (1). But since the boundary conditions are not homogenous, we can not use separation of variables. We introduce a reference function $w(r)$ which need to satisfy the nonhomogeneous boundary conditions only. Let $w(r) = Br$. When $r = R$ then $Ru_0 = BR$ or $B = u_0$ When $r = 0$ then $w = 0$ which is bounded. Hence

$$w(r) = u_0 r$$

Therefore, the solution now can be written as

$$U(r,t) = v(r,t) + u_0 r \tag{3}$$

Where $v(r,t)$ now satisfies the PDE but with homogenous B.C. Substituting (3) into (2) gives

$$\begin{aligned} v_t &= k\frac{\partial^2}{\partial r^2}(v(r,t) + u_0 r) \\ v_t &= kv_{rr}(r,t) \end{aligned} \tag{4}$$

We need to solve the above but with homogenous boundary conditions

$$v(R,t) = 0$$
$$|v(0,t)| < \infty$$

This is standard PDE, who can be solved by separation of variables. let $v = F(r)T(t)$, hence (4) becomes

$$T'F = kF''T$$
$$k\frac{T'}{T} = \frac{F''}{F} = -\lambda^2$$

Which gives

$$F'' + \lambda^2 F = 0$$

Due to boundary conditions only $\lambda > 0$ is eigenvalues. Hence solution is

$$F(r) = A\cos(\lambda r) + B\sin(\lambda r)$$

At $r = 0$, since bounded, say 0, then we can take $A = 0$, leaving the solution

$$F(r) = B\sin(\lambda r)$$

At $r = R$

$$0 = B\sin(\lambda R)$$

For nontrivial solution

$$\lambda R = n\pi \qquad n = 1,2,3,\cdots$$
$$\lambda_n = \frac{n\pi}{R}$$

Hence eigenfunctions are

$$F_n(r) = \sin\left(\frac{n\pi}{R}r\right) \qquad n = 1,2,3,\cdots$$

The time ODE is therefore $T' + \lambda^2 kT = 0$ with solution $T_n(t) = A_n e^{-\left(\frac{n\pi}{R}\right)^2 kt}$. Hence the solution to (4) is

$$v(r,t) = \sum_{n=1}^{\infty} A_n e^{-\left(\frac{n\pi}{R}\right)^2 kt} \sin\left(\frac{n\pi}{R}r\right)$$

Therefore from (3)

$$U(r,t) = \left(\sum_{n=1}^{\infty} A_n e^{-\left(\frac{n\pi}{R}\right)^2 kt} \sin\left(\frac{n\pi}{R}r\right)\right) + u_0 r$$

But $U(r,t) = ru(r,t)$, hence

$$u(r,t) = \left(\frac{1}{r}\sum_{n=1}^{\infty} A_n e^{-\left(\frac{n\pi}{R}\right)^2 kt} \sin\left(\frac{n\pi}{R}r\right)\right) + u_0 \tag{5}$$

Now we find A_n from initial conditions. At $t=0$

$$0 = u_0 + \frac{1}{r}\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{R}r\right)$$

$$-ru_0 = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{R}r\right)$$

Therefore A_n are the Fourier series coefficients of $-ru_0$

$$\begin{aligned}
\frac{R}{2}A_n &= -\int_0^R ru_0 \sin\left(\frac{n\pi}{R}r\right) dr \\
A_n &= -\frac{2u_0}{R}\int_0^R r\sin\left(\frac{n\pi}{R}r\right) dr \\
&= -\frac{2u_0}{R}(-1)^{n+1}\frac{R^2}{n\pi} \\
&= (-1)^n \frac{2R}{n\pi}u_0
\end{aligned}$$

Hence the solution (5) becomes

$$\begin{aligned}
u(r,t) &= u_0 + u_0\frac{2R}{r\pi}\sum_{n=1}^{\infty}(-1)^n \frac{1}{n}e^{-k\left(\frac{n\pi}{R}\right)^2 t}\sin\left(\frac{n\pi}{R}r\right) \\
&= u_0\left(1 + \frac{2R}{r\pi}\sum_{n=1}^{\infty}(-1)^n \frac{1}{n}e^{-k\left(\frac{n\pi}{R}\right)^2 t}\sin\left(\frac{n\pi}{R}r\right)\right) \qquad (7)
\end{aligned}$$

Verification of solution

Verification that (7) satisfies the PDE $u_t = k\nabla^2 u$. Taking time derivative of (7) gives

$$u_t = -u_0\frac{2R}{r\pi}k\sum_{n=1}^{\infty}(-1)^n \frac{1}{n}\left(\frac{n\pi}{R}\right)^2 e^{-k\left(\frac{n\pi}{R}\right)^2 t}\sin\left(\frac{n\pi}{R}r\right) \qquad (8)$$

And taking space derivatives of (7) gives

$$\begin{aligned}
u_x &= u_0\frac{2R}{r\pi}\sum_{n=1}^{\infty}(-1)^n \frac{1}{n}e^{-k\left(\frac{n\pi}{R}\right)^2 t}\frac{n\pi}{R}\cos\left(\frac{n\pi}{R}r\right) \\
u_{xx} &= -u_0\frac{2R}{r\pi}\sum_{n=1}^{\infty}(-1)^n \frac{1}{n}e^{-k\left(\frac{n\pi}{R}\right)^2 t}\left(\frac{n\pi}{R}\right)^2\sin\left(\frac{n\pi}{R}r\right)
\end{aligned}$$

Hence ku_{xx} becomes

$$ku_{xx} = -u_0\frac{2R}{r\pi}k\sum_{n=1}^{\infty}(-1)^n \frac{1}{n}e^{-k\left(\frac{n\pi}{R}\right)^2 t}\left(\frac{n\pi}{R}\right)^2\sin\left(\frac{n\pi}{R}r\right) \qquad (9)$$

Comparing (8) and (9) shows they are the same expressions.

Verification that (7) satisfies the boundary condition.

When $r=R$, therefore (7) gives, when replacing r by R

$$
\begin{aligned}
u(R,t) &= u_0\left(1+\frac{2R}{R\pi}\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}e^{-k\left(\frac{n\pi}{R}\right)^2 t}\sin\left(\frac{n\pi}{R}R\right)\right)\\
&= u_0\left(1+\frac{2R}{R\pi}\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}e^{-k\left(\frac{n\pi}{R}\right)^2 t}\sin(n\pi)\right)\\
&= u_0(1+0)\\
&= u_0
\end{aligned}
$$

But n is integer. Hence $\sin(n\pi)=0$ for all n. And the above becomes

$$
\begin{aligned}
u(R,t) &= u_0(1+0)\\
&= u_0
\end{aligned}
$$

Verified.

Verification that (7) satisfies the initial conditions $u(r,0)=0$ for $r<R$.

At $t=0$ (7) becomes

$$
\begin{aligned}
u(r,0) &= u_0\left(1+\frac{2R}{r\pi}\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}\sin\left(\frac{n\pi}{R}r\right)\right)\\
&= u_0+\frac{2R}{r\pi}u_0\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\left(\frac{n\pi}{R}r\right)\\
&= u_0+\frac{2R}{r\pi}u_0\left(-\sin\left(\frac{\pi}{R}r\right)+\frac{1}{2}\sin\left(\frac{2\pi}{R}r\right)-\frac{1}{3}\sin\left(\frac{3\pi}{R}r\right)+\frac{1}{4}\sin\left(\frac{4\pi}{R}r\right)-\cdots\right)
\end{aligned}
$$

I could not simplify the above by hand, but using the computer, I verified numerically it is zero for $0<r<R$ for a given R and given u_0.

```
In[]:= ClearAll[R, r]
       R = 1; (*radius*)
       u0 = 10; (*B.C. value*)
       s = Sum[(-1)^n 1/n Sin[n Pi/R r], {n, 1, Infinity}] (*obtain sum*)
       Table[Chop[u0 + (2 R)/(r Pi) u0*s], {r, 0.05, R, .05}]

Out[]= -1/2 i (-Log[1 + e^(i π r)] + Log[e^(-i π r) (1 + e^(i π r))])

Out[]= {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

Figure 6: Obtaining the sum using the computer

4 Problem 4

Consider the Helmholtz equation

$$\nabla^2 u(r,\theta) + k^2 u(r,\theta) = 0 \tag{1}$$

inside the circle $r = r_0$ with the boundary condition $u(r_0,\theta) = f(\theta)$. The solution can be written in the form $u(r,\theta) = \int_0^{2\pi} f(\theta') G(r,\theta;\theta') d\theta'$. Find the Green function G.

solution

I will solve (1) directly and then compare the solution obtain to $u(r,\theta) = \int_0^{2\pi} f(\theta') G(r,\theta;\theta') d\theta'$ in order to read off the Green function expression. (1) in polar coordinates becomes

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + k^2 u = 0$$

Writing $u(r,\theta) = R(r)\Theta(\theta)$, the above PDE becomes

$$\begin{aligned} R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}\Theta'' R + k^2 R\Theta &= 0 \\ \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + k^2 &= 0 \\ r^2\frac{R''}{R} + r\frac{R'}{R} + r^2k^2 &= -\frac{\Theta''}{\Theta} = m \end{aligned}$$

Where m is the separation constant. The eigenvalue problem is taken as

$$\Theta'' + m\Theta = 0$$

Due to periodicity of the solution on the disk, then $\Theta(-\pi) = \Theta(\pi)$ and $\Theta'(-\pi) = \Theta'(\pi)$. These boundary conditions restrict m to only positive integer values. Hence let $m = n^2$ and the solution to the above becomes

$$\Theta_\alpha(\theta) = A_n \cos(n\theta) + B_n \sin(n\theta)$$

Now the radial ODE is

$$\begin{aligned} r^2\frac{R''}{R} + r\frac{R'}{R} + r^2k^2 &= \alpha^2 \\ r^2R'' + rR' + \left(r^2k^2 - n^2\right)R &= 0 \\ R'' + \frac{1}{r}R' + \left(k^2 - \frac{n^2}{r^2}\right)R &= 0 \end{aligned}$$

This is Bessel ODE whose solutions are (since n are integers) is

$$R_\alpha(r) = C_n J_n(kr) + E_n Y_n(kr)$$

But $Y_n(kr)$ blows up at $r = 0$, hence it is rejected leaving solution $R_n(r) = C_n J_n(kr)$. Hence the final solution is

$$u(r,\theta)=\sum_{m=1}^{\infty}\left(A_n\cos(n\theta)+B_n\sin(n\theta)\right)J_n(kr) \tag{2}$$

Where the constant C_n is merged with the other two constants. Now, at $r=r_0$ we are told that $u(r_0,\theta)=f(\theta)$. Hence the above becomes

$$f(\theta)=\sum_{m=1}^{\infty}\left(A_n\cos(n\theta)+B_n\sin(n\theta)\right)J_n(kr_0)$$

By orthogonality of $\cos(n\theta),\sin(n\theta)$ we find the Fourier cosine and Fourier sine coefficients A_n,B_n as

$$\begin{aligned}A_nJ_n(kr_0)\frac{1}{\pi}&=\int_0^{2\pi}f(\theta)\cos(n\theta)\,d\theta\\ B_nJ_n(kr_0)\frac{1}{\pi}&=\int_0^{2\pi}f(\theta)\sin(n\theta)\,d\theta\end{aligned}$$

Substituting the above back into the solution found in (2) results in

$$\begin{aligned}u(r,\theta)&=\sum_{m=1}^{\infty}\left[\left(\frac{\pi}{J_n(kr_0)}\int_0^{2\pi}f(\theta')\cos(n\theta')\,d\theta'\right)\cos(n\theta)+\left(\frac{\pi}{J_n(kr_0)}\int_0^{2\pi}f(\theta')\sin(n\theta')\,d\theta'\right)\sin(n\theta)\right]J_n(kr)\\ &=\sum_{m=1}^{\infty}\frac{\pi}{J_n(kr_0)}\left(\int_0^{2\pi}f(\theta')\cos(n\theta')\cos(n\theta)\,d\theta'+\int_0^{2\pi}f(\theta')\sin(n\theta')\sin(n\theta)\,d\theta'\right)J_n(kr)\end{aligned} \tag{3}$$

Using trig relations

$$\begin{aligned}\cos A\cos B&=\frac{1}{2}\left(\cos(A+B)+\cos(A-B)\right)\\ \sin A\sin B&=\frac{1}{2}\left(\cos(A-B)-\cos(A+B)\right)\end{aligned}$$

Then (3) becomes

$$u(r,\theta)=\sum_{m=1}^{\infty}\frac{\pi}{2J_n(kr_0)}\left(\int_0^{2\pi}f(\theta')\left(\cos(n(\theta'+\theta))+\cos(n(\theta'-\theta))\right)d\theta'+\int_0^{2\pi}f(\theta')\left(\cos(n(\theta'-\theta))-\cos(n(\theta'+\theta))\right)d\theta'\right)J_n(k$$

Which is simplified to, after combining both integrals to one

$$\begin{aligned}u(r,\theta)&=\sum_{m=1}^{\infty}\frac{\pi}{2J_n(kr_0)}\left(\int_0^{2\pi}f(\theta')\left(\cos(n(\theta'+\theta))+\cos(n(\theta'-\theta))+\cos(n(\theta'-\theta))-\cos n(\theta'+\theta)\right)d\theta'\right)J\\ &=\sum_{m=1}^{\infty}\frac{\pi}{2J_n(kr_0)}\left[\int_0^{2\pi}f(\theta')\,2\cos(\theta'-\theta)\,d\theta'\right]J_n(kr)\\ &=\sum_{m=1}^{\infty}\left[\int_0^{2\pi}f(\theta')\frac{\pi}{J_n(kr_0)}\cos(\theta'-\theta)\,d\theta'\right]J_n(kr)\end{aligned}$$

Exchanging integration with summation gives

$$u(r,\theta)=\int_0^{2\pi}f(\theta')\left(\sum_{m=1}^{\infty}\frac{\pi}{J_n(kr_0)}\cos(\theta'-\theta)J_n(kr)\right)d\theta'$$

Comparing the above to

$$u(r,\theta) = \int_0^{2\pi} f(\theta')\, G(r,\theta;\theta')\, d\theta'$$

Shows that Green function is

$$G(r,\theta;\theta') = \sum_{m=1}^{\infty} \frac{\pi}{J_n(kr_0)} \cos(\theta' - \theta)\, J_n(kr)$$

Where r_0 is radius of disk. It is symmetric in θ as expected.