① Start with $J_m'' + \frac{1}{x} J_m' + \left(1 - \frac{m^2}{x^2}\right) J_m = 0$

and $J_n'' + \frac{1}{x} J_n' + \left(1 - \frac{n^2}{x^2}\right) J_n = 0$

Multiply the first by $x^l J_n$ and the second by $x^l J_m$ and subtract to get

$$x^l \left(J_m J_n'' - J_n J_m''\right) + x^{l-1}\left(J_m J_n' - J_n J_m'\right) +$$
$$+ \; x^{l-2}\left(m^2 - n^2\right) J_m J_n = 0$$

The integral we are looking for suggests that we choose $l = 1$. Then

$$(m^2 - n^2)\,\frac{J_m J_n}{x} = x\left(J_n J_m'' - J_m J_n''\right) + \left(J_n J_m' - J_m J_n'\right)$$

$$= \frac{d}{dx}\left[x\left(J_n J_m' - J_m J_n'\right)\right]$$ which is a total derivative!

Then $$(m^2 - n^2) \int_0^\infty J_m(x)\, J_n(x)\, \frac{dx}{x} =$$

$$= \int_0^\infty \frac{d}{dx}\left[x\left(J_n J_m' - J_m J_n'\right)\right] dx =$$

$$= x\left(J_n J_m' - J_m J_n'\right)\Big|_0^\infty$$

Because $J_\ell(x) \sim x^\ell$ as $x \to 0$ the lower limit of integration will contribute zero as long as $m+n > 0$. For the upper limit we use the asymptotic expressions

$$J_\ell(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \ell\frac{\pi}{2} - \frac{\pi}{4}\right)$$

$$J_\ell'(x) \sim -\sqrt{\frac{2}{\pi x}} \sin\left(x - \ell\frac{\pi}{2} - \frac{\pi}{4}\right) \quad \text{to get}$$

$$x\left(J_n J_m' - J_m J_n'\right)\Big|_0^\infty = \frac{2}{\pi}\left[\cos\theta_m \sin\theta_n - \sin\theta_m \cos\theta_n\right]$$

$$= \frac{2}{\pi}\sin(\theta_n - \theta_m) \quad \text{where} \quad \theta_\ell = x - \ell\frac{\pi}{2} - \frac{\pi}{4}.$$

$$= \frac{2}{\pi}\sin\left[(m-n)\frac{\pi}{2}\right]$$

Finally

$$\boxed{\int_0^\infty J_m(x)\, J_n(x)\, \frac{dx}{x} = \frac{2}{\pi}\, \frac{\sin\left[(m-n)\frac{\pi}{2}\right]}{m^2 - n^2}}$$

This is valid for $m+n > 0$.

(2) $J_n(z)$ satisfies $z^2 J_n''(z) + z J_n'(z) + (z^2-n^2) J_n(z) = 0$

Let $z = ax^k$. Then $\frac{d}{dx} J_n(z) = J_n'(z) \cdot \frac{dz}{dx} = akx^{k-1} J_n'(z)$

$$\frac{d^2}{dx^2} J_n(z) = \frac{d}{dx}\left[akx^{k-1} J_n'(z)\right] = ak(k-1)x^{k-2} J_n'(z) +$$

$$+ \; a^2k^2 x^{2(k-1)} J_n''(z)$$

$$\Rightarrow \quad J_n'(z) = \frac{1}{ak} x^{-k+1} \frac{d}{dx} J_n(z)$$

and
$$J_n''(z) = \frac{1}{a^2k^2} x^{-2(k-1)} \frac{d^2}{dx^2} J_n(z) - \frac{k-1}{ak} \frac{1}{x^k} \underbrace{J_n'(z)}_{\frac{1}{ak} x^{-k+1} \frac{d}{dx} J_n(z)}$$

$$= \frac{1}{a^2k^2} x^{-2k} \left[x^2 \frac{d^2}{dx^2} J_n(z) - (k-1)\, x \frac{d}{dx} J_n(z) \right]$$

Now let $\boxed{\psi = x^m J_n(z),}$ $\quad \frac{d\psi}{dx} = m x^{m-1} J_n(z) + x^m \frac{d}{dx} J_n(z)$

$\Rightarrow \boxed{J_n(z) = x^{-m} \psi}$ and

$$\frac{d}{dx} J_n(z) = x^{-m} \frac{d\psi}{dx} - \frac{m}{x} J_n(z) = x^{-m} \frac{d\psi}{dx} - m x^{-(m+1)} \psi$$

$$\parallel$$

$$akx^{k-1} J_n'(z) \quad \Rightarrow \quad \boxed{J_n'(z) = \frac{1}{ak} x^{-(m+k-1)} \frac{d\psi}{dx} - \frac{m}{ak} x^{-(m+k)} \psi}$$

$$\frac{d^2}{dx^2} J_n(z) = \frac{d}{dx}\left[x^{-m}\frac{d\psi}{dx} - m\, x^{-(m+1)}\psi \right]$$

$$\parallel$$

$$a^2 h^2 x^{2(h-1)} J_n''(z) + a h (h-1)\, x^{h-2} J_n'(z)$$

$$= x^{-m}\frac{d^2\psi}{dx^2} - 2m\, x^{-(m+1)}\frac{d\psi}{dx} + m(m+1)\, x^{-(m+2)}\psi$$

Solve for $J_n''(z)$.

substitute for this too

$$J_n''(z) = \frac{1}{a^2h^2} x^{-2(h-1)}\Big\{ -ah(h-1)\, x^{h-2} J_n'(z)$$

$$+ x^{-m}\frac{d^2\psi}{dx^2} - 2m\, x^{-(m+1)}\frac{d\psi}{dx} + m(m+1)\, x^{-(m+2)}\psi \Big\}$$

$$= \frac{1}{a^2h^2} x^{-2(h-1)}\Big\{ -ah(h-1)\, x^{h-2}\Big[\frac{1}{ah} x^{-(m+h-1)}\frac{d\psi}{dx}$$

$$- \frac{m}{ah} x^{-(m+h)}\psi\Big] + x^{-m}\frac{d^2\psi}{dx^2} - 2m\, x^{-(m+1)}\frac{d\psi}{dx}$$

$$+ m(m+1)\, x^{-(m+2)}\psi \Big\}$$

$$\boxed{\begin{aligned} J_n''(z) &= \frac{1}{a^2h^2} x^{-(2h+m-2)}\frac{d^2\psi}{dx^2} - \frac{(2m+h-1)}{a^2h^2} x^{-(2h+m-1)}\frac{d\psi}{dx} \\ &\quad + \frac{m(m+h)}{a^2h^2} x^{-(2h+m)}\psi \end{aligned}}$$

Now substitute the above into

$$z^2 J_n''(z) + z J_n'(z) + (z^2 - n^2) J_n(z) = 0$$

$$a^2 x^{2k} \left\{ \frac{1}{a^2 k^2} x^{-(2k+m-2)} \psi''(x) - \frac{(2m+k-1)}{a^2 k^2} x^{-(2k+m-1)} \psi'(x) \right.$$

$$\left. + \frac{m(m+k)}{a^2 k^2} x^{-(2k+m)} \psi \right\} + a x^k \left\{ \frac{1}{ak} x^{-(m+k-1)} \psi'(x) \right.$$

$$\left. - \frac{m}{ak} x^{-(m+k)} \psi(x) \right\} + (a^2 x^{2k} - n^2) x^{-m} \psi(x) = 0$$

Multiply by $k^2 x^m$ and group terms to get

$$\boxed{x^2 \psi''(x) + (1-2m) x \psi'(x) + \left[m^2 + k^2(a^2 x^{2k} - n^2)\right] \psi(x) = 0}$$

when $m \to 0$ and $a \to 1$ and $k \to 1$ we recover the standard form of Bessel's equation.

We can solve $\psi'' + x^2 \psi = 0$ by choosing

$m = \frac{1}{2}$, $k = 2$, $a = \frac{1}{2}$, $n = \frac{1}{4}$

$$\Rightarrow \boxed{\psi = \sqrt{x}\, J_{\frac{1}{4}}\left(\tfrac{1}{2} x^2\right)}$$

③ Use the representation

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - x\sin\theta)\, d\theta$$

Since $|\cos\phi| \le 1$ we have

$$|J_n(x)| \le \frac{1}{\pi}\int_0^{\pi} d\theta = 1$$

$$\boxed{|J_n(x)| \le 1}$$

④ $${}_2F_1(a,b;c;x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tx)^{-a}\,dt$$

$${}_2F_1(1,1;2;x) = \underbrace{\frac{\Gamma(2)}{\Gamma(1)\Gamma(1)}}_{1} \int_0^1 \frac{dt}{1-tx} = -\frac{1}{x}\ln(1-tx)\Big|_0^1$$

$$= -\frac{\ln(1-x)}{x}$$

$$\boxed{{}_2F_1(1,1;2;x) = -\frac{1}{x}\ln(1-x)}$$

$${}_2F_1(a,1;1;x) = \frac{\Gamma(1)}{\Gamma(1)\Gamma(0)} \int_0^1 (1-t)^{-1}(1-tx)^{-a}\,dt$$

But $\Gamma(0)$ is infinite so the integral representation is not well defined. Instead use

$${}_2F_1(a,b;c;x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{x^n}{n!}$$

to get $${}_2F_1(a,1;1;x) = \frac{\Gamma(1)}{\Gamma(a)\Gamma(1)} \sum_{n=0}^{\infty} \Gamma(a+n)\frac{x^n}{n!} =$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(a)} \frac{x^n}{n!} = 1 + ax + \frac{a(a+1)}{2!}x^2 + \frac{a(a+1)(a+2)}{3!}x^3 + \cdots$$

$$\boxed{{}_2F_1(a,1;1;x) = (1-x)^{-a}}$$