# HW 10
# Physics 5041 Mathematical Methods for Physics
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# University of Minnesota, Twin Cities

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# Contents

# 1 Problem 1

Problem Show that

$$\int_0^\infty \frac{1}{x} J_m(x) J_n(x)\,dx = \frac{2}{\pi}\frac{\sin\left((m-n)\frac{\pi}{2}\right)}{m^2-n^2}$$

Solution

$$x^2 J_n''(x) + x J_n'(x) + \left(x^2 - n^2\right) J_n(x) = 0$$
$$x^2 J_m''(x) + x J_m'(x) + \left(x^2 - m^2\right) J_m(x) = 0$$

Dividing both equations by $x^2$ gives

$$J_n''(x) + \frac{1}{x} J_n'(x) + \left(1 - \frac{n^2}{x^2}\right) J_n(x) = 0$$
$$J_m''(x) + \frac{1}{x} J_m'(x) + \left(1 - \frac{m^2}{x^2}\right) J_m(x) = 0$$

Multiplying the first ODE by $xJ_m(x)$ and the second by $xJ_n(x)$ gives (multiplying by just $x$ did not lead to a result that I could use).

$$xJ_m J_n'' + J_m J_n' + x\left(1 - \frac{n^2}{x^2}\right) J_m J_n = 0$$
$$xJ_n J_m'' + J_n J_m' + x\left(1 - \frac{m^2}{x^2}\right) J_n J_m = 0$$

Subtracting gives

$$\left(xJ_m J_n'' + J_m J_n' + x\left(1 - \frac{n^2}{x^2}\right) J_m J_n\right) - \left(xJ_n J_m'' + J_n J_m' + x\left(1 - \frac{m^2}{x^2}\right) J_n J_m\right) = 0$$
$$x\left(J_m J_n'' - J_n J_m''\right) + J_m J_n' - J_n J_m' - xJ_m J_n\left(\left(1 - \frac{n^2}{x^2}\right) - \left(1 - \frac{m^2}{x^2}\right)\right) = 0$$

Or

$$x\left(J_m J_n'' - J_n J_m''\right) + J_m J_n' - J_n J_m' = xJ_m J_n\left(\left(1 - \frac{m^2}{x^2}\right) - \left(1 - \frac{n^2}{x^2}\right)\right) \tag{1}$$

But the LHS above is complete differential[1]

$$x\left(J_m J_n'' - J_n J_m''\right) + J_m J_n' - J_n J_m' = \left(x\left(J_m J_n' - J_n J_m'\right)\right)' \tag{2}$$

Hence using (2) in (1), then (1) simplifies to

$$\begin{aligned}
\left(x\left(J_m J_n' - J_n J_m'\right)\right)' &= xJ_m J_n\left(\left(1 - \frac{m^2}{x^2}\right) - \left(1 - \frac{n^2}{x^2}\right)\right)\\
&= xJ_m J_n\left(\frac{n^2}{x^2} - \frac{m^2}{x^2}\right)\\
&= \frac{J_m J_n}{x}\left(n^2 - m^2\right)
\end{aligned}$$

Integrating both sides above gives

$$\left[x\left(J_m J_n' - J_n J_m'\right)\right]_0^\infty = \left(n^2 - m^2\right)\int_0^\infty \frac{J_m J_n}{x}\,dx$$

Therefore

$$\int_0^\infty \frac{J_m(x) J_n(x)}{x}\,dx = \frac{1}{\left(m^2 - n^2\right)}\left[x\left(J_n(x) J_m'(x) - J_m(x) J_n'(x)\right)\right]_0^\infty \tag{3}$$

[1]

$$\begin{aligned}
\left(x\left(J_m J_n' - J_n J_m'\right)\right)' &= \left(J_m J_n' - J_n J_m'\right) + x\left(J_m J_n' - J_n J_m'\right)'\\
&= J_m J_n' - J_n J_m' + x\left(J_m' J_n' + J_m J_n'' - J_n' J_m' - J_n J_m''\right)\\
&= J_m J_n' - J_n J_m' + x\left(J_m J_n'' - J_n J_m''\right)
\end{aligned}$$

At $x=0$ the expression $x\left(J_n(x)J_m'(x)-J_m(x)J_n'(x)\right)=0$. And at $x=\infty$ we can use the asymptotic approximation given by

$$J_n(x)=\sqrt{\frac{2}{\pi x}}\cos\left(x-\frac{n\pi}{2}-\frac{\pi}{4}\right)$$
$$J_n'(x)=-\sqrt{\frac{2}{\pi x}}\sin\left(x-\frac{n\pi}{2}-\frac{\pi}{4}\right)-\frac{1}{\sqrt{2\pi}}\left(\frac{1}{x}\right)^{\frac{3}{2}}\cos\left(x-\frac{n\pi}{2}-\frac{\pi}{4}\right)$$

And similarly for $J_m(x)$

$$J_m(x)=\sqrt{\frac{2}{\pi x}}\cos\left(x-\frac{m\pi}{2}-\frac{\pi}{4}\right)$$
$$J_m'(x)=-\sqrt{\frac{2}{\pi x}}\sin\left(x-\frac{m\pi}{2}-\frac{\pi}{4}\right)-\frac{1}{\sqrt{2\pi}}\left(\frac{1}{x}\right)^{\frac{3}{2}}\cos\left(x-\frac{m\pi}{2}-\frac{\pi}{4}\right)$$

Therefore

$$J_n(x)J_m'(x)=\sqrt{\frac{2}{\pi x}}\cos\left(x-\frac{n\pi}{2}-\frac{\pi}{4}\right)\left(-\sqrt{\frac{2}{\pi x}}\sin\left(x-\frac{m\pi}{2}-\frac{\pi}{4}\right)-\frac{1}{\sqrt{2\pi}}\left(\frac{1}{x}\right)^{\frac{3}{2}}\cos\left(x-\frac{m\pi}{2}-\frac{\pi}{4}\right)\right)$$

Let $x-\frac{n\pi}{2}-\frac{\pi}{4}=\alpha$, and let $x-\frac{m\pi}{2}-\frac{\pi}{4}=\beta$, then the above becomes

$$\begin{aligned}
J_n(x)J_m'(x)&=\sqrt{\frac{2}{\pi x}}\cos(\alpha)\left(-\sqrt{\frac{2}{\pi x}}\sin(\beta)-\frac{1}{\sqrt{2\pi}}\left(\frac{1}{x}\right)^{\frac{3}{2}}\cos(\beta)\right)\\
&=-\frac{2}{\pi x}\cos(\alpha)\sin(\beta)-\sqrt{\frac{2}{\pi x}}\sqrt{\frac{1}{2\pi}}\left(\frac{1}{x}\right)^{\frac{3}{2}}\cos(\alpha)\cos(\beta)\\
&=-\frac{2}{\pi x}\cos(\alpha)\sin(\beta)-\frac{1}{\pi}\left(\frac{1}{x}\right)^{\frac{1}{2}}\left(\frac{1}{x}\right)^{\frac{3}{2}}\cos(\alpha)\cos(\beta)\\
&=-\frac{2}{\pi x}\cos(\alpha)\sin(\beta)-\frac{1}{\pi}\left(\frac{1}{x}\right)^{2}\cos(\alpha)\cos(\beta) \qquad (4)
\end{aligned}$$

Similarly

$$J_m(x)J_n'(x)=-\frac{2}{\pi x}\cos(\beta)\sin(\alpha)-\frac{1}{\pi}\left(\frac{1}{x}\right)^{2}\cos(\beta)\cos(\alpha) \qquad (5)$$

Substituting (4,5) into (3) gives (only the term as $x\to\infty$ remains)

$$\begin{aligned}
\int_0^\infty \frac{J_m(x)J_n(x)}{x}dx&=\frac{1}{(m^2-n^2)}\left[x\left(J_n(x)J_m'(x)-J_m(x)J_n'(x)\right)\right]_0^\infty\\
&=\frac{x}{(m^2-n^2)}\left(\left(-\frac{2}{\pi x}\cos(\alpha)\sin(\beta)-\frac{1}{\pi}\left(\frac{1}{x}\right)^2\cos(\alpha)\cos(\beta)\right)-\left(-\frac{2}{\pi x}\cos(\beta)\sin(\alpha)-\frac{1}{\pi}\left(\frac{1}{x}\right)^2\cos(\beta)\cos(\alpha)\right)\right)\\
&=\frac{x}{(m^2-n^2)}\left(-\frac{2}{\pi x}\cos(\alpha)\sin(\beta)-\frac{1}{\pi}\left(\frac{1}{x}\right)^2\cos(\alpha)\cos(\beta)+\frac{2}{\pi x}\cos(\beta)\sin(\alpha)+\frac{1}{\pi}\left(\frac{1}{x}\right)^2\cos(\beta)\cos(\alpha)\right)\\
&=\frac{x}{(m^2-n^2)}\left(-\frac{2}{\pi x}\cos(\alpha)\sin(\beta)+\frac{2}{\pi x}\cos(\beta)\sin(\alpha)\right)\\
&=\frac{2}{\pi}\frac{1}{(m^2-n^2)}\left(\sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta)\right) \qquad (6)
\end{aligned}$$

But

$$
\begin{aligned}
\sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta) &= \sin(\alpha - \beta) \\
&= \sin\left(\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right) - \left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right)\right) \\
&= \sin\left(x - \frac{n\pi}{2} - \frac{\pi}{4} - x + \frac{m\pi}{2} + \frac{\pi}{4}\right) \\
&= \sin\left(\frac{m\pi}{2} - \frac{n\pi}{2}\right) \\
&= \sin\left((m-n)\frac{\pi}{2}\right)
\end{aligned}
$$

Using the above in (6) gives

$$
\int_0^\infty \frac{J_m(x) J_n(x)}{x} dx = \frac{2}{\pi} \frac{\sin\left((m-n)\frac{\pi}{2}\right)}{\left(m^2 - n^2\right)}
$$

Which is the result required to show. QED.

# 2 Problem 2

Problem What linear second order ODE does the function $x^m J_n\left(ax^k\right)$ solves? Are there any required relationships among $m, n, k$? Use this to solve $y'' + x^2 y = 0$

Solution

## 2.1 Part (a)

We know that the Bessel ODE

$$t^2 z''(t) + t z'(t) + \left(t^2 - \left(\frac{\alpha}{\beta}\right)^2\right) z(t) = 0 \tag{1}$$

I am using the order as $\frac{\alpha}{\beta}$ instead of $n$ to make it more general. At the end, $\frac{\alpha}{\beta}$ can always be replaced back by $n$.

The ODE above has solution

$$z(t) = J_{\frac{\alpha}{\beta}}(t)$$

Hence using the transformation

$$t = ax^k \tag{2}$$

The solution $y(x) \equiv z\left(ax^k\right)$ will becomes

$$y(x) = J_{\frac{\alpha}{\beta}}\left(ax^k\right)$$

Therefore the question now is, how does ODE (1) transforms under (2)? From (2)

$$x = \left(\frac{t}{a}\right)^{\frac{1}{k}}$$

Hence

$$\begin{aligned}\frac{dx}{dt} &= \frac{1}{k}\left(\frac{t}{a}\right)^{\frac{1}{k}-1} \\ &= \frac{1}{ak}\left(\frac{t}{a}\right)^{\frac{1}{k}-1}\end{aligned} \tag{3}$$

Now

$$\begin{aligned}\frac{dz}{dt} &= \frac{dz}{dx}\frac{dx}{dt} \\ &= \frac{dz}{dx}\frac{1}{ak}\left(\frac{t}{a}\right)^{\frac{1}{k}-1}\end{aligned} \tag{5}$$

And

$$\begin{aligned}\frac{d^2z}{dt^2} &= \frac{d}{dt}\left(\frac{dz}{dt}\right) \\ &= \frac{d}{dt}\left(\frac{dz}{dx}\frac{1}{ak}\left(\frac{t}{a}\right)^{\frac{1}{k}-1}\right) \\ &= \frac{d^2z}{dx^2}\frac{dx}{dt}\left(\frac{1}{ak}\left(\frac{t}{a}\right)^{\frac{1}{k}-1}\right) + \frac{dz}{dx}\frac{d}{dt}\left(\frac{1}{ak}\left(\frac{t}{a}\right)^{\frac{1}{k}-1}\right) \\ &= \frac{d^2z}{dx^2}\left(\frac{1}{ak}\left(\frac{t}{a}\right)^{\frac{1}{k}-1}\right)^2 + \frac{dz}{dx}\left(\frac{1}{a^2k}\left(\frac{1}{k}-1\right)\left(\frac{t}{a}\right)^{\frac{1}{k}-2}\right)\end{aligned} \tag{6}$$

Using (5,6) then ODE (1) becomes

$$t^2\left(z''\left(ax^k\right)\left(\frac{1}{ak}\left(\frac{t}{a}\right)^{\frac{1}{k}-1}\right)^2 + z'\left(ax^k\right)\frac{1}{a^2k}\left(\frac{1}{k}-1\right)\left(\frac{t}{a}\right)^{\frac{1}{k}-2}\right) + t\left(z'\left(ax^k\right)\frac{1}{ak}\left(\frac{t}{a}\right)^{\frac{1}{k}-1}\right) + \left(t^2 - \left(\frac{\alpha}{\beta}\right)^2\right) z\left(ax^k\right) = 0$$

Writing $y(x) \equiv z\left(ax^k\right)$ so we do not have to keep writing $z\left(ax^k\right)$, the above becomes

$$t^2\left(y''(x)\left(\frac{1}{ak}\left(\frac{t}{a}\right)^{\frac{1}{k}-1}\right)^2+y'(x)\frac{1}{a^2k}\left(\frac{1}{k}-1\right)\left(\frac{t}{a}\right)^{\frac{1}{k}-2}\right)+t\left(y'(x)\frac{1}{ak}\left(\frac{t}{a}\right)^{\frac{1}{k}-1}\right)+\left(t^2-\left(\frac{\alpha}{\beta}\right)^2\right)y(x)=0$$

But $t = ax^k$ and the above becomes

$$a^2x^{2k}\left(y''(x)\left(\frac{1}{ak}\left(\frac{ax^k}{a}\right)^{\frac{1}{k}-1}\right)^2+y'(x)\frac{1}{a^2k}\left(\frac{1}{k}-1\right)\left(\frac{ax^k}{a}\right)^{\frac{1}{k}-2}\right)+ax^k\left(y'(x)\frac{1}{ak}\left(\frac{ax^k}{a}\right)^{\frac{1}{k}-1}\right)+\left(a^2x^{2k}-\left(\frac{\alpha}{\beta}\right)^2\right)y(x)=0$$

Which is simplified more as follows

$$\begin{aligned}
a^2x^{2k}\left(y''(x)\left(\frac{1}{ak}\left(\frac{x}{x^k}\right)\right)^2+y'(x)\frac{1}{a^2k}\left(\frac{1}{k}-1\right)\frac{x}{x^{2k}}\right)+ax^k\left(y'(x)\frac{1}{ak}\frac{x}{x^k}\right)+\left(a^2x^{2k}-\left(\frac{\alpha}{\beta}\right)^2\right)y(x)&=0\\
a^2x^{2k}\left(y''(x)\frac{1}{a^2k^2}\left(\frac{x^2}{x^{2k}}\right)+y'(x)\frac{1}{a^2k}\left(\frac{1}{k}-1\right)\frac{x}{x^{2k}}\right)+ax^k\left(y'(x)\frac{1}{ak}\frac{x}{x^k}\right)+\left(a^2x^{2k}-\left(\frac{\alpha}{\beta}\right)^2\right)y(x)&=0\\
x^{2k}\left(y''(x)\frac{1}{k^2}\left(\frac{x^2}{x^{2k}}\right)+y'(x)\frac{1}{k}\left(\frac{1}{k}-1\right)\frac{x}{x^{2k}}\right)+y'(x)\frac{x}{k}+\left(a^2x^{2k}-\left(\frac{\alpha}{\beta}\right)^2\right)y(x)&=0\\
\frac{x^2}{k^2}y''(x)+y'(x)\frac{1}{k}\left(\frac{1}{k}-1\right)x+y'(x)\frac{x}{k}+\left(a^2x^{2k}-\left(\frac{\alpha}{\beta}\right)^2\right)y(x)&=0\\
x^2y''(x)+y'(x)k\left(\frac{1}{k}-1\right)x+y'(x)kx+\left(k^2a^2x^{2k}-k^2\left(\frac{\alpha}{\beta}\right)^2\right)y(x)&=0\\
x^2y''(x)+xy'(x)+\left(k^2a^2x^{2k}-\frac{k^2\alpha^2}{\beta^2}\right)y(x)&=0
\end{aligned} \tag{7}$$

We know that the above ODE has one solution as $y(x) = J_{\frac{\alpha}{\beta}}\left(ax^k\right)$ because this is how the above was constructed. Now assuming that

$$\begin{aligned}
w(x) &= x^m y(x)\\
&= x^m J_{\frac{\alpha}{\beta}}\left(ax^k\right)
\end{aligned}$$

Then $w(x)$ is the solution we want. This means we need to express (7) in terms of $w(x)$ instead of $y(x)$ in order to find the ODE whose solution is $x^m J_{\frac{\alpha}{\beta}}\left(ax^k\right)$.

Since $y(x) = w(x)x^{-m}$ then

$$\begin{aligned}
y'(x) &= \frac{d}{dx}\left(x^{-m}w\right)\\
&= -mx^{-m-1}w + x^{-m}w'
\end{aligned}$$

And

$$\begin{aligned}
y''(x) &= \frac{d}{dx}\left(-mx^{-m-1}w + x^{-m}w'\right)\\
&= -m(-m-1)x^{-m-2}w - mx^{-m-1}w' - mx^{-m-1}w' + x^{-m}w''\\
&= m(m+1)x^{-m-2}w - 2w'mx^{-m-1} + x^{-m}w''
\end{aligned}$$

Substituting the above results back into (7) gives

$$x^2\left(m(m+1)x^{-m-2}w-2w'mx^{-m-1}+x^{-m}w''\right)+x\left(-mx^{-m-1}w+x^{-m}w'\right)+\left(k^2a^2x^{2k}-\frac{k^2\alpha^2}{\beta^2}\right)wx^{-m}=0$$

Dividing by $x^{-m}$

$$x^2\left(m(m+1)x^{-2}w - 2w'mx^{-1} + w''\right) + x\left(-mx^{-1}w + w'\right) + \left(k^2a^2x^{2k} - \frac{k^2\alpha^2}{\beta^2}\right)w = 0$$

$$m(m+1)w - 2xw'm + x^2w'' - mw + xw' + \left(k^2a^2x^{2k} - \frac{k^2\alpha^2}{\beta^2}\right)w = 0$$

$$x^2w'' + w'(-2xm + x) + \left(k^2a^2x^{2k} + m(m+1) - m - \frac{k^2\alpha^2}{\beta^2}\right)w = 0$$

$$x^2w'' + (1-2m)xw' + \left(k^2a^2x^{2k} + m^2 - \frac{k^2\alpha^2}{\beta^2}\right)w = 0 \quad (8)$$

Hence the above ODE (8) will have the solution $x^m J_{\frac{\alpha}{\beta}}\left(ax^k\right)$. We can now let $n = \frac{\alpha}{\beta}$ and the above ODE becomes

$$x^2w'' + (1-2m)xw' + \left(k^2a^2x^{2k} + m^2 - k^2n^2\right)w = 0 \quad (9)$$

Has the required solution $x^m J_n\left(ax^k\right)$.

To answer the final part about the relation between $n, m, k$. One restriction is that $m = \frac{1}{2}$. One relation between the order $n$ and $k$ is that $m^2 - k^2n^2$ being a rational number. This means

$$m^2 - k^2n^2 = \frac{N}{M}$$

Where $N, M$ are integers.

## 2.2 Part (b)

$$y''(x) + x^2y(x) = 0 \quad (1)$$

Comparing this ODE to one found in part (a), written below again, now using $y(x)$ to make it easier to compare

$$x^2y''(x) + (1-2m)xy'(x) + \left(k^2a^2x^{2k} + m^2 - k^2n^2\right)y(x) = 0$$

$$y''(x) + \frac{(1-2m)}{x}y'(x) + \frac{1}{x^2}\left(k^2a^2x^{2k} + m^2 - k^2n^2\right)y(x) = 0 \quad (2)$$

To make (2) same as (1), we want $(1-2m) = 0$ or $m = \frac{1}{2}$. Also need $2k = 4$ or $k = 2$. Using these the above reduces to

$$y''(x) + \left(4a^2x^2 + \frac{\frac{1}{4} - 4n^2}{x^2}\right)y(x) = 0$$

Therefore, we need also that $n^2 = \frac{1}{16}$ in order to cancel extra term above. Hence $n = \frac{1}{4}$. Now the above becomes

$$y''(x) + 4a^2x^2y(x) = 0$$

Finally, if we let $a^2 = \frac{1}{4}$ or $a = \frac{1}{2}$, then the above becomes

$$y''(x) + x^2y(x) = 0$$

Therefore, we found that

$$n = \frac{1}{4}$$
$$a = \frac{1}{2}$$
$$k = 2$$
$$m = \frac{1}{2}$$

Hence the following solves the ODE

$$\begin{aligned} y(x) &= x^m J_n\left(ax^k\right) \\ &= \sqrt{x} J_{\frac{1}{4}}\left(\frac{1}{2}x^2\right) \end{aligned}$$

## 2.3 Appendix

To verify the above result, it is solved again directly. We first need to convert this ODE to Bessel ODE. Let

$$y = x^{\frac{1}{2}} z(x)$$

Then

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2}x^{-\frac{1}{2}}z + x^{\frac{1}{2}}z' \\ \frac{d^2y}{dx^2} &= -\frac{1}{4}x^{-\frac{3}{2}}z + \frac{1}{2}x^{-\frac{1}{2}}z' + \frac{1}{2}x^{-\frac{1}{2}}z' + x^{\frac{1}{2}}z'' \\ &= -\frac{1}{4}x^{-\frac{3}{2}}z + x^{-\frac{1}{2}}z' + x^{\frac{1}{2}}z'' \end{aligned}$$

Substituting the above into (1) gives

$$\begin{aligned} \left(-\frac{1}{4}x^{-\frac{3}{2}}z + x^{-\frac{1}{2}}z' + x^{\frac{1}{2}}z''\right) + x^2x^{\frac{1}{2}}z &= 0 \\ x^{\frac{1}{2}}z'' + x^{-\frac{1}{2}}z' + \left(x^{\frac{5}{2}} - \frac{1}{4}x^{-\frac{3}{2}}\right)z &= 0 \end{aligned}$$

Multiplying both sides by $x^{\frac{3}{2}}$ gives

$$x^2z'' + xz' + \left(x^4 - \frac{1}{4}\right)z = 0 \tag{2}$$

Where the derivatives above is with respect to $x$. Now let $t = \frac{x^2}{2}$. Then

$$\frac{dz}{dx} = \frac{dz}{dt}\frac{dt}{dx} = x\frac{dz}{dt}$$

And

$$\begin{aligned} \frac{d^2z}{dx^2} &= \frac{d^2z}{dt^2}\left(\frac{dt}{dx}\right)(x) + \frac{dz}{dt} \\ &= \frac{d^2z}{dt^2}x^2 + \frac{dz}{dt} \end{aligned}$$

Substituting the above into (2) gives

$$x^2\left(x^2z'' + z'\right) + x(xz') + \left(x^4 - \frac{1}{4}\right)z = 0$$

Where the derivatives above is with respect to $t$ now. This simplifies to

$$x^4z'' + 2x^2z' + \left(x^4 - \frac{1}{4}\right)z = 0$$

But $t = \frac{x^2}{2}$, hence the above becomes

$$\begin{aligned} 4t^2z'' + 4tz' + \left(4t^2 - \frac{1}{4}\right)z &= 0 \\ t^2z'' + tz' + \left(t^2 - \frac{1}{16}\right)z &= 0 \end{aligned}$$

This now in the form of Bessel ODE

$$t^2z'' + tz' + \left(t^2 - n^2\right)z = 0$$

Where $n = \frac{1}{4}$. Hence one solution is

$$\begin{aligned} z(t) &= J_n(t) \\ &= J_{\frac{1}{4}}(t) \end{aligned}$$

But $y(x) = \sqrt{x}z(x)$ and $t = \frac{x^2}{2}$, therefore the above becomes

$$y(x) = \sqrt{x}J_{\frac{1}{4}}\left(\frac{x^2}{2}\right) \tag{3}$$

Which is the same as found in part (b)

## 3 Problem 3

Problem Prove that $|J_n(x)| \leq 1$ for all integers $n$

Solution

From the integral representation of $J_n(x)$ for integer $n$

$$J_n(x) = \frac{1}{\pi} \int_0^{\pi} \cos(n\theta - x \sin\theta)\, d\theta$$

Then

$$\begin{aligned}
|J_n(x)| &\leq \frac{1}{\pi} \left| \int_0^{\pi} \cos(n\theta - x\sin\theta)\, d\theta \right|_{\max} \\
&\leq \frac{1}{\pi} \int_0^{\pi} |\cos(n\theta - x\sin\theta)|_{\max}\, d\theta \\
&= \frac{1}{\pi} |M|_{\max} \int_0^{\pi} d\theta \\
&= \frac{1}{\pi} |M|_{\max}\, \pi \\
&= |M|_{\max}
\end{aligned}$$

Where $|M|_{\max} = |\cos(n\theta - x\sin\theta)|_{\max}$ over $\theta = 0 \cdots \pi$. But this is 1 for the cosine function. Hence

$$|J_n(x)| \leq 1$$

# 4 Problem 4

Problem Starting with the integral formula for hypergeometric function, express the following in terms of elementary functions ${}_2F_1(1,1,2;x)$ and ${}_2F_1(a,1,1;x)$

Solution

$$
\begin{aligned}
{}_2F_1(a,b,c;x) &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tx)^{-a}\,dt &(1)\\
&= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)}\frac{x^n}{n!} &(2)
\end{aligned}
$$

## 4.1 Part (a)

Here $a=1,b=1,c=2$. Therefore, using (1) representation gives

$$
\begin{aligned}
{}_2F_1(1,1,2;x) &= \frac{\Gamma(2)}{\Gamma(1)\Gamma(2-1)}\int_0^1 t^{1-1}(1-t)^{2-1-1}(1-tx)^{-1}\,dt\\
&= \frac{\Gamma(2)}{\Gamma(1)\Gamma(1)}\int_0^1 \frac{dt}{1-tx}
\end{aligned}
$$

But $\Gamma(2)=1,\Gamma(1)=0$, therefore the above becomes

$$
\begin{aligned}
{}_2F_1(1,1,2;x) &= \int_0^1 \frac{dt}{1-tx}\\
&= \left[\frac{-\ln(1-tx)}{x}\right]_0^1\\
&= -\left(\frac{\ln(1-x)}{x} - \frac{-\ln(1-0)}{x}\right)\\
&= -\frac{\ln(1-x)}{x}
\end{aligned}
$$

## 4.2 Part (b)

Here $a=a,b=1,c=1$. Therefore (2) representation gives

$$
\begin{aligned}
{}_2F_1(a,1,1;x) &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)}\frac{x^n}{n!}\\
&= \frac{1}{\Gamma(a)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(1+n)}{\Gamma(1+n)}\frac{x^n}{n!}\\
&= \sum_{n=0}^{\infty}\frac{\Gamma(a+n)}{\Gamma(a)}\frac{x^n}{n!}
\end{aligned}
$$

Looking at few values

| $n$ | ${}_2F_1(a,1,1;x)$ |
|---|---|
| 0 | $\frac{\Gamma(a)}{\Gamma(a)}=1$ |
| 1 | $\frac{\Gamma(a+1)}{\Gamma(a)}x$ |
| 2 | $\frac{\Gamma(a+2)}{\Gamma(a)}\frac{x^2}{2!}$ |
| 3 | $\frac{\Gamma(a+3)}{\Gamma(a)}\frac{x^3}{3!}$ |
| $\vdots$ | $\vdots$ |

Using the recursive relation $\Gamma(a+1)=a\Gamma(a)$, which works for integer and non integer $a$, then we see that

$$
\Gamma(a+1)=a\Gamma(a)
$$

And

$$\begin{aligned}\Gamma(a+2) &= \Gamma((a+1)+1)\\ &= (a+1)\Gamma(a+1)\\ &= (a+1)a\Gamma(a)\end{aligned}$$

And

$$\begin{aligned}\Gamma(a+3) &= \Gamma((a+2)+1)\\ &= (a+2)\Gamma((a+2))\\ &= (a+2)(a+1)a\Gamma(a)\end{aligned}$$

And so on. Hence the above now becomes

| $n$ | ${}_2F_1(a,1,1;x)$ |
|---|---|
| 0 | 1 |
| 1 | $\frac{a\Gamma(a)}{\Gamma(a)}x = ax$ |
| 2 | $\frac{(a+1)a\Gamma(a)}{\Gamma(a)}\frac{x^2}{2!} = a(a+1)\frac{x^2}{2!}$ |
| 3 | $\frac{(a+2)(a+1)a\Gamma(a)}{\Gamma(a)}\frac{x^3}{3!} = a(a+1)(a+2)\frac{x^3}{3!}$ |
| $\vdots$ | $\vdots$ |

We see from the above the pattern of the sequence is as follows

$$ {}_2F_1(a,1,1;x) = 1 + ax + a(a+1)\frac{x^2}{2!} + a(a+1)(a+2)\frac{x^3}{3!} + \cdots \tag{1}$$

Comparing the above to the Binomial expansion given by

$$(1+z)^n = 1 + nz + n(n-1)\frac{z^2}{2!} + n(n-1)(n-2)\frac{z^3}{3!} + \cdots \tag{2}$$

By replacing $z \to -x$ and $n \to -a$, the above becomes

$$\begin{aligned}(1-x)^{-a} &= 1 + (-a)(-x) + (-a)((-a)-1)\frac{(-x)^2}{2!} + (-a)((-a)-1)((-a)-2)\frac{(-x)^3}{3!} + \cdots\\ &= 1 + ax + (a)(a+1)\frac{x^2}{2!} + (a)(a+1)(a+2)\frac{x^3}{3!} + \cdots\end{aligned}$$

Comparing the above to (1) shows it is the same series. Hence

$$ {}_2F_1(a,1,1;x) = (1-x)^{-a}$$