HW 10
Physics 5041 Mathematical Methods for Physics
Spring 2019
University of Minnesota, Twin Cities

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November 2, 2019   Compiled on November 2, 2019 at 10:31pm   [public]

Contents

1   Problem 1

<u>Problem</u> Show that

$$\int_0^\infty \frac{1}{x} J_m(x) J_n(x)\, dx = \frac{2}{\pi} \frac{\sin\left((m-n)\frac{\pi}{2}\right)}{m^2 - n^2}$$

<u>Solution</u>

$$x^2 J_n''(x) + x J_n'(x) + \left(x^2 - n^2\right) J_n(x) = 0$$
$$x^2 J_m''(x) + x J_m'(x) + \left(x^2 - m^2\right) J_m(x) = 0$$

Dividing both equations by x^2 gives

$$J_n''(x) + \frac{1}{x} J_n'(x) + \left(1 - \frac{n^2}{x^2}\right) J_n(x) = 0$$

$$J_m''(x) + \frac{1}{x} J_m'(x) + \left(1 - \frac{m^2}{x^2}\right) J_m(x) = 0$$

Multiplying the first ODE by $x J_m(x)$ and the second by $x J_n(x)$ gives (multiplying by just x did not lead to a result that I could use).

$$x J_m J_n'' + J_m J_n' + x\left(1 - \frac{n^2}{x^2}\right) J_m J_n = 0$$

$$x J_n J_m'' + J_n J_m' + x\left(1 - \frac{m^2}{x^2}\right) J_n J_m = 0$$

Subtracting gives

$$\left(x J_m J_n'' + J_m J_n' + x\left(1 - \frac{n^2}{x^2}\right) J_m J_n\right) - \left(x J_n J_m'' + J_n J_m' + x\left(1 - \frac{m^2}{x^2}\right) J_n J_m\right) = 0$$

$$x\left(J_m J_n'' - J_n J_m''\right) + J_m J_n' - J_n J_m' - x J_m J_n\left(\left(1 - \frac{n^2}{x^2}\right) - \left(1 - \frac{m^2}{x^2}\right)\right) = 0$$

Or

$$x\left(J_m J_n'' - J_n J_m''\right) + J_m J_n' - J_n J_m' = x J_m J_n\left(\left(1 - \frac{m^2}{x^2}\right) - \left(1 - \frac{n^2}{x^2}\right)\right) \tag{1}$$

But the LHS above is complete differential[1]

$$x\left(J_m J_n'' - J_n J_m''\right) + J_m J_n' - J_n J_m' = \left(x\left(J_m J_n' - J_n J_m'\right)\right)' \tag{2}$$

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[1]

$$\left(x\left(J_m J_n' - J_n J_m'\right)\right)' = \left(J_m J_n' - J_n J_m'\right) + x\left(J_m J_n' - J_n J_m'\right)'$$
$$= J_m J_n' - J_n J_m' + x\left(J_m' J_n' + J_m J_n'' - J_n' J_m' - J_n J_m''\right)$$
$$= J_m J_n' - J_n J_m' + x\left(J_m J_n'' - J_n J_m''\right)$$

Hence using (2) in (1), then (1) simplifies to

$$(x\,(J_m J'_n - J_n J'_m))' = x J_m J_n \left(\left(1 - \frac{m^2}{x^2}\right) - \left(1 - \frac{n^2}{x^2}\right)\right)$$

$$= x J_m J_n \left(\frac{n^2}{x^2} - \frac{m^2}{x^2}\right)$$

$$= \frac{J_m J_n}{x}\left(n^2 - m^2\right)$$

Integrating both sides above gives

$$[x\,(J_m J'_n - J_n J'_m)]_0^\infty = \left(n^2 - m^2\right) \int_0^\infty \frac{J_m J_n}{x}\,dx$$

Therefore

$$\int_0^\infty \frac{J_m(x)\,J_n(x)}{x}\,dx = \frac{1}{\left(m^2 - n^2\right)} [x\,(J_n(x)\,J'_m(x) - J_m(x)\,J'_n(x))]_0^\infty \tag{3}$$

At $x = 0$ the expression $x\,(J_n(x)\,J'_m(x) - J_m(x)\,J'_n(x)) = 0$. And at $x = \infty$ we can use the asymptotic approximation given by

$$J_n(x) = \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right)$$

$$J'_n(x) = -\sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right) - \frac{1}{\sqrt{2\pi}}\left(\frac{1}{x}\right)^{\frac{3}{2}}\cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right)$$

And similarly for $J_m(x)$

$$J_m(x) = \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right)$$

$$J'_m(x) = -\sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right) - \frac{1}{\sqrt{2\pi}}\left(\frac{1}{x}\right)^{\frac{3}{2}}\cos\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right)$$

Therefore

$$J_n(x)\,J'_m(x) = \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right)\left(-\sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right) - \frac{1}{\sqrt{2\pi}}\left(\frac{1}{x}\right)^{\frac{3}{2}}\cos\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right)\right)$$

Let $x - \frac{n\pi}{2} - \frac{\pi}{4} = \alpha$, and let $x - \frac{m\pi}{2} - \frac{\pi}{4} = \beta$, then the above becomes

$$J_n(x) J_m'(x) = \sqrt{\frac{2}{\pi x}} \cos(\alpha) \left(-\sqrt{\frac{2}{\pi x}} \sin(\beta) - \frac{1}{\sqrt{2\pi}} \left(\frac{1}{x}\right)^{\frac{3}{2}} \cos(\beta) \right)$$

$$= -\frac{2}{\pi x} \cos(\alpha) \sin(\beta) - \sqrt{\frac{2}{\pi x}} \sqrt{\frac{1}{2\pi}} \left(\frac{1}{x}\right)^{\frac{3}{2}} \cos(\alpha) \cos(\beta)$$

$$= -\frac{2}{\pi x} \cos(\alpha) \sin(\beta) - \frac{1}{\pi} \left(\frac{1}{x}\right)^{\frac{1}{2}} \left(\frac{1}{x}\right)^{\frac{3}{2}} \cos(\alpha) \cos(\beta)$$

$$= -\frac{2}{\pi x} \cos(\alpha) \sin(\beta) - \frac{1}{\pi} \left(\frac{1}{x}\right)^{2} \cos(\alpha) \cos(\beta) \tag{4}$$

Similarly

$$J_m(x) J_n'(x) = -\frac{2}{\pi x} \cos(\beta) \sin(\alpha) - \frac{1}{\pi} \left(\frac{1}{x}\right)^{2} \cos(\beta) \cos(\alpha) \tag{5}$$

Substituting (4,5) into (3) gives (only the term as $x \to \infty$ remains)

$$\int_0^\infty \frac{J_m(x) J_n(x)}{x} dx = \frac{1}{(m^2 - n^2)} \left[x \left(J_n(x) J_m'(x) - J_m(x) J_n'(x) \right) \right]_0^\infty$$

$$= \frac{x}{(m^2 - n^2)} \left(\left(-\frac{2}{\pi x} \cos(\alpha) \sin(\beta) - \frac{1}{\pi} \left(\frac{1}{x}\right)^{2} \cos(\alpha) \cos(\beta) \right) - \left(-\frac{2}{\pi x} \cos(\beta) \sin(\alpha) - \frac{1}{\pi} \left(\frac{1}{x}\right)^{2} \cos(\beta) \cos(\alpha) \right) \right)$$

$$= \frac{x}{(m^2 - n^2)} \left(-\frac{2}{\pi x} \cos(\alpha) \sin(\beta) - \frac{1}{\pi} \left(\frac{1}{x}\right)^{2} \cos(\alpha) \cos(\beta) + \frac{2}{\pi x} \cos(\beta) \sin(\alpha) + \frac{1}{\pi} \left(\frac{1}{x}\right)^{2} \cos(\beta) \cos(\alpha) \right)$$

$$= \frac{x}{(m^2 - n^2)} \left(-\frac{2}{\pi x} \cos(\alpha) \sin(\beta) + \frac{2}{\pi x} \cos(\beta) \sin(\alpha) \right)$$

$$= \frac{2}{\pi} \frac{1}{(m^2 - n^2)} \left(\sin(\alpha) \cos(\beta) - \cos(\alpha) \sin(\beta) \right) \tag{6}$$

But

$$\sin(\alpha) \cos(\beta) - \cos(\alpha) \sin(\beta) = \sin(\alpha - \beta)$$

$$= \sin\left(\left(x - \frac{n\pi}{2} - \frac{\pi}{4} \right) - \left(x - \frac{m\pi}{2} - \frac{\pi}{4} \right) \right)$$

$$= \sin\left(x - \frac{n\pi}{2} - \frac{\pi}{4} - x + \frac{m\pi}{2} + \frac{\pi}{4} \right)$$

$$= \sin\left(\frac{m\pi}{2} - \frac{n\pi}{2} \right)$$

$$= \sin\left((m - n) \frac{\pi}{2} \right)$$

Using the above in (6) gives

$$\int_0^\infty \frac{J_m(x) J_n(x)}{x} dx = \frac{2}{\pi} \frac{\sin\left((m-n)\frac{\pi}{2}\right)}{\left(m^2 - n^2\right)}$$

Which is the result required to show. QED.

2 Problem 2

<u>Problem</u> What linear second order ODE does the function $x^m J_n \left(a x^k \right)$ solves? Are there any required relationships among m, n, k? Use this to solve $y'' + x^2 y = 0$

<u>Solution</u>

2.1 Part (a)

We know that the Bessel ODE

$$t^2 z'' (t) + t z' (t) + \left[t^2 - \left(\frac{\alpha}{\beta} \right)^2 \right] z (t) = 0 \tag{1}$$

I am using the order as $\frac{\alpha}{\beta}$ instead of n to make it more general. At the end, $\frac{\alpha}{\beta}$ can always be replaced back by n.

The ODE above has solution

$$z (t) = J_{\frac{\alpha}{\beta}} (t)$$

Hence using the transformation

$$t = a x^k \tag{2}$$

The solution $y (x) \equiv z \left(a x^k \right)$ will becomes

$$y (x) = J_{\frac{\alpha}{\beta}} \left(a x^k \right)$$

Therefore the question now is, how does ODE (1) transforms under (2)? From (2)

$$x = \left(\frac{t}{a} \right)^{\frac{1}{k}}$$

Hence

$$\frac{dx}{dt} = \frac{1}{k} \left(\frac{t}{a} \right)^{\frac{1}{k} - 1}$$

$$= \frac{1}{ak} \left(\frac{t}{a} \right)^{\frac{1}{k} - 1} \tag{3}$$

Now

$$\frac{dz}{dt} = \frac{dz}{dx} \frac{dx}{dt}$$

$$= \frac{dz}{dx} \frac{1}{ak} \left(\frac{t}{a} \right)^{\frac{1}{k} - 1} \tag{5}$$

And

$$\frac{d^2z}{dt^2} = \frac{d}{dt}\left(\frac{dz}{dt}\right)$$

$$= \frac{d}{dt}\left(\frac{dz}{dx}\frac{1}{ak}\left(\frac{t}{a}\right)^{\frac{1}{k}-1}\right)$$

$$= \frac{d^2z}{dx^2}\frac{dx}{dt}\left(\frac{1}{ak}\left(\frac{t}{a}\right)^{\frac{1}{k}-1}\right) + \frac{dz}{dx}\frac{d}{dt}\left(\frac{1}{ak}\left(\frac{t}{a}\right)^{\frac{1}{k}-1}\right)$$

$$= \frac{d^2z}{dx^2}\left(\frac{1}{ak}\left(\frac{t}{a}\right)^{\frac{1}{k}-1}\right)^2 + \frac{dz}{dx}\left(\frac{1}{a^2k}\left(\frac{1}{k}-1\right)\left(\frac{t}{a}\right)^{\frac{1}{k}-2}\right) \tag{6}$$

Using (5,6) then ODE (1) becomes

$$t^2\left[z''\left(ax^k\right)\left(\frac{1}{ak}\left(\frac{t}{a}\right)^{\frac{1}{k}-1}\right)^2 + z'\left(ax^k\right)\frac{1}{a^2k}\left(\frac{1}{k}-1\right)\left(\frac{t}{a}\right)^{\frac{1}{k}-2}\right] + t\left[z'\left(ax^k\right)\frac{1}{ak}\left(\frac{t}{a}\right)^{\frac{1}{k}-1}\right] + \left(t^2 - \left(\frac{\alpha}{\beta}\right)^2\right)z\left(ax^k\right) = 0$$

Writing $y(x) \equiv z\left(ax^k\right)$ so we do not have to keep writing $z\left(ax^k\right)$, the above becomes

$$t^2\left[y''(x)\left(\frac{1}{ak}\left(\frac{t}{a}\right)^{\frac{1}{k}-1}\right)^2 + y'(x)\frac{1}{a^2k}\left(\frac{1}{k}-1\right)\left(\frac{t}{a}\right)^{\frac{1}{k}-2}\right] + t\left[y'(x)\frac{1}{ak}\left(\frac{t}{a}\right)^{\frac{1}{k}-1}\right] + \left(t^2 - \left(\frac{\alpha}{\beta}\right)^2\right)y(x) = 0$$

But $t = ax^k$ and the above becomes

$$a^2x^{2k}\left[y''(x)\left(\frac{1}{ak}\left(\frac{ax^k}{a}\right)^{\frac{1}{k}-1}\right)^2 + y'(x)\frac{1}{a^2k}\left(\frac{1}{k}-1\right)\left(\frac{ax^k}{a}\right)^{\frac{1}{k}-2}\right] + ax^k\left[y'(x)\frac{1}{ak}\left(\frac{ax^k}{a}\right)^{\frac{1}{k}-1}\right] + \left(a^2x^{2k} - \left(\frac{\alpha}{\beta}\right)^2\right)y(x) = 0$$

Which is simplified more as follows

$$a^2 x^{2k}\left[y''(x)\left(\frac{1}{ak}\left(\frac{x}{x^k}\right)\right)^2 + y'(x)\frac{1}{a^2 k}\left(\frac{1}{k}-1\right)\frac{x}{x^{2k}}\right] + ax^k\left(y'(x)\frac{1}{ak}\frac{x}{x^k}\right)+\left(a^2 x^{2k}-\left(\frac{\alpha}{\beta}\right)^2\right)y(x)=0$$

$$a^2 x^{2k}\left[y''(x)\frac{1}{a^2 k^2}\left(\frac{x^2}{x^{2k}}\right) + y'(x)\frac{1}{a^2 k}\left(\frac{1}{k}-1\right)\frac{x}{x^{2k}}\right] + ax^k\left(y'(x)\frac{1}{ak}\frac{x}{x^k}\right)+\left(a^2 x^{2k}-\left(\frac{\alpha}{\beta}\right)^2\right)y(x)=0$$

$$x^{2k}\left[y''(x)\frac{1}{k^2}\left(\frac{x^2}{x^{2k}}\right) + y'(x)\frac{1}{k}\left(\frac{1}{k}-1\right)\frac{x}{x^{2k}}\right] + y'(x)\frac{x}{k}+\left(a^2 x^{2k}-\left(\frac{\alpha}{\beta}\right)^2\right)y(x)=0$$

$$\frac{x^2}{k^2}y''(x) + y'(x)\frac{1}{k}\left(\frac{1}{k}-1\right)x + y'(x)\frac{x}{k}+\left(a^2 x^{2k}-\left(\frac{\alpha}{\beta}\right)^2\right)y(x)=0$$

$$x^2 y''(x) + y'(x)k\left(\frac{1}{k}-1\right)x + y'(x)kx+\left(k^2 a^2 x^{2k}-k^2\left(\frac{\alpha}{\beta}\right)^2\right)y(x)=0$$

$$x^2 y''(x) + xy'(x) + \left(k^2 a^2 x^{2k}-\frac{k^2\alpha^2}{\beta^2}\right)y(x)=0$$

$$(7)$$

We know that the above ODE has one solution as $y(x)=J_{\frac{\alpha}{\beta}}\left(ax^k\right)$ because this is how the above was constructed. Now assuming that

$$w(x)=x^m y(x)$$
$$=x^m J_{\frac{\alpha}{\beta}}\left(ax^k\right)$$

Then $w(x)$ is the solution we want. This means we need to express (7) in terms of $w(x)$ instead of $y(x)$ in order to find the ODE whose solution is $x^m J_{\frac{\alpha}{\beta}}\left(ax^k\right)$.

Since $y(x)=w(x)x^{-m}$ then

$$y'(x)=\frac{d}{dx}(x^{-m}w)$$
$$=-mx^{-m-1}w+x^{-m}w'$$

And

$$y''(x)=\frac{d}{dx}\left(-mx^{-m-1}w+x^{-m}w'\right)$$
$$=-m(-m-1)x^{-m-2}w-mx^{-m-1}w'-mx^{-m-1}w'+x^{-m}w''$$
$$=m(m+1)x^{-m-2}w-2w'mx^{-m-1}+x^{-m}w''$$

Substituting the above results back into (7) gives

$$x^2\left(m(m+1)x^{-m-2}w-2w'mx^{-m-1}+x^{-m}w''\right)+x\left(-mx^{-m-1}w+x^{-m}w'\right)+\left(k^2 a^2 x^{2k}-\frac{k^2\alpha^2}{\beta^2}\right)wx^{-m}=0$$

Dividing by x^{-m}

$$x^2\left(m(m+1)x^{-2}w - 2w'mx^{-1} + w''\right) + x\left(-mx^{-1}w + w'\right) + \left(k^2a^2x^{2k} - \frac{k^2\alpha^2}{\beta^2}\right)w = 0$$

$$m(m+1)w - 2xw'm + x^2w'' - mw + xw' + \left(k^2a^2x^{2k} - \frac{k^2\alpha^2}{\beta^2}\right)w = 0$$

$$x^2w'' + w'(-2xm + x) + \left(k^2a^2x^{2k} + m(m+1) - m - \frac{k^2\alpha^2}{\beta^2}\right)w = 0$$

$$x^2w'' + (1-2m)xw' + \left(k^2a^2x^{2k} + m^2 - \frac{k^2\alpha^2}{\beta^2}\right)w = 0 \qquad (8)$$

Hence the above ODE (8) <u>will have the solution</u> $x^m J_{\frac{\alpha}{\beta}}\left(ax^k\right)$. We can now let $n = \frac{\alpha}{\beta}$ and the above ODE becomes

$$x^2w'' + (1-2m)xw' + \left(k^2a^2x^{2k} + m^2 - k^2n^2\right)w = 0 \qquad (9)$$

Has <u>the required solution</u> $x^m J_n\left(ax^k\right)$.

To answer the final part about the relation between n, m, k. One restriction is that $m = \frac{1}{2}$. One relation between the order n and k is that $m^2 - k^2n^2$ being a rational number. This means

$$m^2 - k^2n^2 = \frac{N}{M}$$

Where N, M are integers.

2.2 Part (b)

$$y''(x) + x^2y(x) = 0 \qquad (1)$$

Comparing this ODE to one found in part (a), written below again, now using $y(x)$ to make it easier to compare

$$x^2y''(x) + (1-2m)xy'(x) + \left(k^2a^2x^{2k} + m^2 - k^2n^2\right)y(x) = 0$$

$$y''(x) + \frac{(1-2m)}{x}y'(x) + \frac{1}{x^2}\left(k^2a^2x^{2k} + m^2 - k^2n^2\right)y(x) = 0 \qquad (2)$$

To make (2) same as (1), we want $(1-2m) = 0$ or $m = \frac{1}{2}$. Also need $2k = 4$ or $k = 2$. Using these the above reduces to

$$y''(x) + \left(4a^2x^2 + \frac{\frac{1}{4} - 4n^2}{x^2}\right)y(x) = 0$$

Therefore, we need also that $n^2 = \frac{1}{16}$ in order to cancel extra term above. Hence $n = \frac{1}{4}$. Now the above becomes

$$y''(x) + 4a^2x^2y(x) = 0$$

Finally, if we let $a^2 = \frac{1}{4}$ or $a = \frac{1}{2}$, then the above becomes

$$y''(x) + x^2 y(x) = 0$$

Therefore, we found that

$$n = \frac{1}{4}$$
$$a = \frac{1}{2}$$
$$k = 2$$
$$m = \frac{1}{2}$$

Hence the following solves the ODE

$$y(x) = x^m J_n\left(ax^k\right)$$
$$= \sqrt{x} J_{\frac{1}{4}}\left(\frac{1}{2}x^2\right)$$

2.3 Appendix

To verify the above result, it is solved again directly. We first need to convert this ODE to Bessel ODE. Let

$$y = x^{\frac{1}{2}} z(x)$$

Then

$$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}z + x^{\frac{1}{2}}z'$$
$$\frac{d^2y}{dx^2} = -\frac{1}{4}x^{-\frac{3}{2}}z + \frac{1}{2}x^{-\frac{1}{2}}z' + \frac{1}{2}x^{-\frac{1}{2}}z' + x^{\frac{1}{2}}z''$$
$$= -\frac{1}{4}x^{-\frac{3}{2}}z + x^{-\frac{1}{2}}z' + x^{\frac{1}{2}}z''$$

Substituting the above into (1) gives

$$\left(-\frac{1}{4}x^{-\frac{3}{2}}z + x^{-\frac{1}{2}}z' + x^{\frac{1}{2}}z''\right) + x^2 x^{\frac{1}{2}}z = 0$$

$$x^{\frac{1}{2}}z'' + x^{-\frac{1}{2}}z' + \left(x^{\frac{5}{2}} - \frac{1}{4}x^{-\frac{3}{2}}\right)z = 0$$

Multiplying both sides by $x^{\frac{3}{2}}$ gives

$$x^2 z'' + xz' + \left(x^4 - \frac{1}{4}\right)z = 0 \tag{2}$$

Where the derivatives above is with respect to x. Now let $t = \frac{x^2}{2}$. Then

$$\frac{dz}{dx} = \frac{dz}{dt}\frac{dt}{dx} = x\frac{dz}{dt}$$

And

$$\frac{d^2z}{dx^2} = \frac{d^2z}{dt^2}\left(\frac{dt}{dx}\right)(x) + \frac{dz}{dt}$$

$$= \frac{d^2z}{dt^2}x^2 + \frac{dz}{dt}$$

Substituting the above into (2) gives

$$x^2\left(x^2z'' + z'\right) + x\left(xz'\right) + \left(x^4 - \frac{1}{4}\right)z = 0$$

Where the derivatives above is with respect to t now. This simplifies to

$$x^4z'' + 2x^2z' + \left(x^4 - \frac{1}{4}\right)z = 0$$

But $t = \frac{x^2}{2}$, hence the above becomes

$$4t^2z'' + 4tz' + \left(4t^2 - \frac{1}{4}\right)z = 0$$

$$t^2z'' + tz' + \left(t^2 - \frac{1}{16}\right)z = 0$$

This now in the form of Bessel ODE

$$t^2z'' + tz' + \left(t^2 - n^2\right)z = 0$$

Where $n = \frac{1}{4}$. Hence one solution is

$$z(t) = J_n(t)$$

$$= J_{\frac{1}{4}}(t)$$

But $y(x) = \sqrt{x}z(x)$ and $t = \frac{x^2}{2}$, therefore the above becomes

$$y(x) = \sqrt{x}J_{\frac{1}{4}}\left(\frac{x^2}{2}\right) \tag{3}$$

Which is the same as found in part (b)

3   Problem 3

Problem Prove that $|J_n(x)| \leq 1$ for all integers n

Solution

From the integral representation of $J_n(x)$ for integer n

$$J_n(x) = \frac{1}{\pi} \int_0^\pi \cos(n\theta - x\sin\theta)\, d\theta$$

Then

$$|J_n(x)| \leq \frac{1}{\pi} \left| \int_0^\pi \cos(n\theta - x\sin\theta)\, d\theta \right|_{\max}$$

$$\leq \frac{1}{\pi} \int_0^\pi |\cos(n\theta - x\sin\theta)|_{\max}\, d\theta$$

$$= \frac{1}{\pi} |M|_{\max} \int_0^\pi d\theta$$

$$= \frac{1}{\pi} |M|_{\max} \pi$$

$$= |M|_{\max}$$

Where $|M|_{\max} = |\cos(n\theta - x\sin\theta)|_{\max}$ over $\theta = 0 \cdots \pi$. But this is 1 for the cosine function. Hence

$$|J_n(x)| \leq 1$$

4 Problem 4

Problem Starting with the integral formula for hypergeometric function, express the following in terms of elementary functions $_2F_1\left(1,1,2;x\right)$ and $_2F_1\left(a,1,1;x\right)$

Solution

$$_2F_1\left(a,b,c;x\right) = \frac{\Gamma\left(c\right)}{\Gamma\left(b\right)\Gamma\left(c-b\right)}\int_0^1 t^{b-1}\left(1-t\right)^{c-b-1}\left(1-tx\right)^{-a}dt \tag{1}$$

$$= \frac{\Gamma\left(c\right)}{\Gamma\left(a\right)\Gamma\left(b\right)}\sum_{n=0}^{\infty}\frac{\Gamma\left(a+n\right)\Gamma\left(b+n\right)}{\Gamma\left(c+n\right)}\frac{x^n}{n!} \tag{2}$$

4.1 Part (a)

Here $a=1, b=1, c=2$. Therefore, using (1) representation gives

$$_2F_1\left(1,1,2;x\right) = \frac{\Gamma\left(2\right)}{\Gamma\left(1\right)\Gamma\left(2-1\right)}\int_0^1 t^{1-1}\left(1-t\right)^{2-1-1}\left(1-tx\right)^{-1}dt$$

$$= \frac{\Gamma\left(2\right)}{\Gamma\left(1\right)\Gamma\left(1\right)}\int_0^1\frac{dt}{1-tx}$$

But $\Gamma\left(2\right)=1, \Gamma\left(1\right)=0$, therefore the above becomes

$$_2F_1\left(1,1,2;x\right) = \int_0^1\frac{dt}{1-tx}$$

$$= \left[\frac{-\ln\left(1-tx\right)}{x}\right]_0^1$$

$$= -\left(\frac{\ln\left(1-x\right)}{x} - \frac{-\ln\left(1-0\right)}{x}\right)$$

$$= -\frac{\ln\left(1-x\right)}{x}$$

4.2 Part (b)

Here $a=a, b=1, c=1$. Therefore (2) representation gives

$$_2F_1\left(a,1,1;x\right) = \frac{\Gamma\left(c\right)}{\Gamma\left(a\right)\Gamma\left(b\right)}\sum_{n=0}^{\infty}\frac{\Gamma\left(a+n\right)\Gamma\left(b+n\right)}{\Gamma\left(c+n\right)}\frac{x^n}{n!}$$

$$= \frac{1}{\Gamma\left(a\right)}\sum_{n=0}^{\infty}\frac{\Gamma\left(a+n\right)\Gamma\left(1+n\right)}{\Gamma\left(1+n\right)}\frac{x^n}{n!}$$

$$= \sum_{n=0}^{\infty}\frac{\Gamma\left(a+n\right)}{\Gamma\left(a\right)}\frac{x^n}{n!}$$

Looking at few values

n	$_2F_1\left(a,1,1;x\right)$
0	$\dfrac{\Gamma(a)}{\Gamma(a)}=1$
1	$\dfrac{\Gamma(a+1)}{\Gamma(a)}x$
2	$\dfrac{\Gamma(a+2)}{\Gamma(a)}\dfrac{x^2}{2!}$
3	$\dfrac{\Gamma(a+3)}{\Gamma(a)}\dfrac{x^3}{3!}$
$\vdots$	$\vdots$

Using the recursive relation $\Gamma\left(a+1\right)=a\Gamma\left(a\right)$, which works for integer and non integer a, then we see that

$$\Gamma\left(a+1\right)=a\Gamma\left(a\right)$$

And

$$\begin{aligned}\Gamma\left(a+2\right)&=\Gamma\left(\left(a+1\right)+1\right)\\&=\left(a+1\right)\Gamma\left(a+1\right)\\&=\left(a+1\right)a\Gamma\left(a\right)\end{aligned}$$

And

$$\begin{aligned}\Gamma\left(a+3\right)&=\Gamma\left(\left(a+2\right)+1\right)\\&=\left(a+2\right)\Gamma\left(\left(a+2\right)\right)\\&=\left(a+2\right)\left(a+1\right)a\Gamma\left(a\right)\end{aligned}$$

And so on. Hence the above now becomes

n	$_2F_1\left(a,1,1;x\right)$
0	1
1	$\dfrac{a\Gamma(a)}{\Gamma(a)}x=ax$
2	$\dfrac{(a+1)a\Gamma(a)}{\Gamma(a)}\dfrac{x^2}{2!}=a\left(a+1\right)\dfrac{x^2}{2!}$
3	$\dfrac{(a+2)(a+1)a\Gamma(a)}{\Gamma(a)}\dfrac{x^3}{3!}=a\left(a+1\right)\left(a+2\right)\dfrac{x^3}{3!}$
$\vdots$	$\vdots$

We see from the above the pattern of the sequence is as follows

$$_2F_1\left(a,1,1;x\right)=1+ax+a\left(a+1\right)\frac{x^2}{2!}+a\left(a+1\right)\left(a+2\right)\frac{x^3}{3!}+\cdots \tag{1}$$

Comparing the above to the Binomial expansion given by

$$\left(1+z\right)^n=1+nz+n\left(n-1\right)\frac{z^2}{2!}+n\left(n-1\right)\left(n-2\right)\frac{z^3}{3!}+\cdots \tag{2}$$

By replacing $z \to -x$ and $n \to -a$, the above becomes

$$(1-x)^{-a} = 1 + (-a)(-x) + (-a)((-a)-1)\frac{(-x)^2}{2!} + (-a)((-a)-1)((-a)-2)\frac{(-x)^3}{3!} + \cdots$$

$$= 1 + ax + (a)(a+1)\frac{x^2}{2!} + (a)(a+1)(a+2)\frac{x^3}{3!} + \cdots$$

Comparing the above to (1) shows it is the same series. Hence

$$_2F_1(a,1,1;x) = (1-x)^{-a}$$