① $x^2 y' + y^2 = x y y'$ $\quad \dim(y) = \dim(x)$

Try $y = xv \rightarrow \frac{dv}{dx} = \frac{v}{v-1} \frac{1}{x}$ separable

$x = \frac{c}{v} e^{v} \rightarrow \boxed{y = c e^{y/x}}$

② $y' = \frac{a^2}{(x+y)^2}$ Try $u = x+y$

Leads to $dx = \frac{du}{1 + \frac{a^2}{u^2}}$ separable

$$\Rightarrow \quad x - c = \int du \left[1 - \frac{a^2}{u^2+a^2} \right] = u - a \tan^{-1}\left(\frac{u}{a}\right)$$
$$= x+y - a\tan^{-1}\left(\frac{x+y}{a}\right)$$

$$\boxed{y = a \tan\left(\frac{y+c}{a}\right) - x}$$

③ $y'' + y'^2 + 1 = 0$ Lacks 2 variables.

$p = y'$ $\quad p' = -(p^2+1)$ separable

$p = -\tan(x+c) = \frac{dy}{dx}$

$$\boxed{y = a + \ln\left[\cos(x+c)\right]}$$

(4) $xy' + y + x^4 y^4 e^x = 0$

Let $v = xy$ $\quad \frac{dv}{dx} = -v^4 e^x$ separable

$$v^3 = \frac{1}{3(e^x + c)} \qquad \boxed{y = \frac{1}{x} \frac{1}{[3(e^x + c)]^{1/3}}}$$

(5) $x^2 y'^2 - 2(xy-4)y' + y^2 = 0$

Differentiate: $2xy'^2 + 2x^2 y' y'' - 2(y + xy')y'$

$- 2(xy-4)y'' + 2yy' = 0$

Factorize: $y''\left[x^2 y' - xy + 4\right] = 0$

(i) $y'' = 0 \Rightarrow y = ax + b$ Substitute into original equation which is solved only if $a = -\frac{1}{8}b^2$

$$\boxed{y = -\frac{1}{8}b^2 x + b}$$

(ii) $y' - \frac{1}{x}y = \frac{-4}{x^2}$ Solve by method of integrating factor.

$y = \frac{2}{x} + cx$

Substitute into original equation which is solved only if $c = 0$

$$\boxed{y = \frac{2}{x}}$$

⑥ $A^2(x)\, y'' + A(x) A'(x)\, y' + y = 0$

Notice that dim (A) = dim (x)

Look for solution of the form

$$y = c_1 \cos f(x) + c_2 \sin f(x)$$ 2 linearly independent solutions

$$y' = -c_1 f' \sin f + c_2 f' \cos f$$
$$y'' = \cos f\left[-c_1 f'^2 + c_2 f''\right] + \sin f\left[-c_1 f'' - c_2 f'^2\right]$$

Original equation becomes

$$\cos f\left\{c_1\left[1 - A^2 f'^2\right] + c_2 A\left[A f'' + A' f'\right]\right\}$$
$$+ \sin f\left\{c_2\left[1 - A^2 f'^2\right] + c_1 A\left[-A f'' - A' f'\right]\right\} = 0$$

satisfied if $f' = \frac{1}{A}$ or $\boxed{f(x) = \int_{x_0}^{x} \frac{dx'}{A(x')}}$

(sign doesn't matter)

$$\boxed{y = c_1 \cos f(x) + c_2 \sin f(x)}$$

(7) $\quad x y'' + \frac{3}{x} y = 1 + x^3$

$\boxed{y = y_H + y_p}$ $\qquad x^2 y_H'' + 3 y_H = 0$

$x = 0$ is a regular singular point. Try $y_H = x^s$

Then $\quad s(s-1) + 3 = 0 \quad \Rightarrow \quad s = \frac{1}{2} \pm \frac{\sqrt{11}}{2} i$

Now $\quad x^{\pm \frac{\sqrt{11}}{2} i} = \exp\left[\ln x^{\pm \frac{\sqrt{11}}{2} i}\right] = \exp\left[\pm \frac{\sqrt{11}}{2} i \ln x\right]$

or $\quad \cos\left(\frac{\sqrt{11}}{2} \ln x\right)$ & $\sin\left(\frac{\sqrt{11}}{2} \ln x\right)$

$$\boxed{y_H = \sqrt{x}\left[a \cos\left(\frac{\sqrt{11}}{2} \ln x\right) + b \sin\left(\frac{\sqrt{11}}{2} \ln x\right)\right]}$$

Try $\quad y_p = \alpha x^4 + \beta x^3 + \gamma x^2 + \rho x$

Substitution $\Rightarrow \quad \alpha = \frac{1}{15} \quad \rho = \frac{1}{3} \quad \beta = \gamma = 0$

$$\boxed{y_p = \frac{1}{15} x^4 + \frac{1}{3} x}$$

(8) $$y'' - \left(\frac{1}{4} + \frac{k}{x}\right) y = 0$$

As $x \to \infty$ neglect $\frac{k}{x}$ compared to $\frac{1}{4}$.

Then $y \to e^{\pm \frac{1}{2} x}$. We want $y \to e^{-\frac{1}{2} x}$.

Now write $y = f(x)\, e^{-\frac{1}{2}x}$

polynomial in x?

$$f'' - f' - \frac{k}{x} f = 0$$

$$f(x) = \sum_{n=1}^{\infty} a_n x^n \qquad f' = \sum_{n=1}^{\infty} n a_n x^{n-1} \qquad f'' = \sum_{n=2}^{\infty} n(n-1) x^{n-2} a_n$$

Substitution: $$\sum_{n=1}^{\infty} \left\{ n(n+1) a_{n+1} - n a_n - k a_n \right\} x^{n-1} = 0$$

$$\Rightarrow \quad a_{n+1} = \frac{n+k}{n(n+1)} a_n \qquad \frac{a_{n+1}}{a_n} \underset{n \to \infty}{\longrightarrow} \frac{1}{n}$$ so this

would lead to a factor e^{x}, and the full

solution would go as $y \underset{x \to \infty}{\longrightarrow} e^{-\frac{1}{2}x} e^{x} = e^{\frac{1}{2}x}$.

Series terminates if $n + k = 0$ for some $n = 1, 2, 3, \ldots$

$$\boxed{k = -1, -2, -3, \ldots}$$