HW 1
Physics 5041 Mathematical Methods for Physics
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Contents

1 Problem 1

Problem Solve $x^2y' + y^2 = xyy'$

Solution

Rewriting the ODE as

$$y'\left(x^2 - xy\right) + y^2 = 0 \tag{1}$$

Dividing by $x^2 \neq 0$ gives

$$\frac{dy}{dx}\left(1 - \frac{y}{x}\right) + \frac{y^2}{x^2} = 0$$

We see this is homogeneous of order 1. This can be confirmed by writing the above as

$$\begin{aligned} dy\left(x^2 - xy\right) + y^2 dx &= 0 \\ dyx^2 - xydy + y^2 dx &= 0 \end{aligned}$$

We want to find if a weight m can be found, so that the substitution $y = vx^m$ makes the above ODE separable. To find m, we assign weight m to both y and dy, and a weight of 1 to both x and dx, and then try to find if there is an m which makes each term sums to the same total weight. (in other words, we want each term units to be the same).

The term $\left(dy\right)\left(x^2\right)$ has total weight of $m+2$ (it is the exponents that we add). And the term $(x)\left(y\right)\left(dy\right)$ has total weight $1+2m$ and the last term $\left(y^2\right)(dx)$ has weight $2m+1$. Therefore we have this result for the weight of each term (there are 3 terms above).

$$\{m+2, 1+2m, 1+2m\}$$

We see that if $m = 1$ then each term will have the same total weight of 3 giving $\{3,3,3\}$. So this is homogenous ODE of order $m = 1$. Now that we know the weight, we use the substitution

$$\boxed{y = vx}$$

Hence $y' = v'x + v$. Substituting these back into (1) gives a new ODE in v which is separable. If it is not separable, it means we made a mistake somewhere.

$$\begin{aligned} (v'x + v)\left(x^2 - x^2 v\right) + v^2x^2 &= 0 \\ v'x^3 - v'vx^3 + vx^2 - x^2v^2 + v^2x^2 &= 0 \\ v'x^3 - v'vx^3 + vx^2 &= 0 \end{aligned}$$

Dividing by x^3 for $x \neq 0$ gives

$$\begin{aligned} v' - v'v + \frac{v}{x} &= 0 \\ v'(1 - v) &= -\frac{v}{x} \\ \frac{dv}{dx}\frac{(1-v)}{v} &= -\frac{1}{x} \\ dv\frac{(1-v)}{v} &= -\frac{1}{x}dx \\ dv\frac{(v-1)}{v} &= \frac{1}{x}dx \end{aligned}$$

Integrating both sides gives

$$\begin{aligned} \int v - \frac{1}{v}dv &= \int \frac{1}{x}dx \\ v - \ln v &= \ln x + C \end{aligned}$$

Taking exponential of both sides gives

$$\begin{aligned} e^{v - \ln v} &= Cx \\ \frac{e^v}{v} &= Cx \end{aligned}$$

But $v = \frac{y}{x}$. Therefore the above becomes

$$
\begin{aligned}
\frac{e^{\frac{y}{x}}}{\frac{y}{x}} &= Cx \\
\frac{e^{\frac{y}{x}}}{y} &= C
\end{aligned}
$$

Hence the solution is

$$
y = C_1 e^{\frac{y}{x}} \qquad x \neq 0 \tag{2}
$$

Where C_1 is the constant of integration. y can not be solved for directly in the above. But we can solve for x in terms of y if needed as follows

$$
\begin{aligned}
\ln y &= \ln C_1 + \frac{y}{x} \\
\ln y - C_2 &= \frac{y}{x} \\
x &= \frac{y}{\ln y - C_2}
\end{aligned} \tag{3}
$$

2 Problem 2

Problem Solve $y' = \frac{a^2}{\left(x+y\right)^2}$

Solution

Let $v(x) = x + y(x)$. Hence $v' = 1 + y'$ or $y' = v' - 1$. Substituting this back into the ODE gives

$$
\begin{aligned}
v' - 1 &= \frac{a^2}{v^2} \\
\frac{dv}{dx} &= \frac{a^2}{v^2} + 1
\end{aligned}
$$

This is separable.

$$
\begin{aligned}
\frac{dv}{\frac{a^2}{v^2} + 1} &= dx \\
\frac{v^2}{a^2 + v^2} dv &= dx
\end{aligned}
$$

By long division $\frac{v^2}{a^2+v^2} = 1 - \frac{a^2}{a^2+v^2}$. The above becomes

$$
\left(1 - \frac{a^2}{a^2 + v^2}\right) dv = dx
$$

Integrating both sides gives

$$
\begin{aligned}
\int \left(1 - \frac{a^2}{a^2 + v^2}\right) dv &= \int dx \\
\int dv - a^2 \int \frac{1}{a^2 + v^2} dv &= \int dx
\end{aligned}
$$

But $\int \frac{1}{a^2+v^2} dv = \frac{1}{a^2} \int \frac{1}{1+\left(\frac{v}{a}\right)^2} dv = \frac{1}{a^2}\left(a \arctan\left(\frac{v}{a}\right)\right) = \frac{1}{a} \arctan\left(\frac{v}{a}\right)$, hence the above becomes

$$
\begin{aligned}
v - a^2 \left(\frac{1}{a} \arctan\left(\frac{v}{a}\right)\right) &= x + C \\
v - a \arctan\left(\frac{v}{a}\right) &= x + C \\
a \arctan\left(\frac{v}{a}\right) &= v - x - C \\
\arctan\left(\frac{v}{a}\right) &= \frac{v - x}{a} + C_1
\end{aligned}
$$

Where $C_1 = \frac{-C}{a}$, a new constant. Taking the $\tan$ of both sides gives

$$
\frac{v}{a} = \tan\left(\frac{v - x}{a} + C_1\right)
$$

But $v = x + y$, and the above becomes

$$
\begin{aligned}
\frac{x + y}{a} &= \tan\left(\frac{\left(x + y\right) - x}{a} + C_1\right) \\
\frac{x + y}{a} &= \tan\left(\frac{y}{a} + C_1\right)
\end{aligned}
$$

Therefore the final solution is

$$
y = a \tan\left(\frac{y}{a} + C_1\right) - x \qquad a \neq 0
$$

Where C_1 is arbitrary constant.

3 Problem 3

Problem Solve $y'' + \left(y'\right)^2 + 1 = 0$

Solution

Since y is missing from the ODE, we can convert this to a first order using $y' = p(x)$. Therefore $y'' = \frac{dp}{dx}$ and the ODE becomes

$$\begin{aligned}
\frac{dp}{dx} + p^2 + 1 &= 0 \\
\frac{dp}{dx} &= -\left(1 + p^2\right) \\
\frac{dp}{1+p^2} &= -dx
\end{aligned}$$

Integrating both sides gives

$$\begin{aligned}
\int \frac{dp}{1+p^2} &= -\int dx \\
\arctan\left(p\right) &= -x + C_1 \\
p &= \tan\left(-x + C_1\right)
\end{aligned}$$

But $p = y'$. Hence we need now to solve $\frac{dy}{dx} = \tan\left(-x + C_1\right)$. Integrating both sides gives

$$\begin{aligned}
y &= \int \tan\left(-x + C_1\right) dx \\
&= \int \frac{\sin\left(-x + C_1\right)}{\cos\left(-x + C_1\right)} dx \\
&= \int \frac{-\sin\left(x - C_1\right)}{\cos\left(x - C_1\right)} dx \\
&= \int \frac{\frac{d}{dx}\left(\cos\left(x - C_1\right)\right)}{\cos\left(x - C_1\right)} dx
\end{aligned}$$

But $\int \frac{V'}{V} dx = \ln\left(V\right)$, hence the above becomes

$$y = \ln\left(\cos\left(x - C_1\right)\right) + C_2$$

Replacing $-C_1$ by new constant C_3, the final solution becomes

$$y = \ln\left(\cos\left(x + C_3\right)\right) + C_2$$

Where C_2, C_3 are constants of integration.

4 Problem 4

Problem Solve $xy' + y + x^4 y^4 e^x = 0$

Solution

Dividing by $x \neq 0$ and rewriting gives

$$y' + \frac{1}{x}y = \left(-x^3 e^x\right) y^4 \tag{1}$$

A Bernoulli ODE has the form $y' + a(x) y = b(x) y^n$ where $n \neq 1$. Comparing the above to Bernoulli ODE form, show it is Bernoulli ODE where $a(x) = \frac{1}{x}, b(x) = -x^3 e^x$. Dividing (1) by y^4 gives

$$\frac{1}{y^4}y' + \frac{1}{x}y^{-3} = -x^3 e^x$$

Letting $v = y^{-3}$ or $\frac{dv}{dx} = -3y^{-4}\frac{dy}{dx}$. Hence $\frac{dy}{dx} = -\frac{dv}{dx}\frac{y^4}{3}$. Substituting this in the above gives

$$\begin{aligned}
\frac{1}{y^4}\left(-\frac{dv}{dx}\frac{y^4}{3}\right) + \frac{1}{x}v &= -x^3 e^x \\
-\frac{1}{3}\frac{dv}{dx} + \frac{1}{x}v &= -x^3 e^x \\
\frac{dv}{dx} - \frac{3}{x}v &= 3x^3 e^x
\end{aligned}$$

This is now linear in v. The integrating factor $\mu = e^{\int -\frac{3}{x}dx} = e^{-3\ln x} = \frac{1}{x^3}$. Multiplying both sides of the above by this integrating factor making the left side complete differential

$$\begin{aligned}
\frac{d}{dx}\left(\frac{1}{x^3}v\right) &= \frac{1}{x^3}3x^3 e^x \\
\frac{d}{dx}\left(\frac{1}{x^3}v\right) &= 3e^x
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\frac{1}{x^3}v &= 3e^x + C \\
v &= 3x^3 e^x + Cx^3 \\
&= x^3\left(3e^x + C\right)
\end{aligned}$$

But $v = y^{-3}$, hence the above becomes

$$\begin{aligned}
\frac{1}{y^3} &= x^3\left(3e^x + C\right) \\
y^3 &= \frac{1}{x^3\left(3e^x + C\right)}
\end{aligned}$$

This shows that there are 3 solutions since the above is a cubic equation. But we can leave the solution in implicit form

$$\begin{aligned}
y &= \sqrt[\frac{1}{3}]{\frac{1}{x^3\left(3e^x + C\right)}} \\
&= \frac{1}{x}\sqrt[\frac{1}{3}]{\frac{1}{3e^x + C}}
\end{aligned}$$

5 Problem 5

Problem Find both a general solution and a singular solution of

$$x^2\left(y'\right)^2-2\left(xy-4\right)y'+y^2=0$$

Solution

Rewriting the ODE as

$$y^2-2xyy'+8y'+x^2\left(y'\right)^2=0$$

Let $y'=p$ and the above becomes

$$y^2+y\left(-2xp\right)+\left(8p+x^2p^2\right)=0$$

This is quadratic in y. Solving for $y=\frac{-b}{2a}\pm\frac{1}{2a}\sqrt{b^2-4ac}$

$$\begin{aligned}
y &= xp\pm\frac{1}{2}\sqrt{4x^2p^2-4\left(8p+x^2p^2\right)}\\
&= xp\pm\sqrt{x^2p^2-8p-x^2p^2}\\
&= xp\pm 2\sqrt{-2p}
\end{aligned}$$

case one

$$\begin{aligned}
y &= xp+2\sqrt{-2p}\\
&= xp+f\left(p\right)
\end{aligned}\tag{1}$$

This can be written as

$$y=G\left(x,p\right)$$

Where $G\left(x,p\right)=xp+f\left(p\right)$. This form of ODE is called the Clairaut ODE. Taking derivative w.r.t. x gives

$$y'=\frac{\partial G}{\partial x}+\frac{\partial G}{\partial p}\frac{dp}{dx}$$

But $y'=p$ and the above becomes

$$p=\frac{\partial G}{\partial x}+\frac{\partial G}{\partial p}\frac{dp}{dx}$$

But $\frac{\partial G}{\partial x}=p$, hence the above reduces to

$$0=\frac{\partial G}{\partial p}\frac{dp}{dx}\tag{2}$$

Then either $\frac{\partial G}{\partial p}=0$ or $\frac{dp}{dx}=0$.

When $\frac{dp}{dx}=0$ or $y''=0$ therefore the solution is

$$y=C_1x+C_2\tag{3}$$

But we are solving a first order ODE. So we expect it to have one constant of integration only. By comparing (3) with equation (1) which is $y=xp+f\left(p\right)$ shows that

$$C_2=f\left(C_1\right)=2\sqrt{-2C_1}$$

Then the solution will now contain one constant of integration C_1. Hence the first solution is

$$y=C_1x+2\sqrt{-2C_1}$$

The second possibility comes from $\frac{\partial G}{\partial p} = 0$. This gives

$$
\begin{aligned}
x + f'\left(p\right) &= 0 \\
x + 2\frac{d}{dp}\left(-2p\right)^{\frac{1}{2}} &= 0 \\
x + 2\frac{1}{2}\left(-2p\right)^{\frac{-1}{2}}(-2) &= 0 \\
x - \frac{2}{\sqrt{-2p}} &= 0 \\
x\sqrt{-2p} &= 2 \\
-2px^2 &= 4 \\
p &= -\frac{2}{x^2}
\end{aligned}
$$

Now that we found p, we substitute it back into (1) given by $y = xp + 2\sqrt{-2p}$. Hence the second solution is found directly as follows

$$
\begin{aligned}
y &= xp + 2\sqrt{-2p} \\
&= -\frac{2}{x} + 2\sqrt{-2\left(-\frac{2}{x^2}\right)} \\
&= -\frac{2}{x} + 2\sqrt{\frac{4}{x^2}} \\
&= -\frac{2}{x} + \frac{4}{x} \\
&= \frac{2}{x}
\end{aligned}
$$

Summary of case one From above we obtained the following two solutions

$$
\begin{aligned}
y_1 &= C_1x + 2\sqrt{-2C_1} \\
y_2 &= \frac{2}{x}
\end{aligned}
$$

Where $y_2(x)$ is the singular solution since it can't be obtained from the first solution with the constants of integrations by changing them to any value.

We now do the same steps for the case of $y = xp - 2\sqrt{-2p}$. This follows the same steps as above as the only difference is the sign and hence the steps will not be repeated. It gives the solution

$$
y_3 = C_1x - 2\sqrt{-2C_1}
$$

With the same singular solution. Therefore there are three solutions to this ODE and these are summarized below

$$
\begin{aligned}
y_1 &= C_1x + 2\sqrt{-2C_1} \\
y_2 &= \frac{2}{x} \\
y_3 &= C_1x - 2\sqrt{-2C_1}
\end{aligned}
$$

With $y_2(x)$ being the singular solution. Singular solutions do not have constant of integration in them and can not be obtained from the general solution by any substitution for constants of integration. The general solution contain constant of integrations in them.

6 Problem 6

Problem

Find the general real solution to the following equation where $A(x)$ is a known function

$$A(x)y'' + A'(x)y' + \frac{y}{A(x)} = 0$$

Solution

Let us first assume $A(x)$ is constant not zero. The above reduces to

$$y'' + \frac{y}{A^2} = 0$$

This is harmonic oscillator It has the form of $y'' + \omega^2 y = 0$ with $\omega = \frac{1}{A}$ being the natural frequency. The solution to this is easily found to be

$$\begin{aligned} y(x) &= C_1 \cos(\omega x) + C_2 \sin(\omega x) \\ &= C_1 \cos\left(\frac{x}{A}\right) + C_2 \sin\left(\frac{x}{A}\right) \end{aligned} \tag{1}$$

Since A is not constant, then we can try a similar solution[1] but use $f(x)$ for the arguments of the trigonometric functions

$$y(x) = C_1 \cos\left(f(x)\right) + C_2 \sin\left(f(x)\right) \tag{2}$$

where $f(x)$ is function of x to be determined. Hence. From now on, we will write f instead of $f(x)$ to simplify notation.

$$\begin{aligned} y' &= -C_1 f' \sin(f) + C_2 f' \cos(f) \\ y'' &= -C_1 f'' \sin(f) - C_1 (f')^2 \cos(f) + C_2 f'' \cos(f) - C_2 (f')^2 \sin(f) \end{aligned}$$

Substituting these back into the original ODE gives

$$\begin{aligned} A^2\left(-C_1 f'' \sin(f) - C_1 (f')^2 \cos(f) + C_2 f'' \cos(f) - C_2 (f')^2 \sin(f)\right) + \\ AA'\left(-C_1 f' \sin(f) + C_2 f' \cos(f)\right) + C_1 \cos(f) + C_2 \sin(f) = 0 \end{aligned}$$

Collecting terms gives

$$\cos(f)\left(-C_1 A^2 (f')^2 + C_2 A^2 f'' + C_2 AA' f' + C_1\right) + \sin(f)\left(-C_1 A^2 f'' - C_2 A^2 (f')^2 - C_1 AA' f' + C_2\right) = 0$$

Since this is zero for all $\sin$ and $\cos$ then

$$\begin{aligned} -C_1 A^2 (f')^2 + C_2 A^2 f'' + C_2 AA' f' + C_1 &= 0 \\ -C_1 A^2 f'' - C_2 A^2 (f')^2 - C_1 AA' f' + C_2 &= 0 \end{aligned}$$

Multiplying the first equation by C_2 and the second by C_1 gives

$$\begin{aligned} -C_2 C_1 A^2 (f')^2 + C_2^2 A^2 f'' + C_2^2 AA' f' + C_1 C_2 &= 0 \\ -C_1^2 A^2 f'' - C_1 C_2 A^2 (f')^2 - C_1^2 AA' f' + C_1 C_2 &= 0 \end{aligned}$$

Subtracting the second equation from the first gives

$$\begin{aligned} \left(-C_2 C_1 A^2 (f')^2 + C_2^2 A^2 f'' + C_2^2 AA' f' + C_1 C_2\right) - \left(-C_1^2 A^2 f'' - C_1 C_2 A^2 (f')^2 - C_1^2 AA' f' + C_1 C_2\right) &= 0 \\ -C_2 C_1 A^2 (f')^2 + C_2^2 A^2 f'' + C_2^2 AA' f' + C_1 C_2 + C_1^2 A^2 f'' + C_1 C_2 A^2 (f')^2 + C_1^2 AA' f' - C_1 C_2 &= 0 \\ C_2^2 A^2 f'' + C_2^2 AA' f' + C_1^2 A^2 f'' + C_1^2 AA' f' &= 0 \\ f''\left(C_2^2 A^2 + C_1^2 A^2\right) + f'\left(C_2^2 AA' + C_1^2 AA'\right) &= 0 \end{aligned}$$

Let us call $C_2^2 A^2 + C_1^2 A^2 = \mu$ and $C_2^2 AA' + C_1^2 AA' = \beta$ for the moment. The above becomes

$$\mu f'' + \beta f' = 0$$

Since f is missing, then we can solve the above by assuming $f' = v$. The above becomes

[1] Thanks to hint from the Professor.

$v' + \frac{\beta}{\mu}v = 0$. This is linear in v. The integrating factor is $I = e^{\int \frac{\beta}{\mu}dx}$. Hence the ode becomes

$$\frac{d}{dx}\left(e^{\int \frac{\beta}{\mu}dx}v\right) = 0$$
$$v = C_3 e^{-\int \frac{\beta}{\mu}dx}$$

Since the proposed solution in (2) contains integration of constants already, we can choose $C_3 = 1$ without affecting the final solution. Hence

$$f'(x) = e^{-\int \frac{\beta}{\mu}dx}$$

Therefore

$$\begin{aligned} f(x) &= \int e^{-\int \frac{\beta}{\mu}dx}dx + C_4 \\ &= \int \left(C_3 e^{-\int \frac{C_2^2 AA' + C_1^2 AA'}{C_2^2 A^2 + C_1^2 A^2}dx} dx\right) dx + C_4 \end{aligned} \tag{3}$$

Again, since the proposed solution in (2) contains integration of constants already, we can choose $C_4 = 0$. The above becomes

$$\begin{aligned} f(x) &= \int e^{-\int \frac{\beta}{\mu}dx}dx \\ &= \int e^{-\int \frac{C_2^2 AA' + C_1^2 AA'}{C_2^2 A^2 + C_1^2 A^2}dx} dx \end{aligned}$$

The expression $\frac{C_2^2 AA' + C_1^2 AA'}{C_2^2 A^2 + C_1^2 A^2}$ can be simplified as follows

$$\frac{C_2^2 AA' + C_1^2 AA'}{C_2^2 A^2 + C_1^2 A^2} = \frac{AA'\left(C_2^2 + C_1^2\right)}{A^2\left(C_2^2 + C_1^2\right)} = \frac{AA'}{A^2}$$

Hence (3) becomes

$$\begin{aligned} f(x) &= \int e^{-\int \frac{AA'}{A^2}dx}dx \\ &= \int e^{-\int \frac{A'}{A}dx}dx \\ &= \int e^{-\ln A}dx \\ &= \int \frac{1}{A(x)}dx \end{aligned}$$

Therefore the solution from (2) is

$$\begin{aligned} y(x) &= C_1 \cos\left(f(x)\right) + C_2 \sin\left(f(x)\right) \\ &= C_1 \cos\left(\int \frac{1}{A(x)}dx\right) + C_2 \sin\left(\int \frac{1}{A(x)}dx\right) \end{aligned} \tag{4}$$

Let us now try to verify this solution by substituting it back into the ODE. From (4), where we now write A instead of $A(x)$ everywhere to simplify the notation

$$\begin{aligned} y'(x) &= -C_1 \sin\left(\int \frac{1}{A}dx\right)\left(\int \frac{1}{A}dx\right)' + C_2 \cos\left(\int \frac{1}{A}dx\right)\left(\int \frac{1}{A}dx\right)' \\ &= -C_1 \sin\left(\int \frac{1}{A}dx\right)\frac{1}{A} + C_2 \cos\left(\int \frac{1}{A}dx\right)\frac{1}{A} \end{aligned}$$

And $y''(x)$ becomes

$$\begin{aligned} y''(x) = -C_1 &\left(\cos\left(\int \frac{1}{A}dx\right)\left(\int \frac{1}{A}dx\right)'\frac{1}{A} + \sin\left(\int \frac{1}{A}dx\right)\left(\frac{-A'}{A^2}\right)\right) \\ &+ C_2\left(-\sin\left(\int \frac{1}{A}dx\right)\left(\int \frac{1}{A}dx\right)'\frac{1}{A} + \cos\left(\int \frac{1}{A}dx\right)\left(\frac{-A'}{A^2}\right)\right) \end{aligned}$$

or

$$
\begin{aligned}
y''(x) &= -C_1\left(\cos\left(\int\frac{1}{A}dx\right)\frac{1}{A^2}-\sin\left(\int\frac{1}{A}dx\right)\frac{A'}{A^2}\right)+C_2\left(-\sin\left(\int\frac{1}{A}dx\right)\frac{1}{A^2}-\cos\left(\int\frac{1}{A}dx\right)\frac{A'}{A^2}\right)\\
&= -C_1\cos\left(\int\frac{1}{A}dx\right)\frac{1}{A^2}+C_1\sin\left(\int\frac{1}{A}dx\right)\frac{A'}{A^2}-C_2\sin\left(\int\frac{1}{A}dx\right)\frac{1}{A^2}-C_2\cos\left(\int\frac{1}{A}dx\right)\frac{A'}{A^2}
\end{aligned}
$$

Substituting the above expression for y, y', y'' into the original ODE $A^2y''+AA'y'+y=0$ gives

$$
\begin{aligned}
&A^2\left(-C_1\cos\left(\int\frac{1}{A}dx\right)\frac{1}{A^2}+C_1\sin\left(\int\frac{1}{A}dx\right)\frac{A'}{A^2}-C_2\sin\left(\int\frac{1}{A}dx\right)\frac{1}{A^2}-C_2\cos\left(\int\frac{1}{A}dx\right)\frac{A'}{A^2}\right)\\
&+AA'\left(-C_1\sin\left(\int\frac{1}{A}dx\right)\frac{1}{A}+C_2\cos\left(\int\frac{1}{A}dx\right)\frac{1}{A}\right)+C_1\cos\left(\int\frac{1}{A(x)}dx\right)+C_2\sin\left(\int\frac{1}{A(x)}dx\right)=0
\end{aligned}
$$

Simplifying gives

$$
\begin{aligned}
&-C_1\cos\left(\int\frac{1}{A}dx\right)+C_1\sin\left(\int\frac{1}{A}dx\right)A'-C_2\sin\left(\int\frac{1}{A}dx\right)-C_2\cos\left(\int\frac{1}{A}dx\right)A'\\
&-C_1A'\sin\left(\int\frac{1}{A}dx\right)+C_2A'\cos\left(\int\frac{1}{A}dx\right)+C_1\cos\left(\int\frac{1}{A}dx\right)+C_2\sin\left(\int\frac{1}{A(x)}dx\right)=0
\end{aligned}
$$

Canceling C_1terms gives

$$
-C_2\sin\left(\int\frac{1}{A}dx\right)-C_2\cos\left(\int\frac{1}{A}dx\right)A'+C_2A'\cos\left(\int\frac{1}{A}dx\right)+C_2\sin\left(\int\frac{1}{A}dx\right)=0
$$

Which simplifies to

$$
-C_2\cos\left(\int\frac{1}{A}dx\right)A'+C_2A'\cos\left(\int\frac{1}{A}dx\right)=0
$$

Or

$$
0=0
$$

Solution (4) has been verified.

7 Problem 7

Problem Find the general real solution to the equation

$$xy'' + \frac{3}{x}y = 1 + x^3$$

Solution

We start by writing the ODE as

$$x^2y'' + 3y = x + x^4 \tag{1}$$

The solution is given by

$$y = y_h + y_p$$

where y_h is solution to homogeneous ODE $x^2y_h'' + 3y_h = 0$ and y_p is a particular solution to $x^2y_p'' + 3y_p = x + x^4$. We start by solving the homogeneous

$$x^2y'' + 3y = 0$$

This is Euler type ODE. Using the standard substitution $y = Ax^r$, then $y' = Arx^{r-1}, y'' = Ar(r-1)x^{r-2}$ and the above becomes

$$\begin{aligned}
x^2Ar(r-1)x^{r-2} + 3Ax^r &= 0\\
Ar(r-1)x^r + 3Ax^r &= 0
\end{aligned}$$

Since $x^r \neq 0$ and $A \neq 0$ then the above simplifies to

$$\begin{aligned}
r(r-1) + 3 &= 0\\
r^2 - r + 3 &= 0
\end{aligned}$$

Hence

$$\begin{aligned}
r &= \frac{-b}{2a} \pm \frac{1}{2a}\sqrt{b^2 - 4ac}\\
&= \frac{1}{2} \pm \frac{1}{2}\sqrt{1-12}\\
&= \frac{1}{2} \pm \frac{1}{2}i\sqrt{11}
\end{aligned}$$

Hence the solution is

$$\begin{aligned}
y &= C_1x^{\frac{1}{2}+\frac{i}{2}\sqrt{11}} + C_2x^{\frac{1}{2}-\frac{i}{2}\sqrt{11}}\\
&= C_1x^{\frac{1}{2}}x^{\frac{i}{2}\sqrt{11}} + C_2x^{\frac{1}{2}}x^{\frac{-i}{2}\sqrt{11}}\\
&= C_1\sqrt{x}e^{\ln x^{\frac{i}{2}\sqrt{11}}} + C_2\sqrt{x}e^{\ln x^{\frac{-i}{2}\sqrt{11}}}\\
&= C_1\sqrt{x}e^{\frac{i}{2}\sqrt{11}\ln x} + C_2\sqrt{x}e^{\frac{-i}{2}\sqrt{11}\ln x}
\end{aligned}$$

Using Euler formula the above can now be written in terms of sin and cos

$$\begin{aligned}
y &= \sqrt{x}\left(C_1e^{\frac{i}{2}\sqrt{11}\ln x} + C_2e^{\frac{-i}{2}\sqrt{11}\ln x}\right)\\
y_h &= \sqrt{x}\left(C_3\cos\left(\frac{1}{2}\sqrt{11}\ln x\right) + C_4\sin\left(\frac{1}{2}\sqrt{11}\ln x\right)\right) \qquad (2)
\end{aligned}$$

Now we find the particular solution using the method of undetermined coefficients. Since the RHS is polynomial $x + x^4$ then we guess

$$y_p = A + Bx + Cx^2 + Dx^3 + Ex^4$$

Then $y' = B + 2Cx + 3Dx^2 + 4Ex^3$ and $y'' = 2C + 6Dx + 12Ex^2$. Substituting these back in (1)

$$\begin{aligned}
x^2\left(2C + 6Dx + 12Ex^2\right) + 3\left(A + Bx + Cx^2 + Dx^3 + Ex^4\right) &= x + x^4\\
2Cx^2 + 6Dx^3 + 12Ex^4 + 3A + 3Bx + 3Cx^2 + 3Dx^3 + 3Ex^4 &= x + x^4\\
3A + x(3B) + x^2(2C + 3C) + x^3(6D + 3D) + (3E + 12E)x^4 &= x + x^4\\
3A + x(3B) + x^2(5C) + x^3(9D) + 15Ex^4 &= x + x^4
\end{aligned}$$

By comparing coefficients the following equations are generated

$$\begin{aligned} A &= 0 \\ 3B &= 1 \\ 5C &= 0 \\ 9D &= 0 \\ 15E &= 1 \end{aligned}$$

Hence $A = 0, B = \frac{1}{3}, C = 0, D = 0, E = \frac{1}{15}$. Therefore

$$y_p = \frac{1}{3}x + \frac{1}{15}x^4$$

Hence the final solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \sqrt{x}\left(C_3 \cos\left(\frac{1}{2}\sqrt{11}\ln x\right) + C_4 \sin\left(\frac{1}{2}\sqrt{11}\ln x\right)\right) + \frac{1}{3}x + \frac{1}{15}x^4 \end{aligned}$$

8 Problem 8

Problem For what values of k does the equation

$$y'' - \left(\frac{1}{4} + \frac{k}{x}\right) y = 0 \tag{1}$$

defined for $0 < x < \infty$ have a solution vanishing at $x = 0$ and at $x = \infty$?

Solution

Let us look what happens at $x \to \infty$, then the term $\frac{1}{4} \gg \frac{k}{x}$ and the ODE simplifies to

$$y'' - \frac{1}{4} y = 0$$

Which has the solutions $y = \left\{ e^{\frac{1}{2}x}, e^{\frac{-1}{2}x} \right\}$. We reject the first one since it does not vanish at $x \to \infty$, and use $y = e^{\frac{-1}{2}x}$. Now we assume the solution to (1) is of the form

$$y = P(x) e^{\frac{-1}{2}x} \tag{2}$$

And we now try to find $P(x)$. Substituting this solution back into (1), given that

$$\begin{aligned}
y' &= P' e^{\frac{-x}{2}} - \frac{1}{2} P e^{\frac{-x}{2}} \\
y'' &= P'' e^{\frac{-x}{2}} - \frac{1}{2} P' e^{\frac{-x}{2}} - \frac{1}{2} P' e^{\frac{-x}{2}} + \frac{1}{4} P e^{\frac{-x}{2}} \\
&= P'' e^{\frac{-x}{2}} - P' e^{\frac{-x}{2}} + \frac{1}{4} P e^{\frac{-x}{2}}
\end{aligned}$$

Substituting the above into (1) and canceling common term $e^{\frac{-x}{2}}$ gives

$$\begin{aligned}
\left(P'' - P' + \frac{1}{4} P\right) - \left(\frac{1}{4} + \frac{k}{x}\right) P &= 0 \\
P'' - P' - \frac{k}{x} P &= 0 \\
x P'' - x P' - k P &= 0
\end{aligned} \tag{3}$$

To solve this for $P(x)$, we use Frobenius series . Assuming

$$\begin{aligned}
P(x) &= \sum_{n=0}^{\infty} c_n x^{n+r} \\
P'(x) &= \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1} \\
P''(x) &= \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2}
\end{aligned}$$

Hence (3) becomes

$$\begin{aligned}
x \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2} - x \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1} - k \sum_{n=0}^{\infty} c_n x^{n+r} &= 0 \\
\sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-1} - \sum_{n=0}^{\infty} (n+r) c_n x^{n+r} - k \sum_{n=0}^{\infty} c_n x^{n+r} &= 0 \\
\sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-1} - \sum_{n=1}^{\infty} (n+r-1) c_{n-1} x^{n+r-1} - k \sum_{n=1}^{\infty} c_{n-1} x^{n+r-1} &= 0
\end{aligned}$$

For $n = 0$, and assuming $c_0 \neq 0$ then

$$\begin{aligned}
(n+r)(n+r-1) c_n &= 0 \\
(r)(r-1) c_0 &= 0 \\
r(r-1) &= 0
\end{aligned}$$

Hence $r = 1$ or $r = 0$.

Case $r = 1$

$$
\begin{aligned}
P &= \sum_{n=0}^{\infty} c_n x^{n+1} \\
&= \sum_{n=1}^{\infty} c_{n-1} x^n
\end{aligned}
$$

Hence

$$
\begin{aligned}
P' &= \sum_{n=1}^{\infty} n c_{n-1} x^{n-1} \\
P'' &= \sum_{n=1}^{\infty} (n)(n-1) c_{n-1} x^{n-2}
\end{aligned}
$$

And now (3) becomes

$$
\begin{aligned}
x \sum_{n=1}^{\infty} (n)(n-1) c_{n-1} x^{n-2} - x \sum_{n=1}^{\infty} n c_{n-1} x^{n-1} - k \sum_{n=1}^{\infty} c_{n-1} x^n &= 0 \\
\sum_{n=1}^{\infty} (n)(n-1) c_{n-1} x^{n-1} - \sum_{n=1}^{\infty} n c_{n-1} x^n - k \sum_{n=1}^{\infty} c_{n-1} x^n &= 0 \\
\sum_{n=2}^{\infty} (n)(n-1) c_{n-1} x^{n-1} - \sum_{n=1}^{\infty} n c_{n-1} x^n - k \sum_{n=1}^{\infty} c_{n-1} x^n &= 0 \\
\sum_{n=1}^{\infty} (n+1)(n) c_n x^n - \sum_{n=1}^{\infty} n c_{n-1} x^n - k \sum_{n=1}^{\infty} c_{n-1} x^n &= 0
\end{aligned}
$$

Hence for $n \geq 1$ we obtain

$$
\begin{aligned}
(n+1)(n) c_n - n c_{n-1} - k c_{n-1} &= 0 \\
c_n &= \frac{(n+k) c_{n-1}}{n(n+1)}
\end{aligned}
$$

For $n = 1$

$$
c_1 = \frac{(k+1) c_0}{2}
$$

For $n = 2$

$$
c_2 = \frac{(k+2) c_1}{2(3)} = \frac{(k+2)}{2(3)} \frac{(k+1)}{2} c_0 = \frac{(k+1)(k+2)}{(2)(2)(3)} c_0
$$

For $n = 3$

$$
c_3 = \frac{(k+3) c_2}{3(4)} = \frac{(k+3)}{(3)(4)} \frac{(k+1)(k+2)}{(2)(2)(3)} c_0 = \frac{(k+1)(k+2)(k+3)}{(2)(2)(3)(3)(4)} c_0
$$

For $n = 4$

$$
c_4 = \frac{(k+4) c_{n-1}}{(4)(5)} = \frac{(k+4) c_3}{(4)(5)} = \frac{(k+1)(k+2)(k+3)(k+4)}{(2)(2)(3)(3)(4)(4)(5)} c_0
$$

And so on. Hence

$$
\begin{aligned}
P(x) &= \sum_{n=1}^{\infty} c_{n-1} x^n \\
&= c_0 x + c_1 x^2 + c_2 x^3 + c_3 x^4 + \cdots \\
&= c_0 \left(x + \frac{(k+1)}{2} x^2 + \frac{(k+1)(k+2)}{(2)(2)(3)} x^3 + \frac{(k+1)(k+2)(k+3)}{(2)(2)(3)(3)(4)} x^4 + \frac{(k+1)(k+2)(k+3)(k+4)}{(2)(2)(3)(3)(4)(4)(5)} x^5 + \cdots \right. \\
&= c_0 \left(x + (k+1) \frac{x^2}{2!} + \frac{(k+1)(k+2)}{2!} \frac{x^3}{3!} + \frac{(k+1)(k+2)(k+3)}{3!} \frac{x^4}{4!} + \frac{(k+1)(k+2)(k+3)(k+4)}{4!} \frac{x^5}{5!} + \right.
\end{aligned} \tag{4}
$$

But $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$. Or $e^x - 1 = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$. So there is an exponential term inside (4). Hence to make (4) vanish at $x \to \infty$, then k needs to be a negative integer. Taking $k = -1$ makes all terms with k in them vanish, leaving

$$
P(x) = c_0 x
$$

So now the solution from (2) becomes

$$
y(x) = c_0 x e^{-\frac{x}{2}}
$$

Which goes to zero as $x \to \infty$ since an exponential decays to zero faster that x going to infinity.

We now need to check if negative k integer value (specifically $k = -1$ which we picked from above) will also make the solution vanish as $x \to 0$. When $x \to 0$ the ODE becomes

$$y'' - \frac{k}{x} y = 0 \tag{5}$$

Since $\frac{k}{x} \gg \frac{1}{4}$ close to $x = 0$. Since k is negative integer -1 then the above becomes

$$y'' + \frac{k}{x} y = 0$$

To see this will go to zero as $x \to 0$, Intuitively since $\frac{k}{x}$ is now positive and very large, then this is like a harmonic oscillator with very large stiffness. (Spring mass system). When the stiffness becomes very large, the solution goes to zero (the natural frequency goes to infinity, since $\omega = \sqrt{\frac{k}{x}}$ which means the period goes to zero since $\omega = 2\pi T$) which implies no motion. So this shows that negative integer value of k found from first part makes the solution vanish at both $x \to \infty$ and at $x \to 0$. Actually for $x \to 0$ we just needed k to be negative in order to change the sign. But for $x \to \infty$ we found we needed k to be a negative integer which we choose -1. So this will work for $x = 0$ and $x = \infty$.

8.1 Appendix

I first tried to solve the give ODE directly using series method. I left this here as an appendix, not to be graded but as a reference.

x is singular point. But it is a regular singular point since $\lim_{x\to 0} x^2 \frac{k}{x} = x$ and hence the limit exist. Therefore assuming solution is Frobenius series

$$y = x^r \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^{n+r}$$

Therefore $y' = \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1}$ and $y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2}$, then (1) becomes

$$\begin{aligned}
\sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2} - \left(\frac{1}{4} + \frac{k}{x}\right) \sum_{n=0}^{\infty} c_n x^{n+r} &= 0 \\
\sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2} - \frac{k}{x} \sum_{n=0}^{\infty} c_n x^{n+r} - \frac{1}{4} \sum_{n=0}^{\infty} c_n x^{n+r} &= 0 \\
\sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2} - k \sum_{n=0}^{\infty} c_n x^{n+r-1} - \frac{1}{4} \sum_{n=0}^{\infty} c_n x^{n+r} &= 0
\end{aligned}$$

But $k \sum_{n=0}^{\infty} c_n x^{n+r-1} = k \sum_{n=1}^{\infty} c_{n-1} x^{n+r-2}$ and $\sum_{n=0}^{\infty} c_n x^{n+r} = \sum_{n=2}^{\infty} c_{n-2} x^{n+r-2}$ and the above becomes

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2} - k \sum_{n=1}^{\infty} c_{n-1} x^{n+r-2} - \frac{1}{4} \sum_{n=2}^{\infty} c_{n-2} x^{n+r-2} = 0 \tag{2}$$

The first step is to obtain the indicial equation. As the nature of the roots will tell us how to proceed. The indicial equation is obtained from $n = 0$ in (2) with the assumption that $c_0 \neq 0$. This leads to

$$\begin{aligned}
(n+r)(n+r-1) c_n &= 0 \\
r(r-1) c_0 &= 0
\end{aligned}$$

c_0 is always taken as non-zero. This leads to

$$r(r-1) = 0$$

With solutions $r_1 = 1$ or $r_2 = 0$. (We take r_1 as the larger root first, since Frobenius series solution can only guarantee solution for the larger root, when the roots differ by an integer as this is the case).

Since $r_1 - r_2$ is an integer, then this tells us we can obtain a first solution $y_1(x)$ associated with $r_1 = 1$ from the Frobenius series

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+1} \tag{3}$$

But to find the second solution $y_2(x)$ associated with $r_2 = 0$ we can try either reduction of order method or use

$$y_2(x) = A y_1(x) \ln(x) + \sum_{n=0}^{\infty} d_n x^n \tag{4}$$

Where A is some constant, which can be zero, and d_n are the coefficients for the second series. We have to do the above when the roots of the indicial equation differ by integer. Otherwise, the second solution would have been found using Frobenius series $y_2(x) \sum_{n=0}^{\infty} c_n x^{n+r_2}$ like with the first solution.

OK, Now we will first find $y_1(x)$ from (3)

<u>case $r_1 = 1$</u>

Using (3)

$$\begin{aligned} y' &= \sum_{n=0}^{\infty} (n+1) c_n x^n \\ y'' &= \sum_{n=0}^{\infty} n(n+1) c_n x^{n-1} \\ &= \sum_{n=1}^{\infty} n(n+1) c_n x^{n-1} \end{aligned}$$

Substituting the above into (1) gives

$$\begin{aligned} \sum_{n=1}^{\infty} n(n+1) c_n x^{n-1} - \left(\frac{1}{4} + \frac{k}{x}\right) \sum_{n=0}^{\infty} c_n x^{n+1} &= 0 \\ \sum_{n=1}^{\infty} n(n+1) c_n x^{n-1} - \frac{1}{4} \sum_{n=0}^{\infty} c_n x^{n+1} - \frac{k}{x} \sum_{n=0}^{\infty} c_n x^{n+1} &= 0 \\ \sum_{n=1}^{\infty} n(n+1) c_n x^{n-1} - \frac{1}{4} \sum_{n=0}^{\infty} c_n x^{n+1} - k \sum_{n=0}^{\infty} c_n x^n &= 0 \\ \sum_{n=0}^{\infty} (n+1)(n+2) c_{n+1} x^n - \frac{1}{4} \sum_{n=1}^{\infty} c_{n-1} x^n - k \sum_{n=0}^{\infty} c_n x^n &= 0 \end{aligned}$$

For $n = 0$

$$\begin{aligned} (1)(2) c_1 - k c_0 &= 0 \\ c_1 &= \frac{k}{2} c_0 \end{aligned}$$

For $n > 0$ we obtain the recursion equation

$$\begin{aligned} (n+1)(n+2) c_{n+1} - \frac{1}{4} c_{n-1} - k c_n &= 0 \\ c_{n+1} &= \frac{\frac{1}{4} c_{n-1} + k c_n}{(n+1)(n+2)} \end{aligned}$$

For $n = 1$

$$c_2 = \frac{\frac{1}{4} c_0 + k c_1}{(2)(3)} = \frac{\frac{1}{4} c_0 + k\left(\frac{k}{2} c_0\right)}{6} = \frac{\frac{1}{4} c_0 + \frac{k^2}{2} c_0}{6} = c_0 \frac{\frac{1}{4} + \frac{k^2}{2}}{6} = c_0 \frac{1 + 2k^2}{24}$$

For $n = 2$

$$\begin{aligned}
c_3 &= \frac{\frac{1}{4}c_1 + kc_2}{(3)(4)} \\
&= \frac{\frac{1}{4}\frac{k}{2}c_0 + kc_0\frac{1+2k^2}{24}}{12} \\
&= \frac{\frac{k}{8}c_0 + kc_0\frac{1+2k^2}{24}}{12} \\
&= c_0\frac{\frac{3k+k+2k^3}{24}}{(12)} \\
&= c_0\frac{4k + 2k^3}{288}
\end{aligned}$$

And so on. Hence

$$\begin{aligned}
y_1(x) &= c_0x + c_1x^2 + c_2x^3 + c_3x^4 + \cdots \\
&= c_0x + \frac{k}{2}c_0x^2 + c_0\frac{1+2k^2}{24}x^3 + c_0\frac{4k+2k^3}{288}x^4 + \cdots \\
&= c_0x\left(1 + \frac{k}{2}x + \frac{1+2k^2}{24}x^2 + \frac{4k+2k^3}{288}x^3 + \cdots\right) \\
&= c_0x\left(1 + \frac{k}{2}x + \left(\frac{1}{24} + \frac{1}{12}k^2\right)x^2 + \frac{k}{288}\left(4 + 2k^2\right)x^3 + \cdots\right)
\end{aligned}$$

I could not find closed form function for the above.

Now that we found $y_1(x)$, then $y_2(x)$ is, from (4), repeated here

$$y_2(x) = Ay_1(x)\ln(x) + \sum_{n=0}^{\infty} d_nx^n \tag{4}$$

Since we want the solution to vanish at $x = 0$ then we set $A = 0$ and $y_2(x)$ simplifies to

$$y_2(x) = \sum_{n=0}^{\infty} d_nx^n \tag{4}$$

Where $d_0 \neq 0$. Hence $y'(x) = \sum_{n=0}^{\infty} nd_nx^{n-1}$ and $y'' = \sum_{n=0}^{\infty} n(n-1)d_nx^{n-2}$. Rewriting the ODE as $xy'' - \left(\frac{x}{4} + k\right)y = 0$ and now substituting the derivatives into this gives

$$\begin{aligned}
x\sum_{n=0}^{\infty} n(n-1)d_nx^{n-2} - \left(\frac{x}{4} + k\right)\sum_{n=0}^{\infty} d_nx^n &= 0 \\
\sum_{n=0}^{\infty} n(n-1)d_nx^{n-1} - \frac{x}{4}\sum_{n=0}^{\infty} d_nx^n - k\sum_{n=0}^{\infty} d_nx^n &= 0 \\
\sum_{n=2}^{\infty} n(n-1)d_nx^{n-1} - \frac{1}{4}\sum_{n=0}^{\infty} d_nx^{n+1} - k\sum_{n=0}^{\infty} d_nx^n &= 0 \\
\sum_{n=1}^{\infty} (n+1)(n)d_{n+1}x^n - \frac{1}{4}\sum_{n=1}^{\infty} d_{n-1}x^n - k\sum_{n=0}^{\infty} d_nx^n &= 0
\end{aligned}$$

For $n = 0$ we obtain $kd_0 = 0$ which implies $d_0 = 0$ since $k \neq 0$.

For $n > 0$

$$\begin{aligned}
(n+1)(n)d_{n+1} - \frac{1}{4}d_{n-1} - kd_n &= 0 \\
d_{n+1} &= \frac{\frac{1}{4}d_{n-1} + kd_n}{(n)(n+1)}
\end{aligned}$$

For $n = 1$

$$d_2 = \frac{\frac{1}{4}d_0 + kd_1}{2} = \frac{k}{2}d_1$$

For $n = 2$

$$d_3 = \frac{\frac{1}{4}d_1 + kd_2}{(2)(3)} = \frac{\frac{1}{4}d_1 + k\left(\frac{k}{2}d_1\right)}{6} = \frac{d_1 + 2k^2 d_1}{32} = d_1\frac{1 + 2k^2}{32}$$

For $n = 3$

$$d_4 = \frac{\frac{1}{4}d_2 + kd_3}{(3)(4)} = \frac{d_2 + 4kd_3}{48} = \frac{\frac{k}{2}d_1 + 4k\left(d_1\frac{1+2k^2}{32}\right)}{48} = d_1\frac{\frac{1}{8}k\left(2k^2 + 5\right)}{48} = d_1\frac{\left(2k^3 + 5k\right)}{384}$$

And so on. Hence the second solution is

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} d_n x^n \\ &= d_0 + d_1 x + d_2 x^2 + d_3 x^3 + d_4 x^4 + \cdots \\ &= d_1 x + \frac{k}{2}d_1 x^2 + d_1\frac{1 + 2k^2}{32}x^3 + d_1\frac{\left(2k^3 + 5k\right)}{384}x^4 + \cdots \\ &= d_1 x\left(1 + \frac{k}{2}x + d_1\frac{1 + 2k^2}{32}x^2 + d_1\frac{\left(2k^3 + 5k\right)}{384}x^3 +\right) \end{aligned}$$

I am not sure if the above solution for $y_2(x)$ is correct. I need to check this again later.