# HW 9
# MATH 4567 Applied Fourier Analysis
# Spring 2019
# University of Minnesota, Twin Cities

Nasser M. Abbasi

November 2, 2019 Compiled on November 2, 2019 at 9:51pm [public]

# Contents

# 1 Section 61, Problem 2

**2.** Suppose that two continuous functions $f(x)$ and $\psi_1(x)$, with positive norms, are linearly independent on an interval $a \le x \le b$; that is, one is not a constant times the other. By determining the linear combination $f + A\psi_1$ of those functions that is orthogonal to $\psi_1$ on the fundamental interval $a < x < b$, obtain an orthogonal pair $\psi_1$, $\psi_2$ where

$$\psi_2(x) = f(x) - \frac{(f, \psi_1)}{\|\psi_1\|^2}\psi_1(x).$$

Interpret this expression geometrically when $f$, $\psi_1$, and $\psi_2$ represent vectors in three dimensional space.

Figure 1: Problem statement

Solution

Let $\psi_2 = f + A\psi_1$ such that $\langle \psi_2, \psi_1 \rangle = 0$. Hence

$$\begin{aligned}
\langle f + A\psi_1, \psi_1 \rangle &= 0\\
\langle f, \psi_1 \rangle + \langle A\psi_1, \psi_1 \rangle &= 0\\
\langle f, \psi_1 \rangle + A \langle \psi_1, \psi_1 \rangle &= 0\\
\langle f, \psi_1 \rangle + A \left\|\psi_1\right\|^2 &= 0\\
A &= -\frac{\langle f, \psi_1 \rangle}{\left\|\psi_1\right\|^2}
\end{aligned}$$

Therefore, since $\psi_2 = f + A\psi_1$ then

$$\psi_2 = f - \frac{\langle f, \psi_1 \rangle}{\left\|\psi_1\right\|^2}\psi_1$$

Geometrically, the term $\frac{\langle \psi_1, f \rangle}{\left\|\psi_1\right\|^2}\psi_1$ represents the projection of $f$ on $\psi_1$. The term $\frac{\psi_1}{\|\psi_1\|}$ makes a unit vector in the direction of $\psi_1$ and the term $\frac{\langle f, \psi_1 \rangle}{\|\psi_1\|}$ is the magnitude of projection $\left\|\psi_1\right\| \cos(\theta)$ where $\theta$ is the inner angle between $f, \psi_1$. The result of $-\frac{\langle f, \psi_1 \rangle}{\left\|\psi_1\right\|^2}\psi_1$ is a vector in the opposite direction of $\psi_1$. Adding this to $f$ gives $\psi_2$ which is now orthogonal to $f$. This process is called Gram Schmidt.

## 2 Section 61, Problem 3

**3.** In Problem 2, suppose that the fundamental interval is $-\pi < x < \pi$ and that

$$f(x) = \cos nx + \sin nx \quad \text{and} \quad \psi_1(x) = \cos nx,$$

where $n$ is a fixed positive integer. Show that the function $\psi_2(x)$ there turns out to be

$$\psi_2(x) = \sin nx.$$

*Suggestion:* One can avoid evaluating any integrals by using the fact that the set in Example 3, Sec. 61, is orthogonal on the interval $-\pi < x < \pi$.

Figure 2: Problem statement

Solution

Let

$$\begin{aligned} f &= \cos nx + \sin nx \\ \psi_1 &= \cos nx \end{aligned}$$

Then by Gram Schmidt process from problem 2 we know that

$$\psi_2 = f - \frac{\langle f, \psi_1 \rangle}{\left\|\psi_1\right\|^2} \psi_1$$

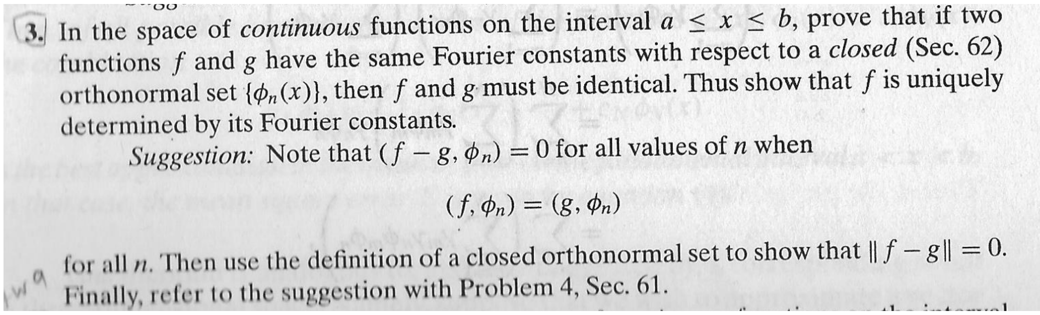
Hence

$$\begin{aligned} \psi_2 &= (\cos nx + \sin nx) - \frac{\int_{-\pi}^{\pi} (\cos nx + \sin nx) \cos nx dx}{\int_{-\pi}^{\pi} \cos^2 (nx) \, dx} \cos nx \\ &= (\cos nx + \sin nx) - \frac{\int_{-\pi}^{\pi} \cos nx \cos nx dx + \int_{-\pi}^{\pi} \sin nx \cos nx dx}{\pi} \cos nx \end{aligned}$$

But $\int_{-\pi}^{\pi} \cos nx \cos nx dx = \int_{-\pi}^{\pi} \cos^2 nx dx = \pi$ and $\int_{-\pi}^{\pi} \sin nx \cos nx dx = 0$ since these are orthogonal. Hence the above simplifies to

$$\begin{aligned} \psi_2 &= (\cos nx + \sin nx) - \cos nx \\ &= \sin nx \end{aligned}$$

# 3 Section 63, Problem 3



3. In the space of *continuous* functions on the interval $a \le x \le b$, prove that if two functions $f$ and $g$ have the same Fourier constants with respect to a *closed* (Sec. 62) orthonormal set $\{\phi_n(x)\}$, then $f$ and $g$ must be identical. Thus show that $f$ is uniquely determined by its Fourier constants.

*Suggestion:* Note that $(f - g, \phi_n) = 0$ for all values of $n$ when

$$(f, \phi_n) = (g, \phi_n)$$

for all $n$. Then use the definition of a closed orthonormal set to show that $\|f - g\| = 0$. Finally, refer to the suggestion with Problem 4, Sec. 61.

Figure 3: Problem statement

<u>Solution</u>

The Fourier coefficients of $f-g$ are given by $\langle f-g, \phi_n \rangle$ by definition. But due to linearity of inner product, this can be written as

$$\langle f-g, \phi_n \rangle = \langle f, \phi_n \rangle - \langle g, \phi_n \rangle$$

But $\langle f, \phi_n \rangle$ are the Fourier coefficients of $f$ and $\langle g, \phi_n \rangle$ are the Fourier coefficients of $g$, and we are told these are the same. Therefore

$$\langle f-g, \phi_n \rangle = 0$$

Which implies that $\left\|f-g\right\| = 0$. Using part(b) in problem 4, section 61, which says that if $\left\|f\right\| = 0$ then $f(x) = 0$ except at possibly finite number of points in the interval, then applying this to $\left\|f-g\right\| = 0$ leads to

$$f-g = 0$$

Which implies $f = g$ which is what required to show.

# 4 Section 63, Problem 4

**4.** Let $\{\phi_n(x)\}$ be an orthonormal set in the space of *continuous* functions on the interval $a \le x \le b$, and suppose that the generalized Fourier series for a function $f(x)$ in that space converges *uniformly* (Sec. 17) to a sum $s(x)$ on that interval.

(*a*) Show that $s(x)$ and $f(x)$ have the same Fourier constants with respect to $\{\phi_n(x)\}$.
(*b*) Use results in part (*a*) and Problem 3 to show that if $\{\phi_n(x)\}$ is *closed* (Sec. 62), then $s(x) = f(x)$ on the interval $a \le x \le b$.

*Suggestion:* Recall from Sec. 17 that the sum of a uniformly convergent series of continuous functions is continuous and that such a series can be integrated term by term.

Figure 4: Problem description

solution

## 4.1 Part (a)

Let the generalized Fourier series of $f(x)$ be

$$f(x) = \sum_{n=1}^{\infty} \langle f(x), \phi_n \rangle \phi_n$$

Let the sum the above converges uniformly to be $s(x)$. Therefore we have, per problem statement the following equality

$$\sum_{n=1}^{\infty} \langle f(x), \phi_n \rangle \phi_n = s(x)$$

Taking the inner product of both sides with respect to $\phi_m$ gives

$$\begin{aligned} \int_a^b \left( \sum_{n=1}^{\infty} \langle f(x), \phi_n \rangle \phi_n \right) \phi_m dx &= \int_a^b s(x) \phi_m dx \\ &= \langle s(x), \phi_m \rangle \end{aligned}$$

Since the sum converges uniformly, then we are allowed to integrate the left side term by term while keeping the equality with the right side. Hence moving the integration inside the sum gives

$$\sum_{n=1}^{\infty} \langle f(x), \phi_n \rangle \int_a^b \phi_n \phi_m dx = \langle s(x), \phi_m \rangle$$

But due to orthogonality of $\phi_n$ and $\phi_m$ and since they are normalized, then $\int_a^b \phi_n \phi_m dx = \langle \phi_n, \phi_m \rangle = 1$ if $n = m$ and zero otherwise. Hence the above simplifies to

$$\langle f(x), \phi_m \rangle = \langle s(x), \phi_m \rangle$$

And since the above is valid for any arbitrary $m = 1 \cdots \infty$, then it shows that $f(x)$ and $s(x)$ have the same generalized Fourier coefficients.

## 4.2 Part (b)

From part (a), we found

$$\langle f, \phi_n \rangle = \langle s, \phi_n \rangle$$

By linearity of inner product, the above is the same as

$$\begin{aligned} \langle f, \phi_n \rangle - \langle s, \phi_n \rangle &= 0 \\ \langle f - s, \phi_n \rangle &= 0 \end{aligned}$$

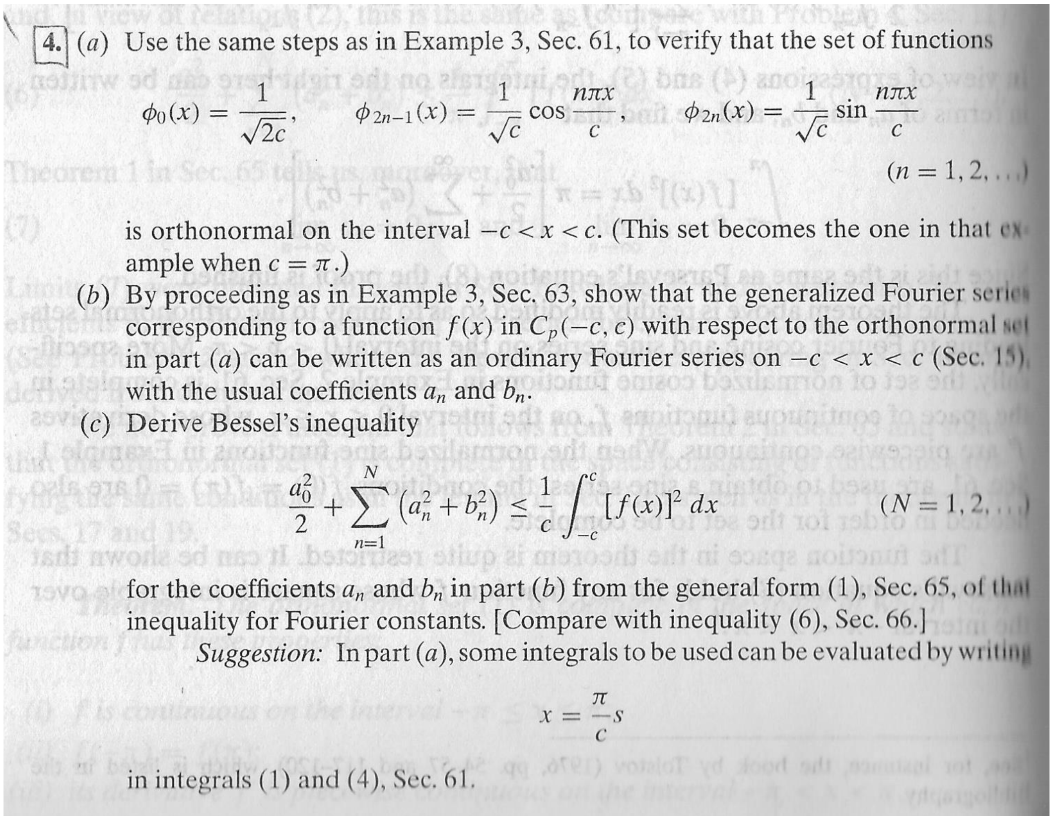
But from problem 3, we know that $\langle f - s, \phi_n \rangle = 0$ implies $\|f - s\| = 0$.

Next, using part(b) in problem 4, section 61, which says that if $\|f\| = 0$ then $f(x) = 0$ except at possibly finite number of points in the interval, then applying this to our case here that $\|f - s\| = 0$ leads to

$$\begin{aligned} f - s &= 0 \\ f &= s \end{aligned}$$

Which is the result required to show.

# 5 Section 66, Problem 4



**4.** (*a*) Use the same steps as in Example 3, Sec. 61, to verify that the set of functions

$$\phi_0(x) = \frac{1}{\sqrt{2c}}, \qquad \phi_{2n-1}(x) = \frac{1}{\sqrt{c}} \cos \frac{n\pi x}{c}, \qquad \phi_{2n}(x) = \frac{1}{\sqrt{c}} \sin \frac{n\pi x}{c} \qquad (n = 1, 2, \ldots)$$

is orthonormal on the interval $-c < x < c$. (This set becomes the one in that example when $c = \pi$.)

(*b*) By proceeding as in Example 3, Sec. 63, show that the generalized Fourier series corresponding to a function $f(x)$ in $C_p(-c, c)$ with respect to the orthonormal set in part (*a*) can be written as an ordinary Fourier series on $-c < x < c$ (Sec. 15), with the usual coefficients $a_n$ and $b_n$.

(*c*) Derive Bessel's inequality

$$\frac{a_0^2}{2} + \sum_{n=1}^{N} (a_n^2 + b_n^2) \leq \frac{1}{c} \int_{-c}^{c} [f(x)]^2 \, dx \qquad (N = 1, 2, \ldots)$$

for the coefficients $a_n$ and $b_n$ in part (*b*) from the general form (1), Sec. 65, of that inequality for Fourier constants. [Compare with inequality (6), Sec. 66.]

*Suggestion:* In part (*a*), some integrals to be used can be evaluated by writing

$$x = \frac{\pi}{c} s$$

in integrals (1) and (4), Sec. 61.

Figure 5: Problem description

solution

## 5.1 Part (a)

We need to find

$$\langle \phi_0, \phi_{2n} \rangle$$
$$\langle \phi_0, \phi_{2n-1} \rangle$$
$$\langle \phi_{2n}, \phi_{2m} \rangle$$
$$\langle \phi_{2n-1}, \phi_{2m-1} \rangle$$
$$\langle \phi_{2m-1}, \phi_{2n} \rangle$$

And also show that

$$\langle \phi_0, \phi_0 \rangle = \|\phi_0\|^2 = 1$$
$$\langle \phi_{2n}, \phi_{2n} \rangle = \|\phi_{2n}\|^2 = 1$$
$$\langle \phi_{2n-1}, \phi_{2n-1} \rangle = \|\phi_{2n-1}\|^2 = 1$$

$\underline{\langle \phi_0, \phi_{2n} \rangle}$

$$\begin{aligned}
\langle \phi_0, \phi_{2n} \rangle &= \int_{-c}^{c} \frac{1}{\sqrt{2c}} \frac{1}{\sqrt{c}} \cos\left(\frac{n\pi}{c}x\right) dx \\
&= \frac{1}{c\sqrt{2}} \left[ \frac{\sin\left(\frac{n\pi}{c}x\right)}{\frac{n\pi}{c}} \right]_{-c}^{c} \\
&= \frac{c}{n\pi c\sqrt{2}} \left[ \sin\left(\frac{n\pi}{c}x\right) \right]_{-c}^{c} \\
&= \frac{1}{n\pi\sqrt{2}} \left[ \sin(n\pi) + \sin(n\pi) \right] \\
&= 0
\end{aligned}$$

Since $n$ is integer.

$\underline{\langle \phi_0, \phi_{2n-1} \rangle}$

$$\begin{aligned}
\langle \phi_0, \phi_{2n-1} \rangle &= \int_{-c}^{c} \frac{1}{\sqrt{2c}} \frac{1}{\sqrt{c}} \sin\left(\frac{n\pi}{c}x\right) dx \\
&= \frac{1}{c\sqrt{2}} \left[ \frac{-\cos\left(\frac{n\pi}{c}x\right)}{\frac{n\pi}{c}} \right]_{-c}^{c} \\
&= \frac{-c}{n\pi c\sqrt{2}} \left[ \cos\left(\frac{n\pi}{c}x\right) \right]_{-c}^{c} \\
&= \frac{-1}{n\pi\sqrt{2}} \left[ \cos(n\pi) - \cos(n\pi) \right] \\
&= 0
\end{aligned}$$

$\underline{\langle \phi_{2n}, \phi_{2m} \rangle}$

$$\begin{aligned}
\langle \phi_{2n}, \phi_{2m} \rangle &= \int_{-c}^{c} \frac{1}{\sqrt{c}} \sin\left(\frac{n\pi}{c}x\right) \frac{1}{\sqrt{c}} \sin\left(\frac{m\pi}{c}x\right) dx \\
&= \frac{1}{c} \int_{-c}^{c} \sin\left(\frac{n\pi}{c}x\right) \sin\left(\frac{m\pi}{c}x\right) dx
\end{aligned}$$

Let $\frac{c}{\pi}s = x$, then $dx = \frac{c}{\pi}ds$. When $x = -c$ then $s = -\pi$ and when $x = c$ then $s = \pi$ and the above becomes

$$\begin{aligned}\langle \phi_{2n}, \phi_{2m} \rangle &= \frac{1}{c} \int_{-\pi}^{\pi} \sin(ns) \sin(ms) \frac{c}{\pi} ds \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(ns) \sin(ms)\, ds\end{aligned}$$

Since the integrand is even, then

$$\langle \phi_{2n}, \phi_{2m} \rangle = \frac{2}{\pi} \int_{0}^{\pi} \sin(ns) \sin(ms)\, ds$$

From equation (1), page 192 we see that

$$\langle \phi_{2n}, \phi_{2m} \rangle = 0$$

Since $n, m$ are different.

<u>$\langle \phi_{2n-1}, \phi_{2m-1} \rangle$</u>

$$\begin{aligned}\langle \phi_{2n-1}, \phi_{2m-1} \rangle &= \int_{-c}^{c} \frac{1}{\sqrt{c}} \cos\left(\frac{n\pi}{c} x\right) \frac{1}{\sqrt{c}} \cos\left(\frac{m\pi}{c} x\right) dx \\ &= \frac{1}{c} \int_{-c}^{c} \cos\left(\frac{n\pi}{c} x\right) \cos\left(\frac{m\pi}{c} x\right) dx\end{aligned}$$

Let $\frac{c}{\pi} s = x$, then $dx = \frac{c}{\pi} ds$. When $x = -c$ then $s = -\pi$ and when $x = c$ then $s = \pi$ and the above becomes

$$\begin{aligned}\langle \phi_{2n-1}, \phi_{2m-1} \rangle &= \frac{1}{c} \int_{-\pi}^{\pi} \cos(ns) \cos(ms) \frac{c}{\pi} ds \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(ns) \cos(ms)\, ds\end{aligned}$$

Since the integrand is even, then

$$\langle \phi_{2n-1}, \phi_{2m-1} \rangle = \frac{2}{\pi} \int_{0}^{\pi} \cos(ns) \cos(ms)\, ds$$

From equation (4), page 192 we see that

$$\langle \phi_{2n-1}, \phi_{2m-1} \rangle = 0$$

Since $n, m$ are different.

<u>$\langle \phi_{2m-1}, \phi_{2n} \rangle$</u>

$$\begin{aligned}\langle \phi_{2m-1}, \phi_{2n} \rangle &= \int_{-c}^{c} \frac{1}{\sqrt{c}} \cos\left(\frac{m\pi}{c} x\right) \frac{1}{\sqrt{c}} \sin\left(\frac{n\pi}{c} x\right) dx \\ &= \frac{1}{c} \int_{-c}^{c} \cos\left(\frac{m\pi}{c} x\right) \sin\left(\frac{n\pi}{c} x\right) dx\end{aligned}$$

Let $\frac{c}{\pi} s = x$, then $dx = \frac{c}{\pi} ds$. When $x = -c$ then $s = -\pi$ and when $x = c$ then $s = \pi$ and the above becomes

$$\begin{aligned}\langle \phi_{2m-1}, \phi_{2n} \rangle &= \frac{1}{c}\int_{-\pi}^{\pi} \cos(ms)\sin(ns)\,\frac{c}{\pi}ds\\ &= \frac{1}{\pi}\int_{-\pi}^{\pi}\cos(ms)\sin(ns)\,ds\end{aligned}$$

Using $\cos(ms)\sin(ns) = \frac{1}{2}(\cos(s(m+n)) + \cos(s(m-n)))$. Hence the above becomes

$$\langle \phi_{2m-1}, \phi_{2n} \rangle = \frac{1}{2\pi}\left(\int_{-\pi}^{\pi}\cos(s(m+n))\,ds + \int_{-\pi}^{\pi}\cos(s(m-n))\,ds\right)$$

Since the integration is over one full period, then each is zero. Hence

$$\langle \phi_{2m-1}, \phi_{2n} \rangle = 0$$

$\underline{\langle \phi_0, \phi_0 \rangle}$

$$\begin{aligned}\langle \phi_0, \phi_0 \rangle &= \int_{-c}^{c}\frac{1}{\sqrt{2c}}\frac{1}{\sqrt{2c}}dx\\ \|\phi_0\|^2 &= \frac{1}{2c}\int_{-c}^{c}dx\\ &= 1\end{aligned}$$

Hence $\|\phi_0\| = 1$.

$\underline{\langle \phi_{2n}, \phi_{2n} \rangle}$

$$\begin{aligned}\langle \phi_{2n}, \phi_{2n} \rangle &= \int_{-c}^{c}\frac{1}{\sqrt{c}}\sin\left(\frac{n\pi}{c}x\right)\frac{1}{\sqrt{c}}\sin\left(\frac{n\pi}{c}x\right)dx\\ &= \frac{1}{c}\int_{-c}^{c}\sin^2\left(\frac{n\pi}{c}x\right)dx\\ &= \frac{1}{c}\int_{-c}^{c}\frac{1}{2} - \frac{1}{2}\cos\left(2\frac{n\pi}{c}x\right)dx\\ &= \frac{1}{2c}\left(\int_{-c}^{c}dx - \int_{-c}^{c}\cos\left(2\frac{n\pi}{c}x\right)dx\right)\\ &= \frac{1}{2c}\left(2c - \left[\frac{\sin\left(2\frac{n\pi}{c}x\right)}{2\frac{n\pi}{c}}\right]_{-c}^{c}\right)\\ &= \frac{1}{2c}\left(2c - \frac{c}{2n\pi}\left[\sin\left(2\frac{n\pi}{c}x\right)\right]_{-c}^{c}\right)\\ &= \frac{1}{2c}(2c)\\ &= 1\end{aligned}$$

Hence $\|\phi_{2n}\| = 1$.

$\underline{\langle \phi_{2n-1}, \phi_{2n-1} \rangle}$

$$
\begin{aligned}
\langle \phi_{2n-1}, \phi_{2n-1} \rangle &= \int_{-c}^{c} \frac{1}{\sqrt{c}} \cos\left(\frac{n\pi}{c}x\right) \frac{1}{\sqrt{c}} \cos\left(\frac{n\pi}{c}x\right) dx \\
\left\|\phi_{2n-1}\right\|^2 &= \frac{1}{c} \int_{-c}^{c} \cos^2\left(\frac{n\pi}{c}x\right) dx \\
&= \frac{1}{c} \int_{-c}^{c} \frac{1}{2} + \frac{1}{2} \sin\left(2\frac{n\pi}{c}x\right) dx \\
&= \frac{1}{2c} \left( \int_{-c}^{c} dx + \int_{-c}^{c} \sin\left(2\frac{n\pi}{c}x\right) dx \right) \\
&= \frac{1}{2c} \left( 2c - \left[ \frac{\cos\left(2\frac{n\pi}{c}x\right)}{2\frac{n\pi}{c}} \right]_{-c}^{c} \right) \\
&= \frac{1}{2c} \left( 2c - \frac{c}{2n\pi} \left[ \cos\left(2\frac{n\pi}{c}x\right) \right]_{-c}^{c} \right) \\
&= \frac{1}{2c} \left( 2c - \frac{c}{2n\pi} \left[ \cos(2n\pi) - \cos(2n\pi) \right] \right) \\
&= \frac{1}{2c} 2c \\
&= 1
\end{aligned}
$$

Hence $\left\|\phi_{2n-1}\right\| = 1$ .

## 5.2 Part (b)

$$
\begin{aligned}
\phi_0(x) &= \frac{1}{\sqrt{2c}} \\
\phi_{2n-1}(x) &= \frac{1}{\sqrt{c}} \cos\left(\frac{n\pi x}{c}\right) \\
\phi_{2n}(x) &= \frac{1}{\sqrt{c}} \sin\left(\frac{n\pi x}{c}\right)
\end{aligned}
$$

On $-c < x < c$. The generalized Fourier series for $f(x)$ in $C_p(-c, c)$ is

$$
\sum_{n=0}^{\infty} c_n \phi_n(x) = c_0 \phi_0(x) + \sum_{n=1}^{\infty} \left( c_{2n-1} \phi_{2n-1}(x) + c_{2n} \phi_{2n}(x) \right)
$$

That is

$$
f(x) \sim c_0 \frac{1}{\sqrt{2c}} + \sum_{n=1}^{\infty} \left( \frac{c_{2n-1}}{\sqrt{c}} \cos\left(\frac{n\pi x}{c}\right) + \frac{c_{2n}}{\sqrt{c}} \sin\left(\frac{n\pi x}{c}\right) \right) \tag{1}
$$

Where

$$
c_0 = \langle f, \phi_0(x) \rangle = \frac{1}{\sqrt{2c}} \int_{-c}^{c} f(x)\, dx
$$

And

$$c_{2n-1} = \langle f, \phi_{2n-1}(x) \rangle = \frac{1}{\sqrt{c}} \int_{-c}^{c} f(x) \cos\left(\frac{n\pi x}{c}\right) dx \qquad n = 1, 2, \cdots$$

$$c_{2n} = \langle f, \phi_{2n}(x) \rangle = \frac{1}{\sqrt{c}} \int_{-c}^{c} f(x) \sin\left(\frac{n\pi x}{c}\right) dx \qquad n = 1, 2, \cdots$$

If we write

$$a_0 = 2\frac{c_0}{\sqrt{2c}}, a_n = \frac{c_{2n-1}}{\sqrt{c}}, b_n = \frac{c_{2n}}{\sqrt{c}} \qquad n = 1, 2, \cdots$$

Then (1) becomes

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{c}\right) + b_n \sin\left(\frac{n\pi x}{c}\right)$$

Where

$$a_n = \frac{1}{c} \int_{-c}^{c} f(x) \cos\left(\frac{n\pi x}{c}\right) dx \qquad n = 1, 2, \cdots$$

$$b_n = \frac{1}{c} \int_{-c}^{c} f(x) \sin\left(\frac{n\pi x}{c}\right) dx \qquad n = 1, 2, \cdots$$

This is the ordinary Fourier series on $-c < x < c$.

## 5.3 Part (c)

From (1) section 65

$$\sum_{n=0}^{N} c_n^2 \leq \|f\|^2 \tag{1}$$

But from part (b) we found that

$$a_0 = 2\frac{c_0}{\sqrt{2c}}, a_n = \frac{c_{2n-1}}{\sqrt{c}}, b_n = \frac{c_{2n}}{\sqrt{c}} \qquad n = 1, 2, \cdots$$

Hence

$$c_0 = \frac{a_0}{2}\sqrt{2c}$$

$$c_{2n-1} = a_n \sqrt{c}$$

$$c_{2n} = b_n \sqrt{c}$$

Substituting the above into (1) gives

$$
\begin{aligned}
c_0^2 + \sum_{n=1}^{N} c_{2n-1}^2 + \sum_{n=1}^{N} c_{2n}^2 &\leq \|f\|^2 \\
\left(\frac{a_0}{2}\sqrt{2c}\right)^2 + \sum_{n=1}^{N} \left(a_n\sqrt{c}\right)^2 + \sum_{n=1}^{N} \left(b_n\sqrt{c}\right)^2 &\leq \int \left[f(x)\right]^2 dx \\
\left(\frac{a_0^2}{4}2c\right) + \sum_{n=1}^{N} a_n^2 c + \sum_{n=1}^{N} b_n^2 c &\leq \int \left[f(x)\right]^2 dx \\
\frac{a_0^2}{2} + \sum_{n=1}^{N} \left(a_n^2 + b_n^2\right) &\leq \frac{1}{c}\int \left[f(x)\right]^2 dx
\end{aligned}
$$

# 6 Section 66, Problem 5

HW4

**5.** Let $s_N(x)$ $(N = 1, 2, \ldots)$ be a sequence of functions defined on the interval $0 \le x \le 1$ by means of the equations

$$s_N(x) = \begin{cases} 0 & \text{when } x = 1, \dfrac{1}{2}, \ldots, \dfrac{1}{N}, \\ 1 & \text{when } x \neq 1, \dfrac{1}{2}, \ldots, \dfrac{1}{N}. \end{cases}$$

Show that this sequence converges in the mean to the function $f(x) = 1$ in $C_p(0, 1)$ but that for each positive integer $p$,

$$\lim_{N\to\infty} s_N\left(\frac{1}{p}\right) = 0.$$

*Suggestion:* Observe that

$$s_N\left(\frac{1}{p}\right) = 0 \quad \text{when} \quad N \ge p.$$

Figure 6: Problem description

<u>solution</u>

The function $S_N(x)$ is almost 1 everywhere as can be seen from this diagram

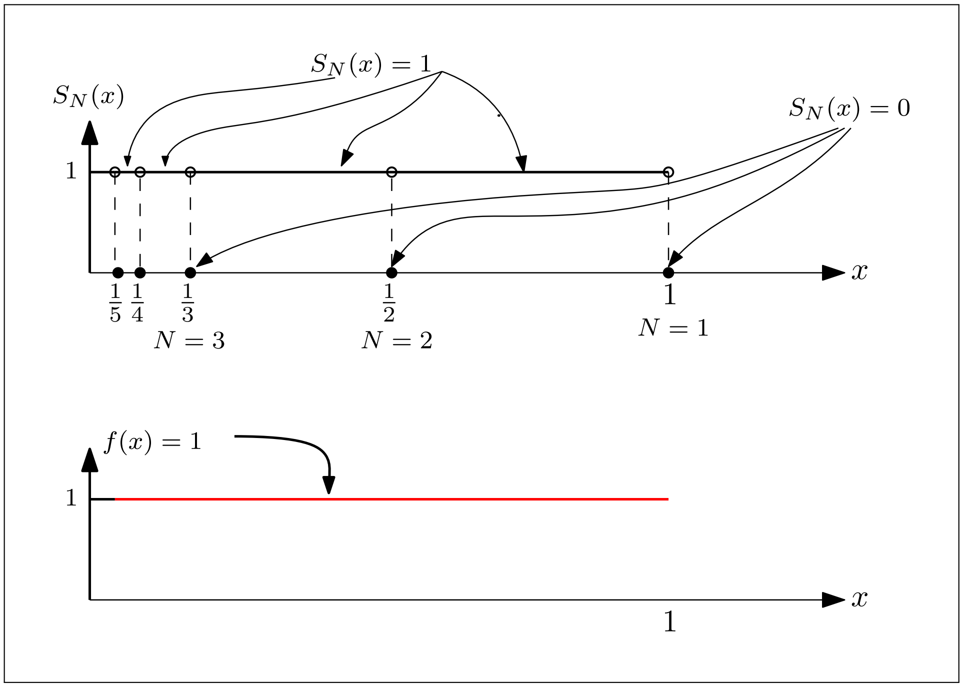



Figure 7: Showing the function $S_N(x)$ and $f(x)$

And the problem is asking us to show that $S_N(x) \to f(x)$ in the mean. This means we need to show the following is true

$$\lim_{N\to\infty} \left\| S_N(x) - f(x) \right\| = 0$$

Except at possibly finite number of points $x$. But this is the case here. Looking at $S_N(x)$ we see it is equal to $f(x) = 1$ everywhere except at the points $x = 1, \frac{1}{2}, \frac{1}{3}, \cdots$ and compared to all the points between 0 and 1, then $S_N(x) = f(x) = 1$ almost everywhere. Even though as $N \to \infty$ the number of points where $S_N(x) \neq 1$ increases, it is still finitely many compared to the number of points where $S_N(x) = f(x) = 1$.

To answer the second part: Since $S_N(x) = 0$ at any $x$ value which can written as $\frac{1}{p}$ where $p$ is an integer (this by definition given), then $S_N\left(\frac{1}{p}\right) = 0$. Then it clearly follows that $\lim_{N\to\infty} S_N\left(\frac{1}{p}\right) = 0$.