HW 7
MATH 4567 Applied Fourier Analysis
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Contents

1 Section 45, Problem 4

4. A string, stretched between the points 0 and π on the x axis and initially at rest, is released from the position $y = f(x)$. Its motion is opposed by air resistance, which is proportional to the velocity at each point (Sec. 28). Let the unit of time be chosen so that the equation of motion becomes

$$y_{tt}(x,t) = y_{xx}(x,t) - 2\beta y_t(x,t) \qquad (0 < x < \pi,\ t > 0),$$

where β is a positive constant. Assuming that $0 < \beta < 1$, derive the expression

$$y(x,t) = e^{-\beta t}\sum_{n=1}^{\infty} B_n\left(\cos\alpha_n t + \frac{\beta}{\alpha_n}\sin\alpha_n t\right)\sin nx,$$

where

$$\alpha_n = \sqrt{n^2-\beta^2}, \qquad B_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx \qquad (n = 1, 2, \ldots),$$

for the transverse displacements.

Figure 1: Problem statement

<u>Solution</u>

Solve for $y(x,t)$ in

$$y_{tt} = y_{xx} - 2\beta y_t \qquad (t > 0, 0 < x < \pi) \tag{1}$$

Boundary conditions

$$\begin{aligned} y(0,t) &= 0 \\ y(\pi,t) &= 0 \end{aligned}$$

Initial conditions

$$\begin{aligned} y(x,0) &= f(x) \\ y_t(x,0) &= 0 \end{aligned}$$

Let $y = XT$. Substituting in (1) gives

$$T''X = X''T - 2\beta T'X$$

Dividing by $XT \neq 0$

$$
\begin{aligned}
\frac{T''}{T} &= \frac{X''}{X} - 2\beta\frac{T'}{T}\\
\frac{T''}{T} + 2\beta\frac{T'}{T} &= \frac{X''}{X} = -\lambda
\end{aligned}
$$

Where λ is separation constant. Due to nature of boundary conditions being both homogeneous, then we know $\lambda > 0$ is only possible case from earlier HW's. The eigenvalue problem is

$$X'' + \lambda X = 0$$

Which we know has eigenvalues $\lambda = n^2$ for $n = 1, 2, \cdots$ with corresponding eigenfunctions

$$X_n = \sin(nx) \tag{1}$$

Now we solve the time ODE using these eigenvalues.

$$
\begin{aligned}
\frac{T''}{T} + 2\beta\frac{T'}{T} &= -n^2\\
T'' + 2\beta T' + n^2 T &= 0
\end{aligned}
$$

This is standard second order ODE with positive damping β and since n^2 is positive. The characteristic equation is

$$r^2 + 2\beta r + n^2 = 0$$

The roots are

$$
\begin{aligned}
r &= -\frac{b}{2a} \pm \frac{1}{2a}\sqrt{b^2 - 4ac}\\
&= -\frac{2\beta}{2} \pm \frac{1}{2}\sqrt{4\beta^2 - 4n^2}\\
&= -\beta \pm \sqrt{\beta^2 - n^2}\\
&= -\beta \pm i\sqrt{n^2 - \beta^2}
\end{aligned}
$$

Hence the solution is

$$
\begin{aligned}
T_n(t) &= A_n e^{r_1 t} + B_n e^{r_2 t}\\
&= A_n e^{\left(-\beta + i\sqrt{n^2-\beta^2}\right)t} + B_n e^{\left(-\beta - i\sqrt{n^2-\beta^2}\right)t}\\
&= e^{-\beta t}\left(A_n e^{i\sqrt{n^2-\beta^2}t} + B_n e^{-i\sqrt{n^2-\beta^2}t}\right)
\end{aligned}
$$

But the above can be rewritten using Euler relation as (the constants A_n, B_n will be different, but kept them the same names for simplicity)

$$T_n(t) = e^{-\beta t}\left(A_n \cos\left(\sqrt{n^2 - \beta^2}t\right) + B_n \sin\left(\sqrt{n^2 - \beta^2}t\right)\right)$$

Let $\alpha_n = \sqrt{n^2 - \beta^2}$, then the above becomes

$$T_n(t) = e^{-\beta t}\left(A_n \cos(\alpha_n t) + B_n \sin(\alpha_n t)\right) \tag{2}$$

Since the PDE is linear and homogenous, then by superposition we obtain the final solution as

$$\begin{aligned} y(x,t) &= \sum_{n=1}^{\infty} X_n T_n \\ &= \sum_{n=1}^{\infty} e^{-\beta t}\left(A_n \cos(\alpha_n t) + B_n \sin(\alpha_n t)\right)\sin(nx) \qquad (3) \end{aligned}$$

Now initial conditions are applied to determine A_n, B_n. At $t = 0$

$$f(x) = \sum_{n=1}^{\infty} A_n \sin(nx)$$

Hence A_n are the Fourier sine coefficient of the representation of $f(x)$ which implies

$$A_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx \qquad (4)$$

Taking time derivative of (3) gives

$$y_t(x,t) = \sum_{n=1}^{\infty}\left[-\beta e^{-\beta t}\left(A_n\cos(\alpha_n t) + B_n\sin(\alpha_n t)\right) + e^{-\beta t}\left(-\alpha_n A_n \sin(\alpha_n t) + \alpha_n B_n \cos(\alpha_n t)\right)\right]\sin(nx)$$

At $t = 0$ the above becomes (since released from rest)

$$0 = \sum_{n=1}^{\infty}\left(-\beta A_n + \alpha_n B_n\right)\sin(nx)$$

Therefore

$$-\beta A_n + \alpha_n B_n = 0$$

Hence $B_n = \frac{\beta A_n}{\alpha_n}$. Therefore (3) becomes

$$\begin{aligned} y(x,t) &= \sum_{n=1}^{\infty} e^{-\beta t}\left(A_n\cos(\alpha_n t) + \frac{\beta A_n}{\alpha_n}\sin(\alpha_n t)\right)\sin(nx) \\ &= e^{-\beta t}\sum_{n=1}^{\infty} A_n\left(\cos(\alpha_n t) + \frac{\beta}{\alpha_n}\sin(\alpha_n t)\right)\sin(nx) \end{aligned}$$

Where $A_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$ and $\alpha_n = \sqrt{n^2 - \beta^2}$. Which is the result required to show (Book used B in place A, but it is the same thing, just different name for a constant).

2 Section 46, Problem 2

2. Let a, b, and ω denote nonzero constants. The general solution of the ordinary differential equation

$$y''(t) + a^2 y(t) = b \sin \omega t$$

is of the form $y = y_c + y_p$, where y_c is the general solution of the complementary equation $y''(t) + a^2 y(t) = 0$ and y_p is any particular solution of the original nonhomogeneous equation.†

(*a*) Suppose that $\omega \neq a$. After substituting

$$y_p = A \cos \omega t + B \sin \omega t,$$

where A and B are constants, into the given differential equation, determine values of A and B such that y_p is a solution. Thus, derive the general solution

$$y(t) = C_1 \cos at + C_2 \sin at + \frac{b}{a^2 - \omega^2} \sin \omega t$$

of that equation.

(*b*) Suppose that $\omega = a$ and find constants A and B such that

$$y_p = At \cos \omega t + Bt \sin \omega t$$

is a particular solution of the given differential equation. Thus obtain the general solution

$$y(t) = C_1 \cos at + C_2 \sin at - \frac{b}{2a} t \cos at.$$

Figure 2: Problem statement

Solution

2.1 Part a

suppose $\omega \neq a$. Let

$$y_p = A \cos \omega t + B \sin \omega t \tag{1}$$

Then

$$\begin{aligned} y_p' &= -A\omega \sin \omega t + B\omega \cos \omega t \\ y_p'' &= -A\omega^2 \cos \omega t - B\omega^2 \sin \omega t \end{aligned}$$

Substituting the above back into the given ODE gives

$$\begin{aligned} y_p''(t) + a^2 y_p(t) &= b\sin\omega t \\ \left(-A\omega^2\cos\omega t - B\omega^2\sin\omega t\right) + a^2\left(A\cos\omega t + B\sin\omega t\right) &= b\sin\omega t \\ \cos\omega t\left(-A\omega^2 + a^2 A\right) + \sin\omega t\left(-B\omega^2 + a^2 B\right) &= b\sin\omega t \end{aligned} \tag{2}$$

By comparing coefficients, we see that

$$\begin{aligned} -A\omega^2 + a^2 A &= 0 \\ A\left(a^2 - \omega^2\right) &= 0 \end{aligned}$$

Since $\omega \neq a$ then this implies that $A = 0$. And from (2), we see that

$$\begin{aligned} -B\omega^2 + a^2 B &= b \\ B &= \frac{b}{a^2 - \omega^2} \end{aligned}$$

Therefore (1) becomes

$$y_p = \frac{b}{a^2 - \omega^2}\sin\omega t \tag{3}$$

Now we need to find the complementary solution to

$$y_c'' + a^2 y = 0$$

Since $a^2 > 0$, then the solution is the standard one given by

$$y_c(t) = C_1\cos at + C_2\sin at \tag{4}$$

Adding (3,4) gives the general solution

$$y(t) = C_1\cos at + C_2\sin at + \frac{b}{a^2 - \omega^2}\sin\omega t$$

2.2 Part (b)

Let

$$y_p = At\cos\omega t + Bt\sin\omega t \tag{1}$$

Then

$$\begin{aligned} y_p' &= A\cos\omega t - At\omega\sin\omega t + B\sin\omega t + Bt\omega\cos\omega t \\ y_p'' &= -A\omega\sin\omega t - \left(A\omega\sin\omega t + At\omega^2\cos\omega t\right) + B\omega\cos\omega t + \left(B\omega\cos\omega t - Bt\omega^2\sin\omega t\right) \\ &= \left(-At\omega^2 + 2B\omega\right)\cos\omega t + \left(-2A\omega - Bt\omega^2\right)\sin\omega t \end{aligned}$$

Substituting the above back into the given ODE gives

$$\begin{aligned} y_p''(t) + a^2 y_p(t) &= b\sin\omega t \\ \left(\left(-At\omega^2 + 2B\omega\right)\cos\omega t + \left(-2A\omega - Bt\omega^2\right)\sin\omega t\right) + a^2\left(At\cos\omega t + Bt\sin\omega t\right) &= b\sin\omega t \\ \cos\omega t\left(-At\omega^2 + 2B\omega + a^2 At\right) + \sin\omega t\left(-2A\omega - Bt\omega^2 + a^2 Bt\right) &= b\sin\omega t \end{aligned} \tag{2}$$

By comparing coefficients, we see that

$$-At\omega^2 + 2B\omega + a^2 At = 0$$

$$At\left(-\omega^2 + a^2\right) + B(2\omega) = 0 \tag{3}$$

And from (2), we see also that

$$-2A\omega - Bt\omega^2 + a^2 Bt = b$$

$$A(-2\omega) + Bt\left(-\omega^2 + a^2\right) = b \tag{4}$$

But since $\omega = a$, then (3) becomes

$$B(2\omega) = 0$$

$$B = 0$$

And (4) becomes

$$A(-2\omega) = b$$

$$A = \frac{-b}{2a}$$

Substituting these values we found for A, B, in (1) gives

$$y_p = \frac{-b}{2a} t \cos \omega t$$

But $\omega = a$, therefore

$$y_p = \frac{-b}{2a} t \cos at \tag{5}$$

The complementary solution do not change from part (a). Hence the general solution is

$$y(t) = C_1 \cos at + C_2 \sin at - \frac{b}{2a} t \cos at$$

Which is the result required to show.

3 Section 46, Problem 3

3. Use the general solutions derived in Problem 2 to obtain the following solutions of the initial value problem

$$y''(t) + a^2 y(t) = b \sin \omega t, \qquad y(0) = 0, \qquad y'(0) = 0:$$

$$y(t) = \begin{cases} \dfrac{b}{\omega^2 - a^2}\left(\dfrac{\omega}{a}\sin at - \sin \omega t\right) & \text{when } \omega \neq a, \\ \dfrac{b}{2a}\left(\dfrac{1}{a}\sin at - t\cos at\right) & \text{when } \omega = a. \end{cases}$$

Figure 3: Problem statement

<u>Solution</u>

The general solution from problem 2 is

$$y(t) = \begin{cases} C_1 \cos at + C_2 \sin at + \frac{b}{a^2-\omega^2} \sin \omega t & \omega \neq a \\ C_1 \cos at + C_2 \sin at - \frac{b}{2a} t \cos at & \omega = a \end{cases}$$

We need to find C_1, C_2 when initial conditions are $y(0) = 0, y'(0) = 0$ for each of the above cases.

<u>case $\omega \neq a$</u>

$y(0) = 0$ gives

$$0 = C_1$$

Hence solution now becomes

$$y(t) = C_2 \sin at + \frac{b}{a^2 - \omega^2} \sin \omega t$$

Taking time derivative gives

$$y'(t) = aC_2 \cos at + \frac{\omega b}{a^2 - \omega^2} \cos \omega t$$

At $t = 0$ the above gives

$$\begin{aligned} 0 &= aC_2 + \frac{\omega b}{a^2 - \omega^2} \\ C_2 &= \frac{1}{a} \frac{\omega b}{\omega^2 - a^2} \end{aligned}$$

Using C_1, C_2 found above, the solution becomes

$$\begin{aligned} y(t) &= \frac{1}{a}\frac{\omega b}{\omega^2 - a^2}\sin at + \frac{b}{a^2-\omega^2}\sin\omega t \\ &= \frac{b}{a^2-\omega^2}\left(\frac{\omega}{a}\sin at - \sin\omega t\right) \end{aligned} \tag{1}$$

<u>case $\omega = a$</u>

$y(0) = 0$ gives

$$0 = C_1$$

Hence solution now becomes

$$y(t) = C_2 \sin at - \frac{b}{2a} t\cos at$$

Taking time derivative gives

$$y'(t) = aC_2\cos at - \left(\frac{b}{2a}\cos at - \frac{b}{2a}t^2 \sin at\right)$$

At $t = 0$ the above gives

$$\begin{aligned} 0 &= aC_2 - \frac{b}{2a} \\ C_2 &= \frac{1}{a}\frac{b}{2a} \end{aligned}$$

Using C_1, C_2 found above, the solution becomes

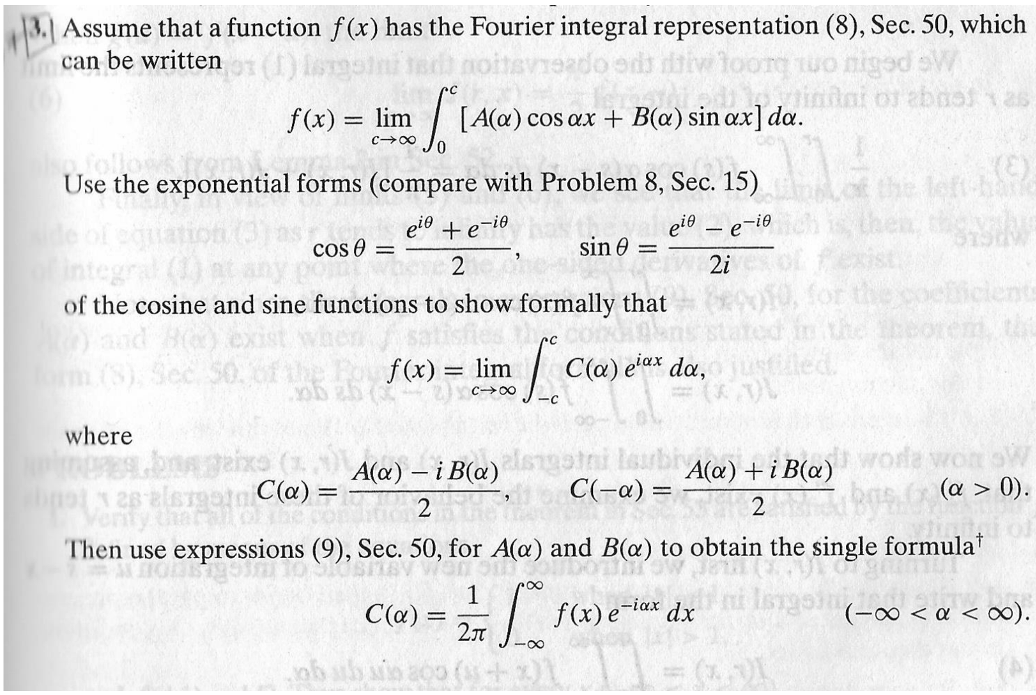
$$\begin{aligned} y(t) &= \frac{1}{a}\frac{b}{2a}\sin at - \frac{b}{2a}t\cos at \\ &= \frac{b}{2a}\left(\frac{1}{a}\sin at - t\cos at\right) \end{aligned} \tag{2}$$

From (1,2) we see that

$$y(t) = \begin{cases} \frac{b}{a^2-\omega^2}\left(\frac{\omega}{a}\sin at - \sin\omega t\right) & \omega \neq a \\ \frac{b}{2a}\left(\frac{1}{a}\sin at - t\cos at\right) & \omega = a \end{cases}$$

Which is the result required to show.

4 Section 52, Problem 3



3. Assume that a function $f(x)$ has the Fourier integral representation (8), Sec. 50, which can be written

$$f(x) = \lim_{c\to\infty} \int_0^c [A(\alpha)\cos\alpha x + B(\alpha)\sin\alpha x]\, d\alpha.$$

Use the exponential forms (compare with Problem 8, Sec. 15)

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}, \qquad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

of the cosine and sine functions to show formally that

$$f(x) = \lim_{c\to\infty} \int_{-c}^c C(\alpha)\, e^{i\alpha x}\, d\alpha,$$

where

$$C(\alpha) = \frac{A(\alpha) - i\,B(\alpha)}{2}, \qquad C(-\alpha) = \frac{A(\alpha)+i\,B(\alpha)}{2} \qquad (\alpha > 0).$$

Then use expressions (9), Sec. 50, for $A(\alpha)$ and $B(\alpha)$ to obtain the single formula†

$$C(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\, dx \qquad (-\infty < \alpha < \infty).$$

Figure 4: Problem statement

Solution

$$\begin{aligned}
f(x) &= \int_0^\infty (A(\alpha)\cos(\alpha x) + B(\alpha)\sin(\alpha x))\, d\alpha \\
&= \int_0^\infty \left(A(\alpha)\left(\frac{e^{i\alpha x}+e^{-i\alpha x}}{2}\right) - iB(\alpha)\left(\frac{e^{i\alpha x}-e^{-i\alpha x}}{2}\right)\right) d\alpha \\
&= \int_0^\infty \left(e^{i\alpha x}\left(\frac{A(\alpha)-iB(\alpha)}{2}\right) + e^{-i\alpha x}\left(\frac{A(\alpha)+iB(\alpha)}{2}\right)\right) d\alpha \\
&= \int_0^\infty e^{i\alpha x}\frac{A(\alpha)-iB(\alpha)}{2} d\alpha + \int_0^\infty e^{-i\alpha x}\frac{A(\alpha)+iB(\alpha)}{2} d\alpha \\
&= \int_0^\infty e^{i\alpha x}\frac{A(\alpha)-iB(\alpha)}{2} d\alpha + \int_0^\infty e^{-i\alpha x}\frac{A(\alpha)+iB(\alpha)}{2} d\alpha \\
&= \int_0^\infty e^{i\alpha x}\frac{A(\alpha)-iB(\alpha)}{2} d\alpha + \int_{-\infty}^0 e^{i\alpha x}\frac{A(\alpha)+iB(\alpha)}{2} d\alpha \\
&= \int_{-\infty}^\infty C(\alpha)\, e^{i\alpha x} d\alpha
\end{aligned}$$

Where

$$C(\alpha) = \frac{A(\alpha) - iB(\alpha)}{2} \quad , \quad C(-\alpha) = \frac{A(\alpha) + iB(\alpha)}{2} \qquad \alpha > 0$$

Expression (9) section (5) is

$$A(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos(\alpha x)\,dx$$
$$B(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin(\alpha x)\,dx$$

Substituting the above in $C(\alpha) = \frac{A(\alpha) - iB(\alpha)}{2}$ gives

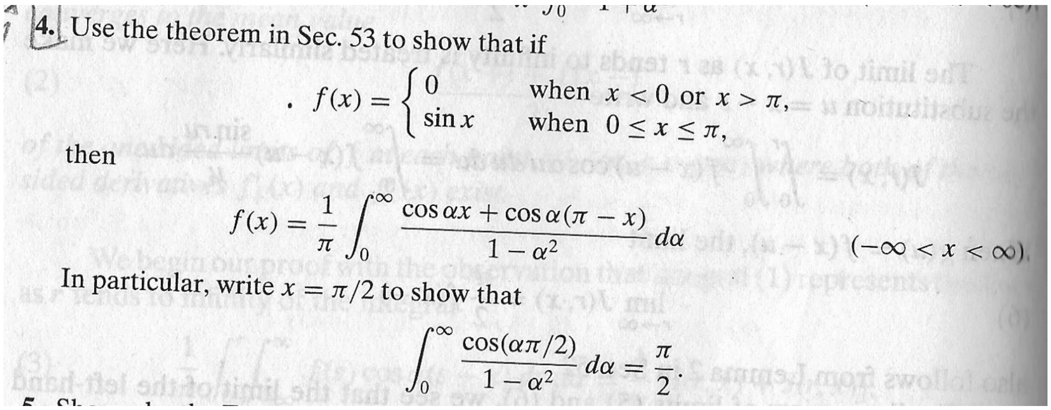
$$\begin{aligned} C(\alpha) &= \frac{1}{2}\left(\frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos(\alpha x)\,dx - i\frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin(\alpha x)\,dx\right) \\ &= \frac{1}{2\pi}\left(\int_{-\infty}^{\infty} f(x)\cos(\alpha x)\,dx - \int_{-\infty}^{\infty} f(x)\,i\sin(\alpha x)\,dx\right) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\left(\cos(\alpha x) - i\sin(\alpha x)\right)dx \end{aligned}$$

But using Euler relation $\cos(\alpha x) - i\sin(\alpha x) = e^{i\alpha x}$ then the above reduces to

$$C(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\,e^{i\alpha x}dx \qquad -\infty < \alpha < \infty$$

Which is what required to show.

5 Section 53, Problem 4



4. Use the theorem in Sec. 53 to show that if

$$f(x) = \begin{cases} 0 & \text{when } x < 0 \text{ or } x > \pi, \\ \sin x & \text{when } 0 \le x \le \pi, \end{cases}$$

then

$$f(x) = \frac{1}{\pi}\int_0^\infty \frac{\cos \alpha x + \cos \alpha(\pi - x)}{1-\alpha^2}\,d\alpha \qquad (-\infty < x < \infty).$$

In particular, write $x = \pi/2$ to show that

$$\int_0^\infty \frac{\cos(\alpha\pi/2)}{1-\alpha^2}\,d\alpha = \frac{\pi}{2}.$$

Figure 5: Problem statement

Solution

Since $f(x)$ is piecewise continuous and absolutely integrable (sine function), then

$$\frac{f(x^+)+f(x^-)}{2} = \frac{1}{\pi}\int_0^\infty \left(\int_{-\infty}^\infty f(s)\cos(\alpha(s-x))\,ds\right)d\alpha$$

Substituting for $f(s)$ inside the integral for the function given gives

$$\frac{f(x^+)+f(x^-)}{2} = \frac{1}{\pi}\int_0^\infty \left(\int_0^\pi \sin(s)\cos(\alpha s-\alpha x)\,ds\right)d\alpha$$

Where we used $\int_0^\pi$ only, since the function is zero everywhere else. Using $2\sin A\cos B = \sin(A+B)+\sin(A-B)$ then the above can be written as

$$\begin{aligned}\frac{f(x^+)+f(x^-)}{2} &= \frac{1}{\pi}\int_0^\infty \left(\frac{1}{2}\int_0^\pi \sin(s+\alpha s-\alpha x)+\sin(s-(\alpha s-\alpha x))\,ds\right)d\alpha \\ &= \frac{1}{2\pi}\int_0^\infty \left(\int_0^\pi \sin(s+\alpha s-\alpha x)+\sin(s-\alpha s+\alpha x)\,ds\right)d\alpha \qquad (1)\end{aligned}$$

But

$$\begin{aligned}\int_0^\pi \sin(s+\alpha s-\alpha x)\,ds &= \left[\frac{-\cos(s+\alpha s-\alpha x)}{1+\alpha}\right]_0^\pi \\ &= \frac{-1}{1+\alpha}\left(\cos(\pi+\alpha\pi-\alpha x)-\cos(-\alpha x)\right) \\ &= \frac{-1}{1+\alpha}\left(\cos(\pi+\alpha(\pi-x))-\cos(\alpha x)\right)\end{aligned}$$

But $\cos(\pi + \alpha(\pi - x)) = -\cos(\alpha(\pi - x))$, and the above becomes

$$\int_0^\pi \sin(s + \alpha s - \alpha x)\, ds = \frac{1}{1+\alpha}\left(\cos(\alpha(\pi - x)) + \cos(\alpha x)\right) \tag{2}$$

Similarly

$$\begin{aligned}
\int_0^\pi \sin(s - \alpha s + \alpha x)\, ds &= \left[\frac{-\cos(s - \alpha s + \alpha x)}{1-\alpha}\right]_0^\pi \\
&= \frac{-1}{1-\alpha}\left(\cos(\pi - \alpha\pi + \alpha x) - \cos(\alpha x)\right) \\
&= \frac{-1}{1-\alpha}\left(\cos(\pi - \alpha(\pi + x)) - \cos(\alpha x)\right) \\
&= \frac{-1}{1-\alpha}\left(-\cos(-\alpha(\pi + x)) - \cos(\alpha x)\right) \\
&= \frac{1}{1-\alpha}\left(\cos(\alpha(\pi + x)) + \cos(\alpha x)\right) \qquad (3)
\end{aligned}$$

Substituting (2,3) back in (1) gives

$$\begin{aligned}
\frac{f(x^+) + f(x^-)}{2} &= \frac{1}{2\pi}\int_0^\infty \left(\frac{1}{1+\alpha}\left(\cos(\alpha(\pi - x)) + \cos(\alpha x)\right) + \frac{1}{1-\alpha}\left(\cos(\alpha(\pi + x)) + \cos(\alpha x)\right)\right) d\alpha \\
&= \frac{1}{2\pi}\int_0^\infty \left(\cos(\alpha(\pi - x))\left(\frac{1}{1+\alpha} + \frac{1}{1-\alpha}\right) + \cos(\alpha x)\left(\frac{1}{1+\alpha} + \frac{1}{1-\alpha}\right)\right) d\alpha \\
&= \frac{1}{2\pi}\int_0^\infty \left(\cos(\alpha(\pi - x))\left(\frac{2}{1-\alpha^2}\right) + \cos(\alpha x)\left(\frac{2}{1-\alpha^2}\right)\right) d\alpha \\
&= \frac{1}{\pi}\int_0^\infty \frac{\cos(\alpha(\pi - x)) + \cos(\alpha x)}{1-\alpha^2} d\alpha
\end{aligned}$$

But $f(x)$ is continuous then $\frac{f(x^+)+f(x^-)}{2} = f(x)$ and the above becomes

$$f(x) = \frac{1}{\pi}\int_0^\infty \frac{\cos(\alpha(\pi - x)) + \cos(\alpha x)}{1-\alpha^2} d\alpha$$

When $x = \frac{\pi}{2}$ the above gives

$$f\left(\frac{\pi}{2}\right) = \frac{1}{\pi}\int_0^\infty \frac{\cos\left(\alpha\left(\pi - \frac{\pi}{2}\right)\right) + \cos\left(\alpha\frac{\pi}{2}\right)}{1-\alpha^2} d\alpha$$

But $f\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1$, hence

$$\begin{aligned}
1 &= \frac{1}{\pi}\int_0^\infty \frac{\cos\left(\alpha\frac{\pi}{2}\right) + \cos\left(\alpha\frac{\pi}{2}\right)}{1-\alpha^2} d\alpha \\
&= \frac{1}{\pi}\int_0^\infty \frac{2\cos\left(\alpha\frac{\pi}{2}\right)}{1-\alpha^2} d\alpha
\end{aligned}$$

Therefore

$$\frac{\pi}{2} = \int_0^\infty \frac{\cos\left(\alpha\frac{\pi}{2}\right)}{1-\alpha^2}d\alpha$$