HW 6
MATH 4567 Applied Fourier Analysis
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Contents

1 Section 40, Problem 1

1 The initial temperature of a slab $0 \le x \le \pi$ is zero throughout, and the face $x = 0$ is kept at that temperature. Heat is supplied through the face $x = \pi$ at a constant rate A $(A > 0)$ per unit area, so that $Ku_x(\pi, t) = A$ (see Sec. 26). Write

$$u(x,t) = U(x,t) + \Phi(x)$$

and use the solution of the problem in Example 2, Sec. 40, to derive the expression

$$u(x,t) = \frac{A}{K}\left\{x + \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^2}\exp\left[-\frac{(2n-1)^2 k}{4}t\right]\sin\frac{(2n-1)x}{2}\right\}$$

for the temperatures in this slab.

Figure 1: Problem statement

<u>Solution</u>

The PDE to solve is

$$u_{tt} = ku_{xx}$$

With boundary conditions

$$\begin{aligned} u(0,t) &= 0 \\ Ku_x(\pi,t) &= A \end{aligned} \tag{1}$$

And initial conditions

$$u(x,0) = 0$$

The solution to example 2 section 40 is

$$U(x,t) = \sum_{n=1}^{\infty} B_{2n-1}\exp\left(\frac{-(2n-1)^2 k}{4}t\right)\sin\left(\frac{(2n-1)x}{2}\right) \tag{2}$$

With

$$B_{2n-1} = \frac{2}{\pi}\int_0^{\pi} f(x)\sin\left(\frac{(2n-1)x}{2}\right)dx$$

Now, in this problem, we start by writing

$$u(x,t) = U(x,t) + \Phi(x) \tag{3}$$

The function $\Phi(x)$ needs to satisfy the nonhomogeneous B.C. (1). Let

$$\Phi(x) = c_1 x + c_2$$

When $x = 0$ this gives $0 = c_2$. Hence $\Phi(x) = c_1 x$. Taking derivative gives $\Phi'(x) = c_1$. But from (1) $K\Phi'(\pi) = A$. Hence $c_1 = \frac{A}{K}$. Therefore

$$\Phi(x) = \frac{A}{K}x$$

Substituting the above back into (3) gives

$$u(x,t) = U(x,t) + \frac{A}{K}x$$

But $U(x,t)$ is given by (2), hence the above becomes

$$u(x,t) = \frac{A}{K}x + \sum_{n=1}^{\infty} B_{2n-1} \exp\left(\frac{-(2n-1)^2 k}{4}t\right) \sin\left(\frac{(2n-1)x}{2}\right) \tag{4}$$

At $t = 0$, the initial conditions is 0. Hence the above becomes

$$-\frac{A}{K}x = \sum_{n=1}^{\infty} B_{2n-1} \sin\left(\frac{(2n-1)x}{2}\right)$$

Hence B_{2n-1} is the Fourier sine series of $-\frac{A}{K}x$ given by

$$\begin{aligned}
B_{2n-1} &= \frac{2}{\pi}\int_0^{\pi} \left(-\frac{A}{K}x\right) \sin\left(\frac{(2n-1)x}{2}\right) dx \\
&= -\frac{2A}{\pi K}\int_0^{\pi} x \sin\left(\frac{(2n-1)x}{2}\right) dx
\end{aligned}$$

Integration by parts. Let $u = x, dv = \sin\left(\frac{(2n-1)x}{2}\right)$, hence $du = 1$ and $v = -\frac{2}{(2n-1)}\cos\left(\frac{(2n-1)x}{2}\right)$ and the above becomes

$$\begin{aligned}
B_{2n-1} &= -\frac{2A}{\pi K}\left(\left[-\frac{2x}{(2n-1)}\cos\left(\frac{(2n-1)x}{2}\right)\right]_0^{\pi} + \int_0^{\pi} \frac{2}{(2n-1)}\cos\left(\frac{(2n-1)x}{2}\right) dx\right) \\
&= -\frac{2A}{\pi K}\left(-\frac{2}{(2n-1)}\left[x\cos\left(\frac{(2n-1)x}{2}\right)\right]_0^{\pi} + \frac{4}{(2n-1)^2}\left[\sin\left(\frac{(2n-1)x}{2}\right)\right]_0^{\pi}\right) \\
&= -\frac{2A}{\pi K}\left(-\frac{2\pi}{(2n-1)}\cos\left(\frac{(2n-1)\pi}{2}\right) + \frac{4}{(2n-1)^2}\sin\left(\frac{(2n-1)\pi}{2}\right)\right)
\end{aligned}$$

Since $2n-1$ is odd, then the cosine terms above vanish and the above simplifies to

$$\begin{aligned}
B_{2n-1} &= -\frac{A}{\pi K}\frac{8(-1)^{n+1}}{(2n-1)^2} \\
&= \frac{A}{\pi K}\frac{8(-1)^{n+2}}{(2n-1)^2} \\
&= \frac{A}{\pi K}\frac{8(-1)^{n}}{(2n-1)^2}
\end{aligned}$$

Substituting the above in (4) gives

$$\begin{aligned} u(x,t) &= \frac{A}{K}x + \sum_{n=1}^{\infty} \frac{A}{\pi K} \frac{8(-1)^n}{(2n-1)^2} \exp\left(\frac{-(2n-1)^2 k}{4} t\right) \sin\left(\frac{(2n-1)x}{2}\right) \\ &= \frac{A}{K}\left\{x + \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^2} \exp\left(\frac{-(2n-1)^2 k}{4} t\right) \sin\left(\frac{(2n-1)x}{2}\right)\right\} \end{aligned}$$

Which is the result required.

2 Section 40, Problem 3

3. Let $v(x, t)$ denote temperatures in a slender wire lying along the x axis. Variations of the temperature over each cross section are to be neglected. At the lateral surface, the linear law of surface heat transfer between the wire and its surroundings is assumed to apply (see Problem 6, Sec. 27). Let the surroundings be at temperature zero; then

$$v_t(x, t) = kv_{xx}(x, t) - bv(x, t),$$

where b is a positive constant. The ends $x = 0$ and $x = c$ of the wire are insulated (Fig. 34), and the initial temperature distribution is $f(x)$. Solve the boundary value problem for v by separation of variables. Then show that

$$v(x, t) = u(x, t)\, e^{-bt}$$

where u is the temperature function found in Sec. 36.

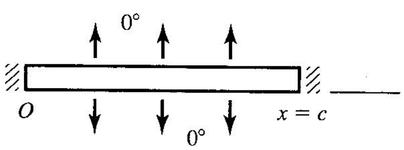


Figure 2: Problem statement

<u>Solution</u>

The PDE is

$$v_t = kv_{xx} - bv$$

With boundary conditions

$$v_x(0, t) = 0$$
$$v_x(c, t) = 0$$

And initial conditions

$$v(x, 0) = f(x)$$

Let $v(x, t) = X(x)\,T(t)$. Substituting into the PDE gives

$$T'X = kX''T - bXT$$

Dividing by $XT \neq 0$ gives

$$\frac{T'}{T} = k\frac{X''}{X} - b$$
$$\frac{T'}{T} + b = k\frac{X''}{X}$$
$$\frac{T'}{kT} + \frac{b}{k} = \frac{X''}{X} = -\lambda$$

Where λ is the separation constant. We obtain the boundary value eigenvalue ODE as

$$X'' + \lambda X = 0 \tag{1}$$
$$X'(0) = 0$$
$$X'(c) = 0$$

And the time ODE as

$$\frac{T'}{kT} + \frac{b}{k} = -\lambda$$
$$T' + \frac{b}{k}kT = -\lambda kT$$
$$T' + \frac{b}{k}kT + \lambda kT = 0$$
$$T' + T(b + \lambda k) = 0$$

Now we solve the space ODE (1) in order to determine the eigenvalues λ.

<u>Case $\lambda < 0$</u>

The solution to (1) becomes

$$X(x) = A\cosh\left(\sqrt{-\lambda}x\right) + B\sinh\left(\sqrt{-\lambda}x\right)$$
$$X' = A\sqrt{-\lambda}\sinh\left(\sqrt{-\lambda}x\right) + B\sqrt{-\lambda}\cosh\left(\sqrt{-\lambda}x\right)$$

Satisfying $X'(0) = 0$ gives

$$0 = B\sqrt{-\lambda}$$

Hence $B = 0$ and the solution becomes $X(x) = A\cosh\left(\sqrt{-\lambda}x\right)$. Therefore $X' = A\sqrt{-\lambda}\sinh\left(\sqrt{-\lambda}x\right)$. Satisfying $X'(c) = 0$ gives

$$0 = A\sqrt{\lambda}\sinh\left(\sqrt{-\lambda}c\right)$$

But sinh is zero only when its argument is zero, which is not the case here since $\lambda \neq 0$. This implies $A = 0$, leading to trivial solution. Therefore $\lambda < 0$ is not possible.

<u>Case $\lambda = 0$</u>

The solution to (1) becomes

$$X(x) = Ax + B$$
$$X' = A$$

Satisfying $X'(0) = 0$ gives

$$0 = A$$

And the solution becomes $X(x) = B$. Therefore $X' = 0$. Satisfying $X'(c) = 0$ gives

$$0 = 0$$

Which is valid for any B. Hence choosing $B = 1$ shows that $\lambda = 0$ is valid eigenvalue with corresponding eigenfunction $X_0(x) = 1$.

<u>Case $\lambda > 0$</u>

The solution to (1) becomes

$$X(x) = A\cos\left(\sqrt{\lambda}x\right) + B\sin\left(\sqrt{\lambda}x\right)$$
$$X' = -A\sqrt{\lambda}\sin\left(\sqrt{\lambda}x\right) + B\sqrt{\lambda}\cos\left(\sqrt{\lambda}x\right)$$

Satisfying $X'(0) = 0$ gives

$$0 = B\sqrt{\lambda}$$

Hence $B = 0$ and the solution becomes $X(x) = A\cos\left(\sqrt{\lambda}x\right)$. Therefore $X' = -A\sqrt{\lambda}\sin\left(\sqrt{\lambda}x\right)$. Satisfying $X'(c) = 0$ gives

$$0 = -A\sqrt{\lambda}\sin\left(\sqrt{\lambda}c\right)$$

For nontrivial solution we want

$$\sin\left(\sqrt{\lambda}c\right) = 0$$
$$\sqrt{\lambda}c = n\pi \qquad n = 1, 2, 3, \cdots$$
$$\lambda_n = \left(\frac{n\pi}{c}\right)^2 \tag{2}$$

And the corresponding eigenfunctions

$$X_n(x) = \cos\left(\sqrt{\lambda_n}x\right) \tag{3}$$

Now that we found λ_n, we can solve the time ODE $T' + T(b + \lambda k) = 0$. The solution is

$$T_n(t) = e^{-(b+\lambda_n k)t} \tag{4}$$

Hence the fundamental solution is

$$v_n(x,t) = X_n(x)T_n(t)$$
$$= \cos\left(\sqrt{\lambda_n}x\right)e^{-(b+\lambda_n k)t}$$

And the general solution is the superposition of all these solutions

$$v(x,t) = A_0X_0T_0 + \sum_{n=1}^{\infty} A_nX_n(x)T_n(t)$$
$$= A_0e^{-bt} + \sum_{n=1}^{\infty} A_n\cos\left(\sqrt{\lambda_n}x\right)e^{-(b+\lambda_n k)t}$$

Which can be written as

$$v(x,t) = u(x,t)e^{-bt}$$

Where $u(x,t)$ is

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n\cos\left(\sqrt{\lambda_n}x\right)e^{-\lambda_n kt}$$

Which is the same as given in section 36, page 106. In the above

$$\lambda_0 = 0$$

$$\lambda_n = \left(\frac{n\pi}{c}\right)^2 \qquad n = 1, 2, 3, \cdots$$

3 Section 41, Problem 3

3. A *hollow* sphere $1 \le r \le 2$ is initially at temperature zero. The interior surface is kept at that temperature, and the outer one is maintained at a constant temperature u_0. Set up the boundary value problem for the temperatures

$$u = u(r,t) \qquad (1 < r < 2, t > 0)$$

and follow these steps to solve it:

(*a*) Write $v(r,t) = ru(r,t)$ to obtain a new boundary value problem for $v(r,t)$. Then put $s = r - 1$ to obtain the problem

$$v_t = kv_{ss} \qquad (0 < s < 1, t > 0),$$

$$v = 0 \text{ when } s = 0, \qquad v = 2u_0 \text{ when } s = 1,$$

$$v = 0 \text{ when } t = 0.$$

(*b*) Use the result in Problem 2, Sec. 40, to write a solution of the boundary value problem reached in part (*a*). Then show how it follows from the substitutions made in part (*a*) that

$$u(r,t) = 2u_0\left[1 - \frac{1}{r} + \frac{2}{\pi r}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-n^2\pi^2 kt}\sin n\pi(r-1)\right].$$

Figure 3: Problem statement

<u>Solution</u>

The heat PDE in spherical coordinates, assuming no dependency on ϕ nor on θ is given by

$$\begin{aligned} u_t &= k\nabla^2 u \qquad (1) \\ &= k\frac{1}{r}(ru)_{rr} \end{aligned}$$

Where $1 < r < 2$ and $t > 0$. With the boundary conditions

$$u(1,t) = 0$$
$$u(2,0) = u_0$$

And initial conditions

$$u(r,0) = 0$$

3.1 Part (a)

Let $v(r,t) = ru(r,t)$. Hence $v_t = ru_t$ and $\frac{1}{r}(ru)_{rr} = \frac{1}{r}v_{rr}$. Substituting these in(1), the PDE simplifies to

$$v_t = kv_{rr} \tag{2}$$

And the boundary conditions $u(1,t) = 0$ becomes $v(1,t) = 0$ and $u(2,0) = u_0$ becomes $v(2,t) = 2u_0$. And initial conditions $u(r,0) = 0$ becomes $v(r,0) = 0$. Hence the new boundary conditions

$$\begin{aligned} v(1,t) &= 0 \\ v(2,t) &= 2u_0 \end{aligned}$$

And new initial conditions

$$v(r,0) = 0$$

Now let $s = r - 1$. Since $\frac{\partial r}{\partial s} = 1$, then the PDE becomes $v_t = kv_{ss}$. When $r = 1$, then $s = 0$ and the boundary conditions $v(1,t) = 0$ becomes $v(0,t) = 0$ and the boundary conditions $v(2,t) = 2u_0$ becomes $v(1,t) = 2u_0$. And initial conditions do not change. Hence the new problem is to solve for $v(s,t)$ in

$$\begin{aligned} v_t &= kv_{ss} \qquad (3) \\ v(1,t) &= 0 \\ v(1,t) &= 2u_0 \\ v(s,0) &= 0 \end{aligned}$$

With $0 < s < 1$ and $t > 0$.

3.2 Part (b)

The PDE (3) in part(a) is now the same as result of problem 2 section 40. Hence we can use that solution for (3) which gives

$$v(s,t) = 2u_0\left[x + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-n^2\pi^2kt}\sin(n\pi s)\right]$$

Replacing s by $r-1$ in the above gives

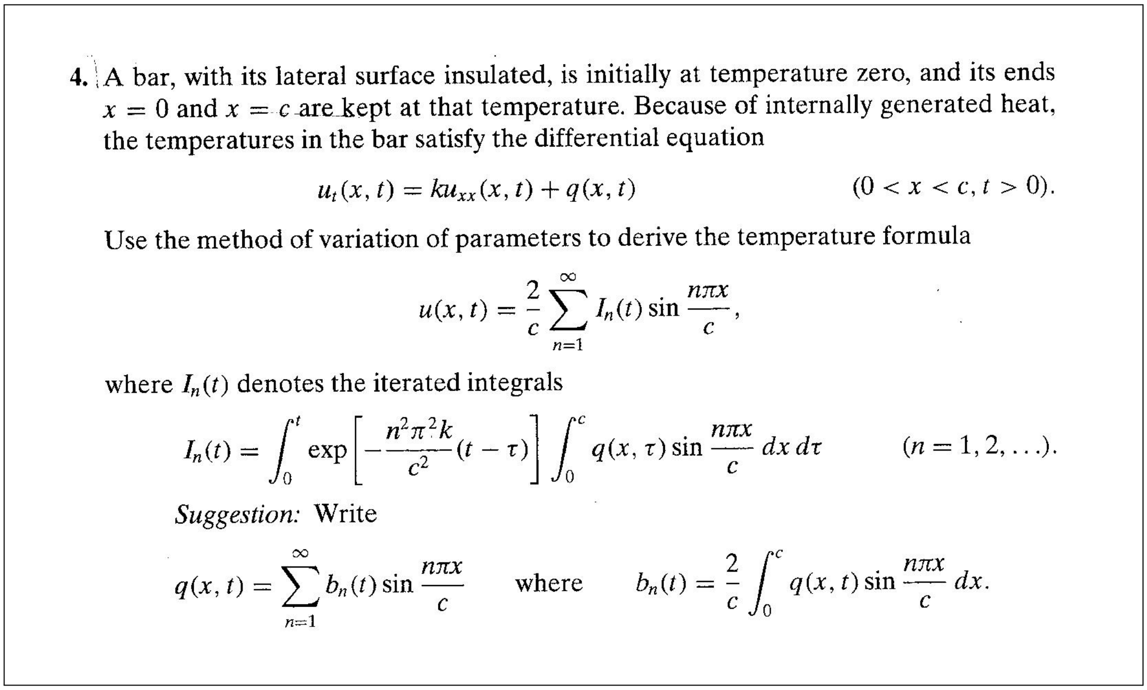
$$v(r,t) = 2u_0\left[(r-1) + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-n^2\pi^2kt}\sin(n\pi(r-1))\right]$$

But $v(r,t) = ru(r,t)$, hence $u(r,t) = \frac{v}{r}$ and therefore

$$\begin{aligned} u(r,t) &= 2u_0\left[\frac{(r-1)}{r} + \frac{2}{\pi r}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-n^2\pi^2kt}\sin(n\pi(r-1))\right] \\ &= 2u_0\left[\left(1 - \frac{1}{r}\right) + \frac{2}{\pi r}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-n^2\pi^2kt}\sin(n\pi(r-1))\right] \end{aligned}$$

Which is the result required.

4 Section 42, Problem 4



4. A bar, with its lateral surface insulated, is initially at temperature zero, and its ends $x = 0$ and $x = c$ are kept at that temperature. Because of internally generated heat, the temperatures in the bar satisfy the differential equation

$$u_t(x,t) = ku_{xx}(x,t) + q(x,t) \qquad (0 < x < c, t > 0).$$

Use the method of variation of parameters to derive the temperature formula

$$u(x,t) = \frac{2}{c}\sum_{n=1}^{\infty} I_n(t)\sin\frac{n\pi x}{c},$$

where $I_n(t)$ denotes the iterated integrals

$$I_n(t) = \int_0^t \exp\left[-\frac{n^2\pi^2 k}{c^2}(t-\tau)\right]\int_0^c q(x,\tau)\sin\frac{n\pi x}{c}\,dx\,d\tau \qquad (n = 1, 2, \ldots).$$

Suggestion: Write

$$q(x,t) = \sum_{n=1}^{\infty} b_n(t)\sin\frac{n\pi x}{c} \quad \text{where} \quad b_n(t) = \frac{2}{c}\int_0^c q(x,t)\sin\frac{n\pi x}{c}\,dx.$$

Figure 4: Problem statement

<u>Solution</u>

Using method of eigenfunction expansion (or method of variation of parameters as the book calls it), we start by assuming the solution to the PDE $u_t = ku_{xx} + q(x,t)$ is given by

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\Phi_n(x) \tag{1}$$

Where $\Phi_n(x)$ are the eigenfunctions associated with the homogeneous PDE $u_t = ku_{xx}$ with the homogeneous boundary conditions $u(0,t) = 0$ and $u(c,t) = 0$. But we solved this homogeneous PDE before. It has eigenvalues and corresponding eigenfunctions

$$\lambda_n = \left(\frac{n\pi}{c}\right)^2 \qquad n = 1, 2, 3, \cdots$$
$$\Phi_n(x) = \sin\left(\sqrt{\lambda_n}x\right)$$

Substituting (1) into the original PDE $u_t = ku_{xx} + q(x,t)$ results in

$$\frac{\partial}{\partial t}\sum_{n=1}^{\infty} a_n(t)\Phi_n(x) = k\frac{\partial^2}{\partial x^2}\sum_{n=1}^{\infty} a_n(t)\Phi_n(x) + q(x,t)$$
$$\sum_{n=1}^{\infty} a_n'(t)\Phi_n(x) = k\sum_{n=1}^{\infty} a_n(t)\Phi_n''(x) + q(x,t)$$

But from the Sturm-Liouville ODE, we know that $\Phi_n''(x) + \lambda_n\Phi_n(x) = 0$. Hence $\Phi_n''(x) =$

$-\lambda_n \Phi_n(x)$ and the above reduces to

$$\sum_{n=1}^{\infty} a_n'(t)\Phi_n(x) = -k\sum_{n=1}^{\infty} a_n(t)\lambda_n\Phi_n(x) + q(x,t) \tag{2}$$

Since the eigenfunctions $\Phi_n(x)$ are complete, we can expand $q(x,t)$ using them. Therefore

$$q(x,t) = \sum_{n=1}^{\infty} b_n(t)\Phi_n(x)$$

Substituting the above back in (2) gives

$$\sum_{n=1}^{\infty} a_n'(t)\Phi_n(x) = -k\sum_{n=1}^{\infty} a_n(t)\lambda_n\Phi_n(x) + \sum_{n=1}^{\infty} b_n(t)\Phi_n(x)$$

Since $\Phi_n(x)$ are never zero, we can simplify the above to

$$\begin{aligned} a_n'(t) &= -ka_n(t)\lambda_n + b_n(t) \\ a_n'(t) + ka_n(t)\lambda_n &= b_n(t) \end{aligned}$$

The above is first order ODE in $I_n(t)$. It is linear ODE. The integrating factor is $\mu = e^{\int k\lambda_n dt} = e^{k\lambda_n t}$. Multiplying the above ODE by this integrating factor gives

$$\frac{d}{dt}\left(a_n(t)e^{k\lambda_n t}\right) = b_n(t)e^{k\lambda_n t}$$

Integrating both sides

$$\begin{aligned} a_n(t)e^{k\lambda_n t} &= \int_0^t b_n(\tau)e^{k\lambda_n\tau}d\tau \\ a_n(t) &= \int_0^t b_n(\tau)e^{-k\lambda_n(t-\tau)}d\tau \end{aligned}$$

Now that we found $a_n(t)$, we substitute it back into (1) which gives

$$u(x,t) = \sum_{n=1}^{\infty}\left(\int_0^t b_n(\tau)e^{-k\lambda_n(t-\tau)}d\tau\right)\Phi_n(x) \tag{3}$$

What is left is to find $b_n(t)$. Since $q(x,t) = \sum_{n=1}^{\infty} b_n(t)\Phi_n(x)$, then by orthogonality we obtain

$$\begin{aligned} \int_0^c q(x,t)\Phi_m(x)\,dx &= \int_0^c \sum_{n=1}^{\infty} b_n(t)\Phi_n(x)\Phi_m(x)\,dx \\ &= \sum_{n=1}^{\infty} b_n(t)\int_0^c \Phi_n(x)\Phi_m(x)\,dx \\ &= b_m(t)\int_0^c \Phi_m^2(x)\,dx \\ &= b_m(t)\frac{c}{2} \end{aligned}$$

Hence

$$b_n(t) = \frac{2}{c}\int_0^c q(x,t)\Phi_m(x)\,dx$$

Substituting this back into (3) gives

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty}\left(\int_0^t e^{-k\lambda_n(t-\tau)}\frac{2}{c}\left(\int_0^c q(x,\tau)\,\Phi_m(x)\,dx\right)d\tau\right)\Phi_n(x) \\ &= \frac{2}{c}\sum_{n=1}^{\infty}\left(\int_0^t e^{-k\lambda_n(t-\tau)}\left(\int_0^c q(x,\tau)\,\Phi_m(x)\,dx\right)d\tau\right)\Phi_n(x) \qquad (4) \end{aligned}$$

If we let

$$I_n(t) = \int_0^t e^{-k\lambda_n(t-\tau)}\left(\int_0^c q(x,\tau)\,\Phi_m(x)\,dx\right)d\tau$$

Then (4) becomes

$$u(x,t) = \frac{2}{c}\sum_{n=1}^{\infty} I_n(t)\,\Phi_n(x)$$

Since $\Phi_n(x) = \sin\left(\frac{n\pi}{c}x\right)$ then the above is

$$u(x,t) = \frac{2}{c}\sum_{n=1}^{\infty} I_n(t)\sin\left(\frac{n\pi}{c}x\right)$$

Which is what required to show.

5 Section 42, Problem 5

5. By writing $c = 1$, $k = 1$, and $q(x, t) = xp(t)$ in the solution found in Problem 4, obtain the solution already found in Problem 1.

Figure 5: Problem statement

Solution

The solution in problem 4 above us

$$u(x,t) = \frac{2}{c}\sum_{n=1}^{\infty} I_n(t)\sin\left(\frac{n\pi}{c}x\right) \tag{1}$$

Where

$$I_n(t) = \int_0^t e^{-k\lambda_n(t-\tau)}\left(\int_0^c q(x,\tau)\sin\left(\frac{n\pi}{c}x\right)dx\right)d\tau$$

And $\lambda_n = \left(\frac{n\pi}{c}\right)^2$. Let $c = 1, k = 1$ and $q(x,t) = xp(t)$, then the above becomes

$$I_n(t) = \int_0^t e^{-n^2\pi^2(t-\tau)}\left(\int_0^1 xp(\tau)\sin(n\pi x)\,dx\right)d\tau$$

Substituting this in (1), using $c = 1$, then (1) becomes

$$\begin{aligned} u(x,t) &= 2\sum_{n=1}^{\infty}\left(\int_0^t e^{-n^2\pi^2(t-\tau)}\left(\int_0^1 xp(\tau)\sin(n\pi x)\,dx\right)d\tau\right)\sin(n\pi x) \\ &= 2\sum_{n=1}^{\infty}\left(\int_0^t p(\tau)e^{-n^2\pi^2(t-\tau)}\left(\int_0^1 x\sin(n\pi x)\,dx\right)d\tau\right)\sin(n\pi x) \end{aligned} \tag{2}$$

But $\int_0^1 x\sin(n\pi x)\,dx$ can now be integrated by parts. Let $u = x, dv = \sin(n\pi x)$, hence $du = 1, v = -\frac{\cos(n\pi x)}{n\pi}$ and therefore

$$\begin{aligned} \int_0^1 x\sin(n\pi x)\,dx &= -\frac{1}{n\pi}\left[x\cos(n\pi x)\right]_0^1 + \frac{1}{n\pi}\int_0^1 \cos(n\pi x)\,dx \\ &= -\frac{1}{n\pi}\cos(n\pi) + \frac{1}{n\pi}\left[\frac{\sin(n\pi x)}{n\pi}\right]_0^1 \\ &= -\frac{1}{n\pi}(-1)^n + \frac{1}{n^2\pi^2}\left[\sin(n\pi)\right] \\ &= \frac{(-1)^{n+1}}{n\pi} \end{aligned}$$

Substituting this back in (2) gives

$$\begin{aligned} u(x,t) &= 2\sum_{n=1}^{\infty}\left(\int_0^t p(\tau)\, e^{-n^2\pi^2(t-\tau)}\left(\frac{(-1)^{n+1}}{n\pi}\right)d\tau\right)\sin(n\pi x) \\ &= \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(n\pi x)\left(\int_0^t p(\tau)\, e^{-n^2\pi^2(t-\tau)}d\tau\right) \end{aligned}$$

Which is the solution for problem 1.

6 Section 42, Problem 8

Sec. 42.

8. Using a series of the form

$$u(x,t) = A_0(t) + \sum_{n=1}^{\infty} A_n(t) \cos \frac{n\pi x}{c}$$

and the expansion (see Example 1 in Sec. 8)

$$x^2 = \frac{c^2}{3} + \frac{4c^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos \frac{n\pi x}{c} \qquad (0 < x < c),$$

solve the following temperature problem for a slab $0 \leq x \leq c$ with insulated faces:

$$u_t(x,t) = ku_{xx}(x,t) + ax^2 \qquad (0 < x < c, t > 0),$$
$$u_x(0,t) = 0, \qquad u_x(c,t) = 0, \qquad u(x,0) = 0,$$

where a is a constant. Thus, show that

$$u(x,t) = ac^2 \left\{ \frac{t}{3} + \frac{4c^2}{\pi^4 k} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} \left[1 - \exp\left(-\frac{n^2\pi^2 k}{c^2} t \right) \right] \cos \frac{n\pi x}{c} \right\}.$$

Figure 6: Problem statement

Solution

The PDE to solve is

$$u_t = ku_{xx} + ax^2$$

With boundary conditions

$$u_x(0,t) = 0$$
$$u_x(c,t) = 0$$

And initial conditions

$$u(x,0) = 0$$

Using method of eigenfunction expansion, we start by assuming the solution to the PDE $u_t = ku_{xx} + ax^2$ is given by

$$u(x,t) = \sum_{n=0}^{\infty} a_n(t) \Phi_n(x) \tag{1}$$

Where $\Phi_n(x)$ are the eigenfunctions associated with the homogeneous PDE $u_t = ku_{xx}$ with the homogeneous boundary conditions $u_x(0,t) = 0$ and $u_x(c,t) = 0$. But we solved this homogeneous PDE before. It has eigenvalues and corresponding eigenfunctions

$$\begin{aligned}
\lambda_0 &= 0\\
\Phi_0(x) &= 1\\
\lambda_n &= \frac{n^2\pi^2}{c^2} \qquad n = 1,2,3,\cdots\\
\Phi_n(x) &= \cos\left(\frac{n\pi}{c}x\right)
\end{aligned}$$

Substituting (1) into the original PDE $u_t = ku_{xx} + ax^2$ results in

$$\begin{aligned}
\frac{\partial}{\partial t}\sum_{n=0}^{\infty} a_n(t)\Phi_n(x) &= k\frac{\partial^2}{\partial x^2}\sum_{n=0}^{\infty} a_n(t)\Phi_n(x) + ax^2\\
\sum_{n=0}^{\infty} a_n'(t)\Phi_n(x) &= k\sum_{n=0}^{\infty} a_n(t)\Phi_n''(x) + ax^2
\end{aligned}$$

But from the Sturm-Liouville ODE, we know that $\Phi_n''(x) + \lambda_n\Phi_n(x) = 0$. Hence $\Phi_n''(x) = -\lambda_n\Phi_n(x)$ and the above reduces to

$$\sum_{n=0}^{\infty} a_n'(t)\Phi_n(x) = -k\sum_{n=0}^{\infty} a_n(t)\lambda_n\Phi_n(x) + ax^2 \tag{2}$$

Since the eigenfunctions $\Phi_n(x)$ are complete, we can expand ax^2 using them. Therefore

$$ax^2 = \sum_{n=0}^{\infty} b_n(x)\Phi_n(x)$$

Substituting the above back in (2) gives

$$\sum_{n=0}^{\infty} a_n'(t)\Phi_n(x) = -k\sum_{n=0}^{\infty} a_n(t)\lambda_n\Phi_n(x) + \sum_{n=0}^{\infty} b_n(x)\Phi_n(x)$$

Since $\Phi_n(x)$ are never zero, we can simplify the above to

$$\begin{aligned}
a_n'(t) &= -ka_n(t)\lambda_n + b_n(x)\\
a_n'(t) + ka_n(t)\lambda_n &= b_n(x)
\end{aligned}$$

The above is first order ODE in $I_n(t)$. It is linear ODE. The integrating factor is $\mu = e^{\int k\lambda_n dt} = e^{k\lambda_n t}$. Multiplying the above ODE by this integrating factor gives

$$\frac{d}{dt}\left(a_n(t)e^{k\lambda_n t}\right) = b_n(x)e^{k\lambda_n t}$$

Integrating both sides

$$\begin{aligned}
a_n(t)e^{k\lambda_n t} &= b_n(x)\int_0^t e^{k\lambda_n\tau}d\tau\\
a_n(t) &= b_n(x)\int_0^t e^{-k\lambda_n(t-\tau)}d\tau \qquad (3)
\end{aligned}$$

What is left is to find $b_n(x)$. Since $ax^2 = \sum_{n=0}^{\infty} b_n(x)\Phi_n(x)$, and from example 1 section 8, we found that

$$b_0(x) = a\frac{c^2}{3}$$
$$b_n(x) = a\frac{4c^2}{\pi^2}\frac{(-1)^n}{n^2} \qquad n = 1, 2, 3, \cdots$$

Hence when $n = 0$, then (3) becomes (since $\lambda_0 = 0$)

$$\begin{aligned} a_0(t) &= a\frac{c^2}{3}\int_0^t d\tau \\ &= \frac{ac^2}{3}t \end{aligned}$$

When $n > 0$ then (3) becomes

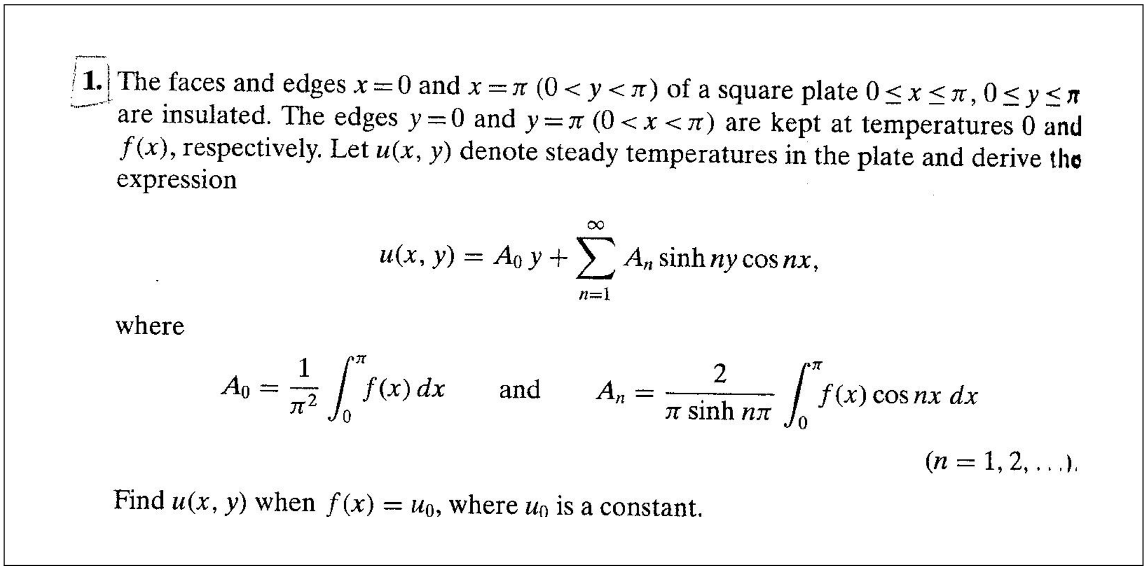
$$\begin{aligned} a_n(t) &= \left(a\frac{4c^2}{\pi^2}\frac{(-1)^n}{n^2}\right)\int_0^t e^{-k\lambda_n(t-\tau)}d\tau \\ &= \frac{(-1)^n}{n^2}\frac{4ac^2}{\pi^2}\int_0^t e^{-k\left(\frac{n\pi}{c}\right)^2(t-\tau)}d\tau \\ &= \frac{(-1)^n}{n^2}\frac{4ac^2}{\pi^2}e^{-k\left(\frac{n\pi}{c}\right)^2 t}\int_0^t e^{k\left(\frac{n\pi}{c}\right)^2\tau}d\tau \\ &= \frac{(-1)^n}{n^2}\frac{4ac^2}{\pi^2}e^{-k\left(\frac{n\pi}{c}\right)^2 t}\left[\frac{e^{k\left(\frac{n\pi}{c}\right)^2\tau}}{k\left(\frac{n\pi}{c}\right)^2}\right]_0^t \\ &= \frac{(-1)^n}{n^2}\frac{4ac^2}{\pi^2}\frac{e^{-k\left(\frac{n\pi}{c}\right)^2 t}}{k\left(\frac{n\pi}{c}\right)^2}\left[e^{k\left(\frac{n\pi}{c}\right)^2 t} - 1\right] \\ &= \frac{(-1)^n}{n^2}\frac{4ac^2}{\pi^2}\frac{1 - e^{-k\left(\frac{n\pi}{c}\right)^2 t}}{k\frac{n^2\pi^2}{c^2}} \\ &= \frac{(-1)^n}{n^4}\frac{4ac^4}{k\pi^4}\left(1 - e^{-k\left(\frac{n\pi}{c}\right)^2 t}\right) \end{aligned}$$

Now that we found $a_n(t)$, we substitute it back into (1) which gives

$$\begin{aligned} u(x,t) &= a_0(t) + \sum_{n=1}^{\infty} a_n(t)\,\Phi_n(x) \\ u(x,t) &= \frac{ac^2}{3}t + \sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}\frac{4ac^4}{k\pi^4}\left(1 - e^{-k\left(\frac{n\pi}{c}\right)^2 t}\right)\cos\left(\frac{n\pi}{c}x\right) \\ &= \frac{ac^2}{3}t + \frac{4ac^4}{k\pi^4}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}\left(1 - e^{-k\left(\frac{n\pi}{c}\right)^2 t}\right)\cos\left(\frac{n\pi}{c}x\right) \\ &= ac^2\left\{\frac{t}{3} + \frac{4c^2}{k\pi^4}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}\left(1 - e^{-k\left(\frac{n\pi}{c}\right)^2 t}\right)\cos\left(\frac{n\pi}{c}x\right)\right\} \end{aligned}$$

Which is the result required to show.

7 Section 43, Problem 1



1. The faces and edges $x = 0$ and $x = \pi$ $(0 < y < \pi)$ of a square plate $0 \le x \le \pi$, $0 \le y \le \pi$ are insulated. The edges $y = 0$ and $y = \pi$ $(0 < x < \pi)$ are kept at temperatures 0 and $f(x)$, respectively. Let $u(x, y)$ denote steady temperatures in the plate and derive the expression

$$u(x, y) = A_0\, y + \sum_{n=1}^{\infty} A_n \sinh ny \cos nx,$$

where

$$A_0 = \frac{1}{\pi^2}\int_0^{\pi} f(x)\,dx \qquad \text{and} \qquad A_n = \frac{2}{\pi \sinh n\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$(n = 1, 2, \ldots).$$

Find $u(x, y)$ when $f(x) = u_0$, where u_0 is a constant.

Figure 7: Problem statement

Solution

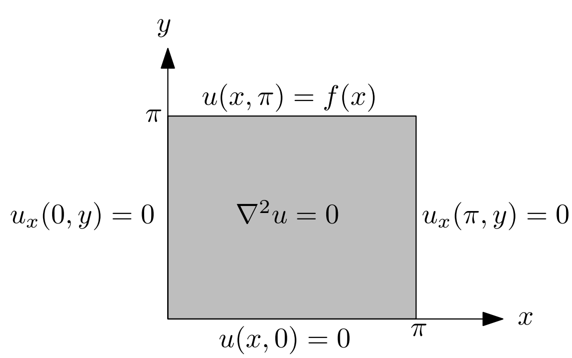


Figure 8: PDE and boundary conditions

Let $u(x,y) = X(x)Y(y)$. The PDE becomes

$$X''Y + Y''X = 0$$
$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$$

Hence the eigenvalue problem is

$$X'' + \lambda X = 0 \tag{1}$$
$$X'(0) = 0$$
$$X'(\pi) = 0$$

And the ODE for $Y(y)$ is

$$Y'' - \lambda Y = 0$$

We start by solving (1) to find the eigenvalues and eigenfunctions.

Case $\lambda < 0$ The solution is

$$X = A\cosh\left(\sqrt{-\lambda}x\right) + B\sinh\left(\sqrt{-\lambda}x\right)$$
$$X' = A\sqrt{-\lambda}\sinh\left(\sqrt{-\lambda}x\right) + B\sqrt{-\lambda}\cosh\left(\sqrt{-\lambda}x\right)$$

At $x = 0$ the above becomes

$$0 = B\sqrt{-\lambda}$$

Hence $B = 0$ and the solution becomes

$$X = A\cosh\left(\sqrt{-\lambda}x\right)$$
$$X' = A\sqrt{-\lambda}\sinh\left(\sqrt{-\lambda}x\right)$$

At $x = \pi$ the above gives

$$0 = A\sqrt{-\lambda}\sinh\left(\sqrt{-\lambda}\pi\right)$$

For nontrivial solution $\sinh\left(\sqrt{-\lambda}\pi\right) = 0$ but this is not possible since sinh is zero only when its argument is zero and this is not the case here. Hence $\lambda < 0$ is not eigenvalue.

Case $\lambda = 0$ The solution is

$$X = Ax + B$$
$$X' = A$$

At $x = 0$ the above becomes

$$0 = A$$

Hence the solution becomes

$$X = B$$
$$X' = 0$$

At $x = \pi$ the above gives

$$0 = 0$$

Therefore $\lambda = 0$ is eigenvalue with $X_0(x) = 1$.

Case $\lambda > 0$ The solution is

$$X = A\cos\left(\sqrt{\lambda}x\right) + B\sin\left(\sqrt{\lambda}x\right)$$
$$X' = -A\sqrt{\lambda}\sin\left(\sqrt{\lambda}x\right) + B\sqrt{\lambda}\cos\left(\sqrt{\lambda}x\right)$$

At $x = 0$ the above becomes

$$0 = B\sqrt{\lambda}$$

Hence $B = 0$ and the solution becomes

$$X = A\cos\left(\sqrt{\lambda}x\right)$$
$$X' = -A\sqrt{\lambda}\sin\left(\sqrt{\lambda}x\right)$$

At $x = \pi$ the above gives

$$0 = -A\sqrt{\lambda}\sin\left(\sqrt{\lambda}\pi\right)$$

For nontrivial solution

$$\sin\left(\sqrt{\lambda}\pi\right) = 0$$
$$\sqrt{\lambda}\pi = n\pi \qquad n = 1, 2, 3, \cdots$$
$$\lambda_n = n^2$$

And the corresponding eigenfunctions $X_n(x) = \cos(nx)$. Therefore in summary we have

eigenvalue		eigenfunction
$\lambda_0 = 0$		1
$\lambda_n = n^2$	$n = 1, 2, 3, \cdots$	$\cos(nx)$

Hence the $Y(y)$ ode becomes

$$Y'' - \lambda_n Y = 0$$
$$Y'' - n^2 Y = 0$$

The solution to the above is, when $n = 0$

$$Y_0 = A_0 y + B_0$$

When $y = 0$ the above gives $0 = B_0$. Hence $Y_0 = A_0 y$.

When $n > 0$

$$Y_n(y) = B_n\cosh(ny) + A_n\sinh(ny)$$

When $y = 0$ the above gives $0 = B_n$, Hence

$$Y_n(y) = A_n\sinh(ny)$$

Hence the fundamental solution is

$$u(x, y) = X_n Y_n$$

And the general solution is the superposition of these solutions

$$u\left(x,y\right) = A_0 X_0 Y_0 + \sum_{n=1}^{\infty} A_n Y_n X_n$$

Therefore

$$u\left(x,y\right) = A_0 y + \sum_{n=1}^{\infty} A_n \sinh\left(ny\right)\cos\left(nx\right) \tag{A}$$

What is left is to determine A_0 and A_n. At $y = \pi$ the above gives

$$f(x) = A_0\pi + \sum_{n=1}^{\infty} A_n \sinh(n\pi)\cos(nx)$$

Multiplying both sides by $\cos(mx)$ and integrating gives

$$\int_0^{\pi} f(x)\cos(mx)\,dx = \int_0^{\pi} A_0\pi\cos(mx)\,dx + \int_0^{\pi}\sum_{n=1}^{\infty} A_n\sinh(n\pi)\cos(nx)\cos(mx)\,dx \tag{1}$$

For $m = 0$, (1) becomes

$$\begin{aligned}\int_0^{\pi} f(x)\,dx &= \int_0^{\pi} A_0\pi dx\\ \int_0^{\pi} f(x)\,dx &= A_0\pi^2\\ A_0 &= \frac{1}{\pi^2}\int_0^{\pi} f(x)\,dx \end{aligned}\tag{2}$$

For $m > 0$, (1) becomes

$$\begin{aligned}\int_0^{\pi} f(x)\cos(mx)\,dx &= \int_0^{\pi}\sum_{n=1}^{\infty} A_n\sinh(n\pi)\cos(nx)\cos(mx)\,dx\\ \int_0^{\pi} f(x)\cos(mx)\,dx &= A_m\sinh(m\pi)\int_0^{\pi}\cos^2(nx)\,dx\\ &= A_m\sinh(m\pi)\frac{\pi}{2}\end{aligned}$$

Hence

$$A_n = \frac{2}{\pi\sinh(n\pi)}\int_0^{\pi} f(x)\cos(nx)\,dx \tag{3}$$

When $f(x) = u_0$ a constant, then (2) becomes

$$\begin{aligned}A_0 &= \frac{1}{\pi^2}\int_0^{\pi} u_0 dx\\ &= \frac{u_0}{\pi}\end{aligned}$$

And (3) becomes

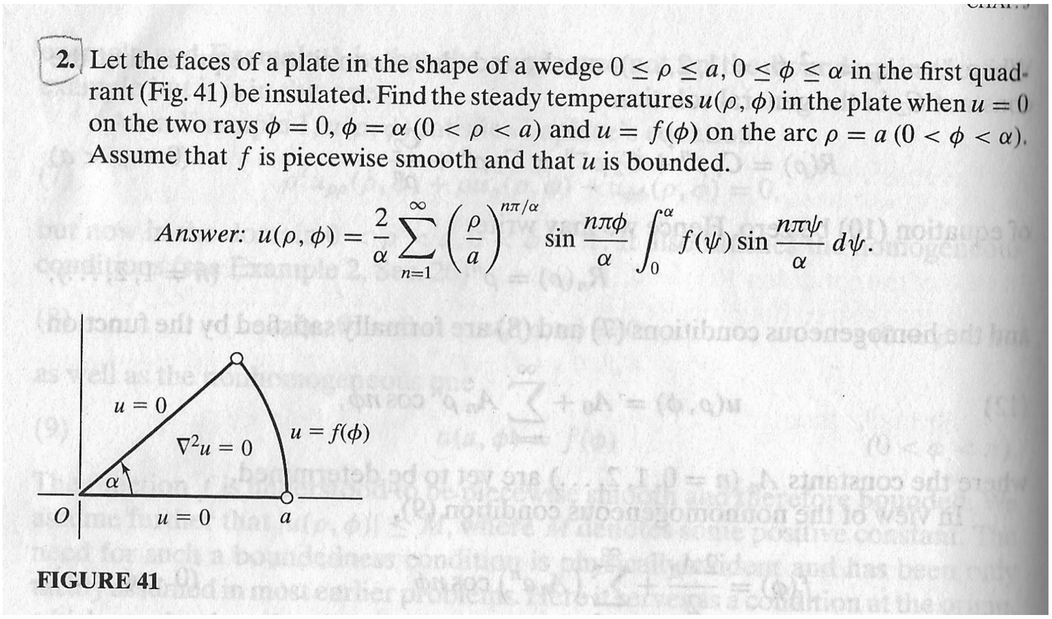
$$\begin{aligned} A_n &= \frac{2}{\pi \sinh(n\pi)} \int_0^{\pi} u_0 \cos(nx)\, dx \\ &= \frac{2u_0}{\pi \sinh(n\pi)} \left[\frac{\sin(nx)}{n} \right]_0^{\pi} \\ &= 0 \end{aligned}$$

Hence the solution (A) becomes

$$\boxed{u(x,y) = u_0 \frac{y}{\pi}}$$

This shows the final solution changes linearly in y. When $y = 0$ then $u(x, 0) = 0$ and when $y = \pi$, then $u(x, \pi) = u_0$.

8 Section 44, Problem 2



2. Let the faces of a plate in the shape of a wedge $0 \leq \rho \leq a, 0 \leq \phi \leq \alpha$ in the first quadrant (Fig. 41) be insulated. Find the steady temperatures $u(\rho, \phi)$ in the plate when $u = 0$ on the two rays $\phi = 0, \phi = \alpha \ (0 < \rho < a)$ and $u = f(\phi)$ on the arc $\rho = a \ (0 < \phi < \alpha)$. Assume that f is piecewise smooth and that u is bounded.

$$\textit{Answer: } u(\rho, \phi) = \frac{2}{\alpha} \sum_{n=1}^{\infty} \left(\frac{\rho}{a}\right)^{n\pi/\alpha} \sin \frac{n\pi\phi}{\alpha} \int_0^{\alpha} f(\psi) \sin \frac{n\pi\psi}{\alpha} \, d\psi.$$

FIGURE 41

Figure 9: Problem statement

<u>Solution</u>

The PDE $\nabla^2 u(\rho, \phi) = 0$ in polar coordinates is

$$u_{\rho\rho} + \frac{1}{\rho} u_{\rho} + \frac{1}{\rho^2} u_{\phi\phi} = 0$$

For $0 < \rho < a$ and $0 < \phi < \alpha$. With boundary conditions

$$\begin{aligned} u(\rho, 0) &= 0 \\ u(\rho, \alpha) &= 0 \\ u(a, \phi) &= f(\phi) \end{aligned}$$

And since u is bounded, then we have an extra condition $u(0, \phi) < \infty$.

Let $u\left(\rho,\phi\right)=R\left(\rho\right)\Phi\left(\phi\right)$. Substituting into the above PDE gives

$$
\begin{aligned}
R''\Phi+\frac{1}{\rho}R'\Phi+\frac{1}{\rho^2}\Phi''R&=0\\
\frac{R''}{R}+\frac{1}{\rho}\frac{R'}{R}+\frac{1}{\rho^2}\frac{\Phi''}{\Phi}&=0\\
\frac{\Phi''}{\Phi}&=-\left(\rho^2\frac{R''}{R}+\rho\frac{R'}{R}\right)=-\lambda
\end{aligned}
$$

Where λ is the separation constant. The above gives the boundary values problem to solve for λ

$$
\begin{aligned}
\Phi''+\lambda\Phi&=0 \qquad (1)\\
\Phi\left(0\right)&=0\\
\Phi\left(\alpha\right)&=0
\end{aligned}
$$

And

$$
\begin{aligned}
\rho^2\frac{R''}{R}+\rho\frac{R'}{R}&=\lambda\\
\rho^2R''+\rho R'-\lambda R&=0 \qquad (2)
\end{aligned}
$$

We start with (1) to find λ then use the result to solve (2). The ODE (1) we solved before, it has the eigenvalues

$$
\lambda_n=\left(\frac{n\pi}{\alpha}\right)^2 \qquad n=1,2,3,\cdots
$$

And corresponding eigenfunctions

$$
\Phi_n\left(\phi\right)=\sin\left(\frac{n\pi}{\alpha}\phi\right) \qquad (3)
$$

Now (2) can be solved. This is a Euler ODE. Using $R\left(\rho\right)=\rho^m$ and substituting into (2) gives

$$
\begin{aligned}
\rho^2m\left(m-1\right)\rho^{m-2}+\rho m\rho^{m-1}-\left(\frac{n\pi}{\alpha}\right)^2\rho^m&=0\\
m\left(m-1\right)\rho^m+m\rho^m-\left(\frac{n\pi}{\alpha}\right)^2\rho^m&=0\\
m\left(m-1\right)+m-\left(\frac{n\pi}{\alpha}\right)^2&=0\\
m^2&=\left(\frac{n\pi}{\alpha}\right)^2
\end{aligned}
$$

Hence

$$
m=\pm\frac{n\pi}{\alpha}
$$

Therefore the solution to (2) is

$$
R_n\left(\rho\right)=A_n\rho^{\frac{n\pi}{\alpha}}+B_n\rho^{\frac{-n\pi}{\alpha}}
$$

We immediately reject the solution $\rho^{\frac{-n\pi}{\alpha}}$ since this blows up at origin where $\rho \to 0$. Hence the above becomes

$$R_n(\rho) = A_n \rho^{\frac{n\pi}{\alpha}} \tag{4}$$

Now that we found $\Phi_n(\phi)$ and $R_n(\rho)$, then we use superposition to obtain the general solution

$$\begin{aligned} u(\rho,\phi) &= \sum_{n=1}^{\infty} R_n(\rho)\,\Phi_n(\phi) \\ &= \sum_{n=1}^{\infty} A_n \rho^{\frac{n\pi}{\alpha}} \sin\left(\frac{n\pi}{\alpha}\phi\right) \end{aligned} \tag{5}$$

At $\rho = a$, $u(a,\phi) = f(\phi)$, hence the above becomes

$$f(\phi) = \sum_{n=1}^{\infty} A_n a^{\frac{n\pi}{\alpha}} \sin\left(\frac{n\pi}{\alpha}\phi\right)$$

By orthogonality we obtain

$$\begin{aligned} \int_0^{\alpha} f(\phi) \sin\left(\frac{m\pi}{\alpha}\phi\right) d\phi &= \int_0^{\alpha} \sum_{n=1}^{\infty} A_n a^{\frac{n\pi}{\alpha}} \sin\left(\frac{n\pi}{\alpha}\phi\right) \sin\left(\frac{m\pi}{\alpha}\phi\right) d\phi \\ &= A_m a^{\frac{m\pi}{\alpha}} \int_0^{\alpha} \sin^2\left(\frac{m\pi}{\alpha}\phi\right) d\phi \\ &= A_m a^{\frac{m\pi}{\alpha}} \frac{\alpha}{2} \end{aligned}$$

Solving for A_n from the above gives

$$A_n = \frac{2}{\alpha} a^{\frac{-n\pi}{\alpha}} \int_0^{\alpha} f(\phi) \sin\left(\frac{n\pi}{\alpha}\phi\right) d\phi$$

Substituting the above in (5) gives the final solution

$$\begin{aligned} u(\rho,\phi) &= \sum_{n=1}^{\infty} \left(\frac{2}{\alpha} a^{\frac{-n\pi}{\alpha}} \int_0^{\alpha} f(\psi) \sin\left(\frac{n\pi}{\alpha}\psi\right) d\psi\right) \rho^{\frac{n\pi}{\alpha}} \sin\left(\frac{n\pi}{\alpha}\phi\right) \\ &= \frac{2}{\alpha} \sum_{n=1}^{\infty} \left(\frac{\rho}{a}\right)^{\frac{n\pi}{\alpha}} \sin\left(\frac{n\pi}{\alpha}\phi\right) \left(\int_0^{\alpha} f(\psi) \sin\left(\frac{n\pi}{\alpha}\psi\right) d\psi\right) \end{aligned}$$

9 Section 49, Problem 2

2. Solve the boundary value problem

$$u_t(x,t) = ku_{xx}(x,t) \qquad (-\pi < x < \pi, t > 0),$$

$$u(-\pi,t) = u(\pi,t), \quad u_x(-\pi,t) = u_x(\pi,t), \quad u(x,0) = f(x).$$

The solution $u(x,t)$ represents, for example, temperatures in an insulated wire of length 2π that is bent into a unit circle and has a given temperature distribution along it. For

convenience, the wire is thought of as being cut at one point and laid on the x axis between $x = -\pi$ and $x = \pi$. The variable x then measures the distance along the wire, starting at the point $x = -\pi$; and the points $x = -\pi$ and $x = \pi$ denote the same point on the circle. The first two boundary conditions in the problem state that the temperatures and the flux must be the same for each of those values of x. This problem was of considerable interest to Fourier himself, and the wire has come to be known as *Fourier's ring*.

Answer: $u(x,t) = A_0 + \sum_{n=1}^{\infty} e^{-n^2kt}(A_n \cos nx + B_n \sin nx)$,

where

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

and

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \qquad B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \qquad (n = 1, 2, \ldots).$$

Figure 10: Problem statement

<u>Solution</u>

$$u_t = ku_{xx}$$

With $-\pi < x < \pi, t > 0$ and periodic boundary conditions

$$u(-\pi,t) = u(\pi,t)$$
$$u_x(-\pi,t) = u_x(\pi,t)$$

And initial conditions

$$u(x,0) = f(x)$$

Normal process of separation of variables leads to eigenvalue problem

$$X'' + \lambda X = 0 \tag{1}$$
$$X(-\pi) = X(\pi)$$
$$X'(-\pi) = X'(\pi)$$

And the time ODE

$$T' + k\lambda T = 0 \tag{2}$$

We start by solving (1) to find the eigenvalues and eigenfunctions.

<u>Case $\lambda < 0$</u>

Solution is

$$X(x) = A\cosh\left(\sqrt{-\lambda}x\right) + B\sinh\left(\sqrt{-\lambda}x\right)$$
$$X'(x) = A\sqrt{-\lambda}\sinh\left(\sqrt{-\lambda}x\right) + B\sqrt{-\lambda}\cosh\left(\sqrt{-\lambda}x\right)$$

The boundary conditions $X(-\pi) = X(\pi)$ results in (using the fact that $\cosh$ is even and $\sinh$ is odd)

$$A\cosh\left(\sqrt{-\lambda}\pi\right) + B\sinh\left(\sqrt{-\lambda}\pi\right) = A\cosh\left(\sqrt{-\lambda}\pi\right) - B\sinh\left(\sqrt{-\lambda}\pi\right)$$
$$B\sinh\left(\sqrt{-\lambda}\pi\right) = -B\sinh\left(\sqrt{-\lambda}\pi\right)$$
$$B\sinh\left(\sqrt{-\lambda}\pi\right) = 0 \tag{3}$$

The boundary conditions $X'(-\pi) = X'(\pi)$ results in (using the fact that $\cosh$ is even and $\sinh$ is odd)

$$A\sqrt{-\lambda}\sinh\left(\sqrt{-\lambda}\pi\right) + B\sqrt{-\lambda}\cosh\left(\sqrt{-\lambda}\pi\right) = -A\sqrt{-\lambda}\sinh\left(\sqrt{-\lambda}\pi\right) + B\sqrt{-\lambda}\cosh\left(\sqrt{-\lambda}\pi\right)$$
$$A\sqrt{-\lambda}\sinh\left(\sqrt{-\lambda}\pi\right) = -A\sqrt{-\lambda}\sinh\left(\sqrt{-\lambda}\pi\right)$$
$$A\sinh\left(\sqrt{-\lambda}\pi\right) = 0 \tag{4}$$

So we obtain (3,4) equations, here they are again

$$B\sinh\left(\sqrt{-\lambda}\pi\right) = 0$$
$$A\sinh\left(\sqrt{-\lambda}\pi\right) = 0$$

There are two possibility, either $\sinh\left(\sqrt{-\lambda}\pi\right) = 0$ or $\sinh\left(\sqrt{-\lambda}\pi\right) \neq 0$. If $\sinh\left(\sqrt{-\lambda}\pi\right) \neq 0$ then this leads to trivial solution, as it implies that both $A = 0$ and $B = 0$. On the other hand, if $\sinh\left(\sqrt{-\lambda}\pi\right) = 0$ then this implies that $\sqrt{-\lambda}\pi = 0$ since $\sinh$ is only zero when its argument is zero which is not the case here. This implies that <u>$\lambda < 0$ is not possible.</u>

<u>Case $\lambda = 0$</u>

The solution now becomes $X(x) = Ax+B$. Satisfying the boundary conditions $X(-\pi) = X(\pi)$ gives

$$\begin{aligned} A\pi + B &= -A\pi + B \\ 2A\pi &= 0 \\ A &= 0 \end{aligned}$$

Hence the solution becomes

$$\begin{aligned} X(x) &= B \\ X' &= 0 \end{aligned}$$

Satisfying the boundary conditions $X'(-\pi) = X'(\pi)$ gives $0 = 0$. Hence $\lambda = 0$ is possible eigenvalue, with corresponding eigenfunction as constant, say 1.

<u>Case $\lambda > 0$</u>

Solution is

$$\begin{aligned} X(x) &= A\cos\left(\sqrt{\lambda}x\right) + B\sin\left(\sqrt{\lambda}x\right) \\ X'(x) &= -A\sqrt{\lambda}\sin\left(\sqrt{\lambda}x\right) + B\sqrt{\lambda}\cos\left(\sqrt{\lambda}x\right) \end{aligned}$$

The boundary conditions $X(-\pi) = X(\pi)$ results in (using the fact that cos is even and sin is odd)

$$\begin{aligned} A\cos\left(\sqrt{\lambda}\pi\right) + B\sin\left(\sqrt{\lambda}\pi\right) &= A\cos\left(\sqrt{\lambda}\pi\right) - B\sin\left(\sqrt{\lambda}\pi\right) \\ B\sin\left(\sqrt{\lambda}\pi\right) &= -B\sin\left(\sqrt{\lambda}\pi\right) \\ B\sin\left(\sqrt{\lambda}\pi\right) &= 0 \qquad (5) \end{aligned}$$

The boundary conditions $X'(-\pi) = X'(\pi)$ results in (using the fact that cosh is even and sinh is odd)

$$\begin{aligned} -A\sqrt{\lambda}\sin\left(\sqrt{\lambda}\pi\right) + B\sqrt{\lambda}\cos\left(\sqrt{\lambda}\pi\right) &= A\sqrt{\lambda}\sin\left(\sqrt{\lambda}\pi\right) + B\sqrt{\lambda}\cos\left(\sqrt{\lambda}\pi\right) \\ -A\sqrt{\lambda}\sin\left(\sqrt{\lambda}\pi\right) &= A\sqrt{\lambda}\sin\left(\sqrt{\lambda}\pi\right) \\ A\sin\left(\sqrt{\lambda}\pi\right) &= 0 \qquad (6) \end{aligned}$$

So we obtain (5,6) equations, here they are again

$$\begin{aligned} B\sin\left(\sqrt{\lambda}\pi\right) &= 0 \\ A\sin\left(\sqrt{\lambda}\pi\right) &= 0 \end{aligned}$$

There are two possibility, either $\sin\left(\sqrt{\lambda}\pi\right) = 0$ or $\sin\left(\sqrt{\lambda}\pi\right) \neq 0$. If $\sin\left(\sqrt{\lambda}\pi\right) \neq 0$ then this leads to trivial solution, as it implies that both $A = 0$ and $B = 0$. If $\sin\left(\sqrt{\lambda}\pi\right) = 0$ then this implies that $\sqrt{\lambda}\pi = n\pi$ where $n = 1, 2, 3, \cdots$. Hence $\lambda > 0$ is possible with eigenvalues and corresponding eigenfunctions given by

$$\lambda_n = n^2 \qquad n = 1, 2, 3, \cdots$$
$$X_n(x) = A_n \cos(nx) + B_n \sin(nx)$$

Now that we solved the eigenvalue problem (1), we use the eigenvalues found to solve the time ODE (2)

$$T' + k\lambda_n T = 0$$

When $\lambda = 0$, this becomes $T' = 0$ or $T_0(t)$ is constant. When $\lambda > 0$ the solution is

$$\begin{aligned} T_n(t) &= e^{-k\lambda_n t} \\ &= e^{-kn^2 t} \end{aligned}$$

Hence the fundamental solution is

$$u_n(x,t) = X_n(x)\, T_n(t)$$

And by superposition, the general solution is

$$u(x,t) = A_0 X_0(x)\, T_0(t) + \sum_{n=1}^{\infty} (A_n \cos(nx) + B_n \sin(nx))\, e^{-kn^2 t}$$

But $X_0(x) = 1$ and $T_0(t)$ is constant. Hence the above simplifies to

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} (A_n \cos(nx) + B_n \sin(nx))\, e^{-kn^2 t}$$

What is left is to find A_0, A_n, B_n. At $t = 0$ the above gives

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos(nx) + B_n \sin(nx) \tag{7}$$

For $n = 0$, by orthogonality we obtain

$$\begin{aligned} \int_{-\pi}^{\pi} f(x)\, dx &= \int_{-\pi}^{\pi} A_0 dx \\ \int_{-\pi}^{\pi} f(x)\, dx &= A_0 (2\pi) \\ A_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx \end{aligned}$$

For $n > 0$. We start by multiplying both sides of (7) by $\cos(mx)$ and integrating both sides. This gives

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) \cos(mx)\, dx &= \int_{-\pi}^{\pi} \left(\sum_{n=1}^{\infty} A_n \cos(nx) \cos(mx) + B_n \sin(nx) \cos(mx) \right) dx \\ &= \sum_{n=1}^{\infty} A_n \int_{-\pi}^{\pi} \cos(nx) \cos(mx)\, dx + \sum_{n=1}^{\infty} B_n \int_{-\pi}^{\pi} \sin(nx) \cos(mx)\, dx \end{aligned}$$

But $\int_{-\pi}^{\pi} \sin(nx) \cos(mx)\, dx = 0$ for all n, m. And $\int_{-\pi}^{\pi} \cos(nx) \cos(mx)\, dx = \int_{-\pi}^{\pi} \cos^2(mx)\, dx$

and zero for all other $n \neq m$. Hence the above simplifies to

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx &= A_m \int_{-\pi}^{\pi} \cos^2(mx)\,dx \\ &= A_m \pi\end{aligned}$$

Therefore

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

To find B_n we do the same, but now we multiply both sides of (7) by $\sin(mx)$ and this leads to

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\sin(mx)\,dx &= \int_{-\pi}^{\pi}\left(\sum_{n=1}^{\infty} A_n\cos(nx)\sin(mx) + B_n\sin(nx)\sin(mx)\right)dx \\ &= \sum_{n=1}^{\infty} A_n\int_{-\pi}^{\pi}\cos(nx)\sin(mx)\,dx + \sum_{n=1}^{\infty} B_n\int_{-\pi}^{\pi}\sin(nx)\sin(mx)\,dx\end{aligned}$$

But $\int_{-\pi}^{\pi}\cos(nx)\sin(mx)\,dx = 0$ for all n, m. And $\int_{-\pi}^{\pi}\sin(nx)\sin(mx)\,dx = \int_{-\pi}^{\pi}\sin^2(mx)\,dx$ and zero for all other $n \neq m$. Hence the above simplifies to

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\sin(mx)\,dx &= B_m \int_{-\pi}^{\pi} \sin^2(mx)\,dx \\ &= B_m \pi\end{aligned}$$

Therefore

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

This completes the solution. The final solution is

$$\begin{aligned}u(x,t) &= A_0 + \sum_{n=1}^{\infty}\left(A_n\cos(nx) + B_n\sin(nx)\right)e^{-kn^2t} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx + \sum_{n=1}^{\infty} e^{-kn^2t}\left[\left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx\right)\cos(nx) + \left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx\right)\sin(nx)\right]\end{aligned}$$