

HW 4
MATH 4567 Applied Fourier Analysis
Spring 2019
University of Minnesota, Twin Cities

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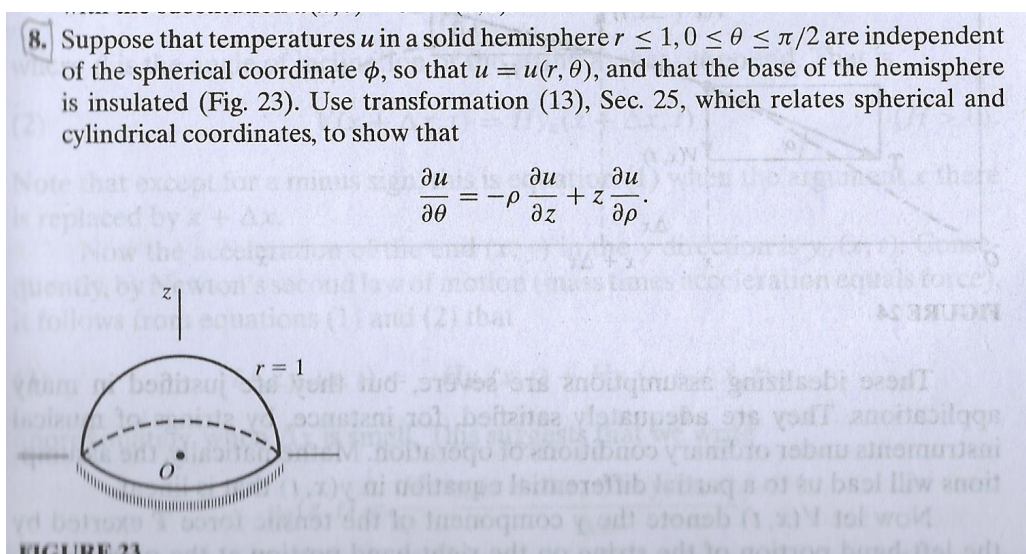
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1 Section 27, Problem 8



Thus show that u must satisfy the boundary condition

$$u_{\theta}\left(r, \frac{\pi}{2}\right) = 0.$$

Figure 1: Problem statement

Solution

The cylindrical and spherical coordinates are defined as given in the textbook figures shown below

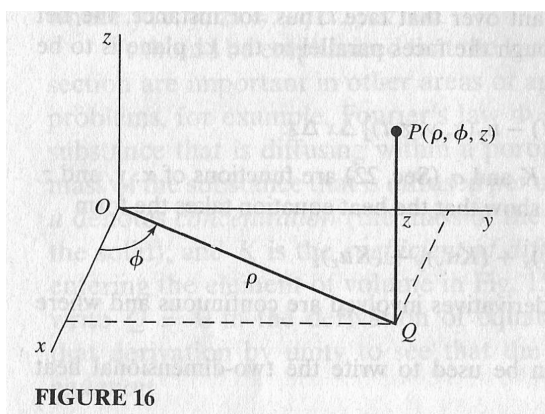


Figure 2: Cylindrical coordinates

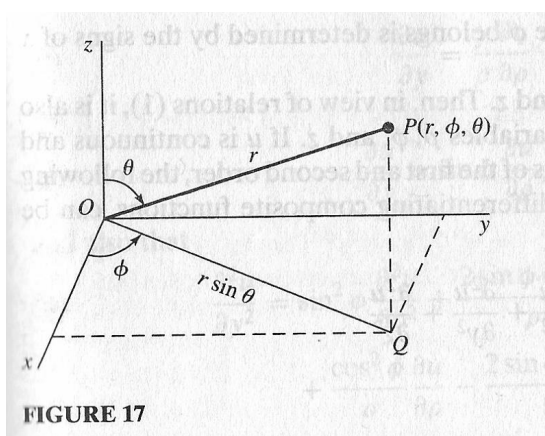


Figure 3: Spherical coordinates

The relation between these is given by (13) in the book

$$z = r \cos \theta \quad (1)$$

$$\rho = r \sin \theta \quad (2)$$

$$\phi = \phi \quad (3)$$

To obtain the required formula, we will use the chain rule. Since in spherical we have $u \equiv u(r, \theta)$ and in cylindrical we have $u \equiv u(\rho, z)$, then by chain rule

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial \rho} \frac{\partial \rho}{\partial \theta} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial \theta}$$

But from (2) $\frac{\partial \rho}{\partial \theta} = r \cos \theta$ and from (1) $\frac{\partial z}{\partial \theta} = -r \sin \theta$, hence the above becomes

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial \rho} (r \cos \theta) + \frac{\partial u}{\partial z} (-r \sin \theta)$$

But $r \cos \theta = z$ and $-r \sin \theta = \rho$, hence the above simplifies to

$$\frac{\partial u}{\partial \theta} = z \frac{\partial u}{\partial \rho} - \rho \frac{\partial u}{\partial z} \quad (4)$$

Which is the result required to show. Now we need to show that $\frac{\partial u}{\partial \theta}$ evaluated at boundary $r = 1, \theta = \frac{\pi}{2}$ is zero. But $\theta = \frac{\pi}{2}$ implies that $z = 0$, since $z = r \cos \theta$. Hence (4) now reduces to

$$\frac{\partial u}{\partial \theta} = -\rho \frac{\partial u}{\partial z} \quad (4)$$

Since $\theta = \frac{\pi}{2}$, then $\frac{\partial u}{\partial z}$ is the directional derivative normal to the base surface. But we are told it is insulated. This implies that $\frac{\partial u}{\partial z} = 0$, since by definition this is what insulated means. Therefore $\frac{\partial u}{\partial \theta} = 0$ at $r = 1, \theta = \frac{\pi}{2}$, which is what we are asked to show.

2 Section 28, Problem 1

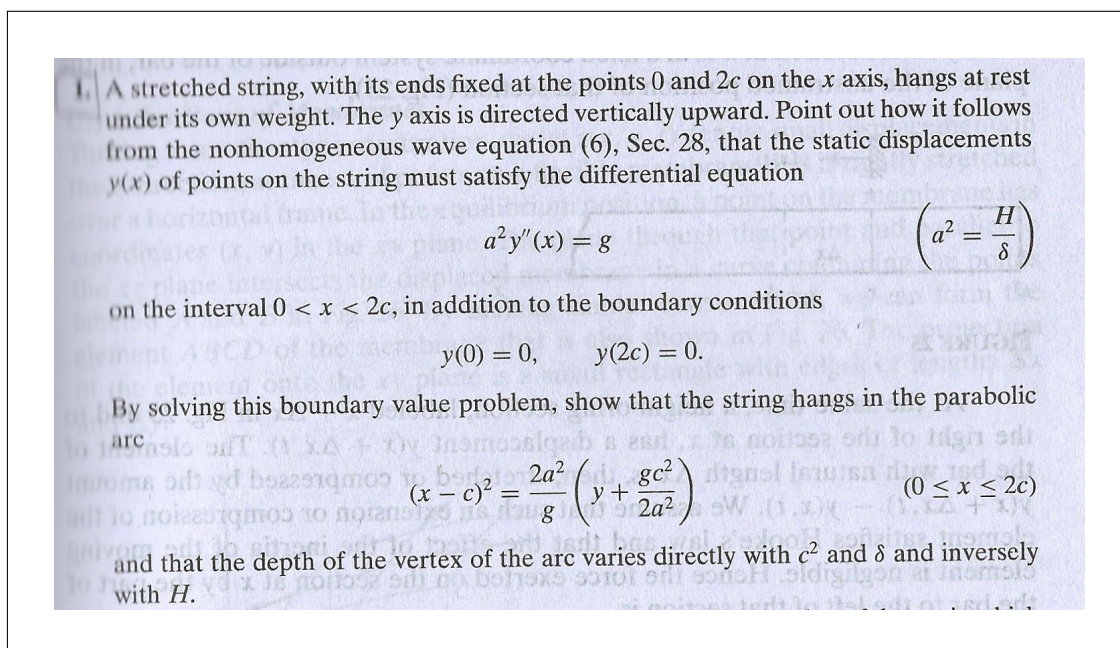


Figure 4: Problem statement

Eq (6) in section 28 is

$$y_{tt}(x, t) = a^2 y_{xx}(x, t) - g$$

At static displacement, by definition, there is no time dependency, hence $y_{tt} = 0$ and the above becomes

$$0 = a^2 y_{xx}(x, t) - g$$

Therefore now this becomes an ODE instead of a PDE since it does not depend on time, and we can write the above as

$$a^2 y''(x) = g \quad (1)$$

The boundary conditions $y(0, t) = 0$ and $y(2x, t) = 0$ now become $y(0) = 0, y(2x) = 0$. Now we need to solve (1) with these boundary conditions. This is an boundary value ODE.

$$y''(x) = \frac{g}{a^2}$$

The RHS is constant. The solution to the homogeneous ODE $y'' = 0$ is $y_h = Ax + B$. Let the particular solution be $y_p = C_3 x^2$, then $y'_p = 2C_3 x$ and $y''_p = 2C_3$. Substituting this in the above ODE gives

$$2C_3 = \frac{g}{a^2}$$

$$C_3 = \frac{g}{2a^2}$$

Hence $y_p(x) = \frac{g}{2a^2} x^2$. Therefore the general solution is

$$y = y_h + y_p$$

$$= Ax + B + \frac{g}{2a^2} x^2 \quad (2)$$

Now we will use the boundary conditions to find A, B above. At $x = 0$, (2) becomes

$$0 = B$$

Hence solution (2) reduces to

$$y(x) = Ax + \frac{g}{2a^2} x^2 \quad (3)$$

At $x = 2c$, the second boundary condition gives

$$\begin{aligned} 0 &= 2cA + \frac{g}{2a^2} (4c^2) \\ A &= \frac{-g (4c^2)}{2a^2 \cdot 2c} \\ &= \frac{-gc}{a^2} \end{aligned}$$

Hence the solution (3) becomes

$$\begin{aligned} y &= \frac{-gc}{a^2} x + \frac{g}{2a^2} x^2 \\ y &= \frac{gx^2 - 2gcx}{2a^2} \end{aligned} \tag{4}$$

To get the result needed, we can manipulate this more as follows. From (4)

$$\begin{aligned} 2a^2y &= gx^2 - 2gcx \\ &= g(x^2 - 2cx) \\ &= g(x - c)^2 - gc^2 \end{aligned}$$

Hence

$$\begin{aligned} g(x - c)^2 &= 2a^2y + gc^2 \\ (x - c)^2 &= \frac{2a^2y}{g} + c^2 \\ &= \frac{2a^2}{g} \left(y + \frac{gc^2}{2a^2} \right) \end{aligned}$$

Now since $a^2 = \frac{H}{\delta}$ then the above becomes

$$\begin{aligned} \frac{g}{2a^2} (x - c)^2 &= y + \frac{gc^2}{2a^2} \\ y &= \frac{1}{2a^2} (g(x - c)^2 - gc^2) \\ &= \frac{g}{2\frac{H}{\delta}} ((x - c)^2 - c^2) \\ &= \frac{\delta g}{H2} ((x - c)^2 - c^2) \end{aligned}$$

We see now that y is directly proportional to δ and c^2 and inversely proportional to H .

3 Section 28, Problem 5

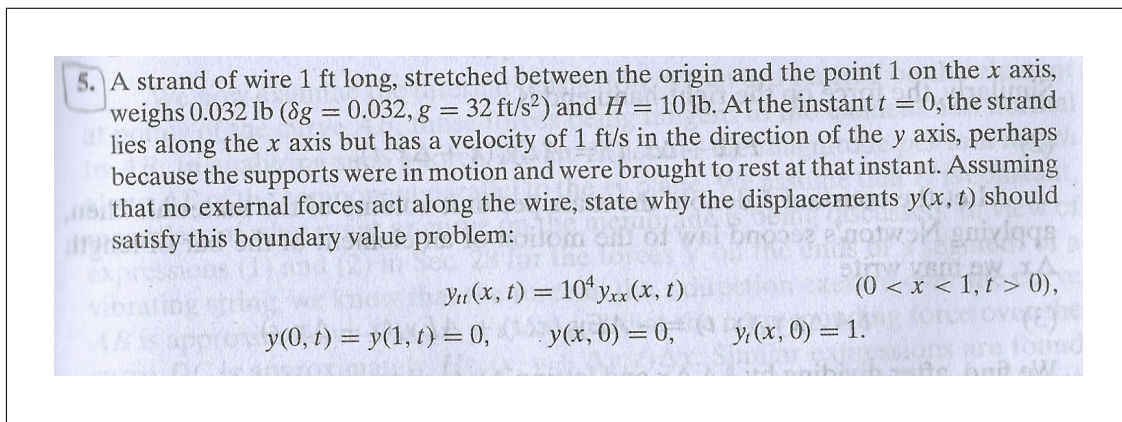


Figure 5: Problem statement

solution

The wave PDE in 1D is given by

$$y_{tt}(x, t) = a^2 y_{xx}(x, t) \quad (1)$$

Where

$$a^2 = \frac{H}{\delta}$$

Where H is the tension in the strand and δ is the mass per unit length of the strand. But $weight = (mass)g$. hence $\delta = \frac{weight}{g}$. We are given that $weight = 0.032$ lb, and that $g = 32$ ft/s². This implies that

$$\delta = \frac{0.032}{32} = \frac{1}{1000}$$

Hence

$$a^2 = \frac{10}{\frac{1}{1000}} = 10^4$$

Therefore (1) becomes

$$y_{tt}(x, t) = 10^4 y_{xx}(x, t) \quad (2)$$

Since at $t = 0$ we are told that strand lies along the x -axis, then $y(x, 0) = 0$ and problem says $y_t(x, 0) = 1$. For boundary conditions, since strand fixed at $x = 0$ and $x = 1$, then this implies $y(0, t) = 0$ and $y(1, t) = 0$. Therefore the PDE is

$$\begin{aligned} y_{tt}(x, t) &= 10^4 y_{xx}(x, t) & 0 < x < 1, t > 0 \\ y(x, 0) &= 0 \\ y_t(x, 0) &= 1 \\ y(0, t) &= 0 \\ y(1, t) &= 0 \end{aligned}$$

4 Section 30, Problem 3

3. Let $y(x, t)$ represent transverse displacements in a long stretched string one end of which is attached to a ring that can slide along the y axis. The other end is so far out on the positive x axis that it may be considered to be infinitely far from the origin. The ring is initially at the origin and is then moved along the y axis (Fig. 27) so that $y = f(t)$ when $x = 0$ and $t \geq 0$, where f is a prescribed continuous function and $f(0) = 0$. We assume that the string is initially at rest on the x axis; thus $y(x, t) \rightarrow 0$ as $x \rightarrow \infty$. The

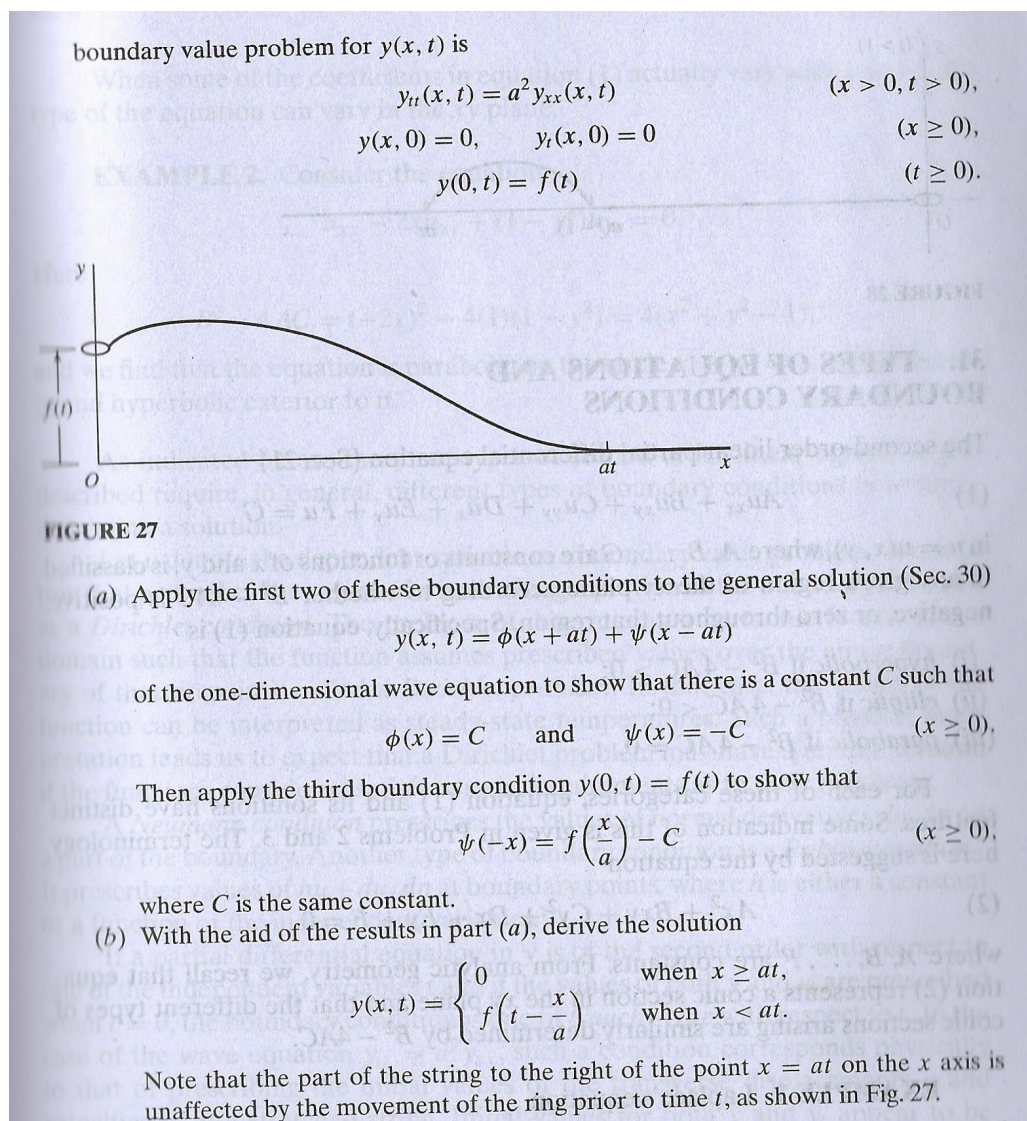


Figure 6: Problem statement

4.1 Part a

Applying the first initial conditions $y(x, 0) = 0$ to the solution

$$y(x, t) = \phi(x + at) + \psi(x - at) \quad (1)$$

Gives

$$0 = \phi(x) + \psi(x) \quad (2)$$

But $y_t = a\phi' - a\psi'$. Hence the second initial conditions at $t = 0$ gives

$$0 = a\phi'(x) - a\psi'(x) \quad (3)$$

Taking derivative of (2) and multiplying the resulting equation by a gives

$$0 = a\phi'(x) + a\psi'(x) \quad (2A)$$

Adding (3,2A) gives

$$\begin{aligned} 2a\phi'(x) &= 0 \\ \phi'(x) &= 0 \end{aligned}$$

Therefore

$$\phi(x) = C \quad (4)$$

Where C is an arbitrary constant. Substituting the above result back in (2) gives

$$\begin{aligned} 0 &= C + \psi(x) \\ \psi(x) &= -C \end{aligned} \quad (5)$$

From (4,5) we see that

$$\begin{aligned} \phi(x) &= C \\ \psi(x) &= -C \end{aligned}$$

Now applying boundary condition $y(0, t) = f(t)$ to (1) gives

$$f(t) = \phi(at) + \psi(-at)$$

But a is the speed of the wave given by $a = \frac{x}{t}$ or $t = \frac{x}{a}$. Hence the above becomes

$$\begin{aligned} f\left(\frac{x}{a}\right) &= \phi(x) + \psi(-x) \\ \psi(-x) &= f\left(\frac{x}{a}\right) - \phi(x) \end{aligned}$$

Since $\phi(x) = C$ from equation (4), then the final result is obtained

$$\psi(-x) = f\left(\frac{x}{a}\right) - C \quad x \geq 0 \quad (6)$$

4.2 Part b

Since the part to the right of $x = at$ is unaffected by the movement of the right, then

$$y(x, t) = 0 \quad x \geq at \quad (1)$$

So now we need to find the solution for $x < at$ and $x \geq 0$. From

$$y(x, t) = \phi(x + at) + \psi(x - at)$$

And using (6) in part (a), we see that $\psi(x - at) = f\left(\frac{-(x-at)}{a}\right) - C$. Therefore the above becomes

$$y(x, t) = \phi(x + at) + f\left(\frac{-(x - at)}{a}\right) - C$$

But also from part (a) $\phi(x + at) = C$. Hence the above simplifies to

$$\begin{aligned} y(x, t) &= c + f\left(\frac{-(x - at)}{a}\right) - C \\ &= f\left(\frac{-x + at}{a}\right) \\ &= f\left(t - \frac{x}{a}\right) \quad x < at \end{aligned} \quad (2)$$

Combining (1) and (2) shows that

$$y(x, t) = \begin{cases} 0 & x \geq at \\ f\left(t - \frac{x}{a}\right) & x < at \end{cases}$$

5 Section 30, Problem 4

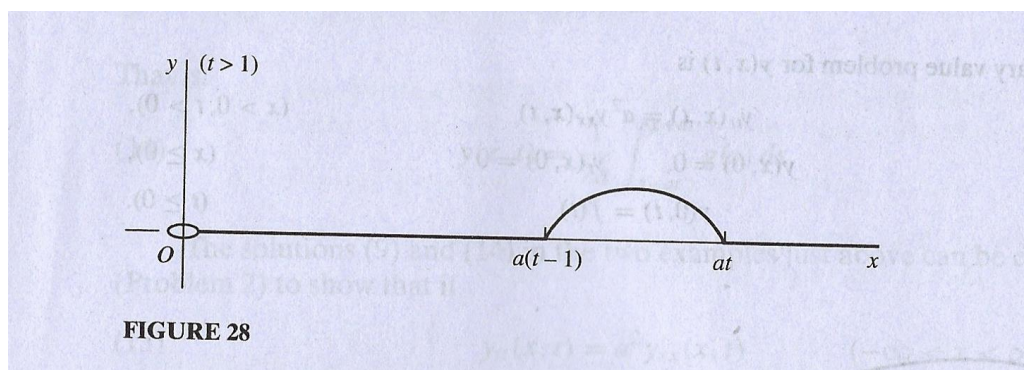
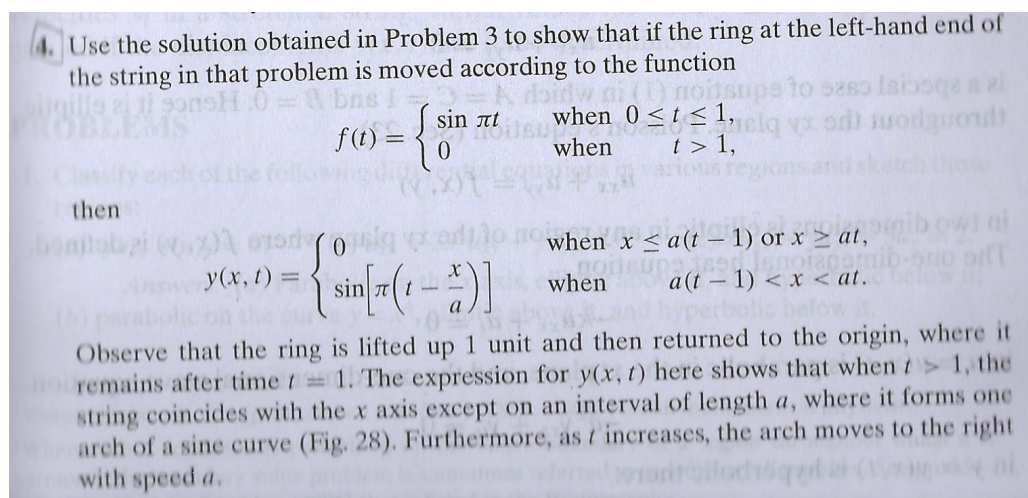


Figure 7: Problem statement

This requires just substitution of the function $f(t)$ given into the solution found above which is

$$y(x, t) = \begin{cases} 0 & x \geq at \\ f\left(t - \frac{x}{a}\right) & x < at \end{cases} \quad (1)$$

But

$$f(t) = \begin{cases} \sin \pi t & 0 \leq t \leq 1 \\ 0 & t > 1 \end{cases} \quad (2)$$

Substituting (2) into (1) gives, after replacing each t in (2) by $t - \frac{x}{a}$ the result needed

$$y(x, t) = \begin{cases} 0 & x \geq at \\ \sin\left(\pi\left(t - \frac{x}{a}\right)\right) & a(t-1) < x < at \end{cases}$$

6 Section 31, Problem 2

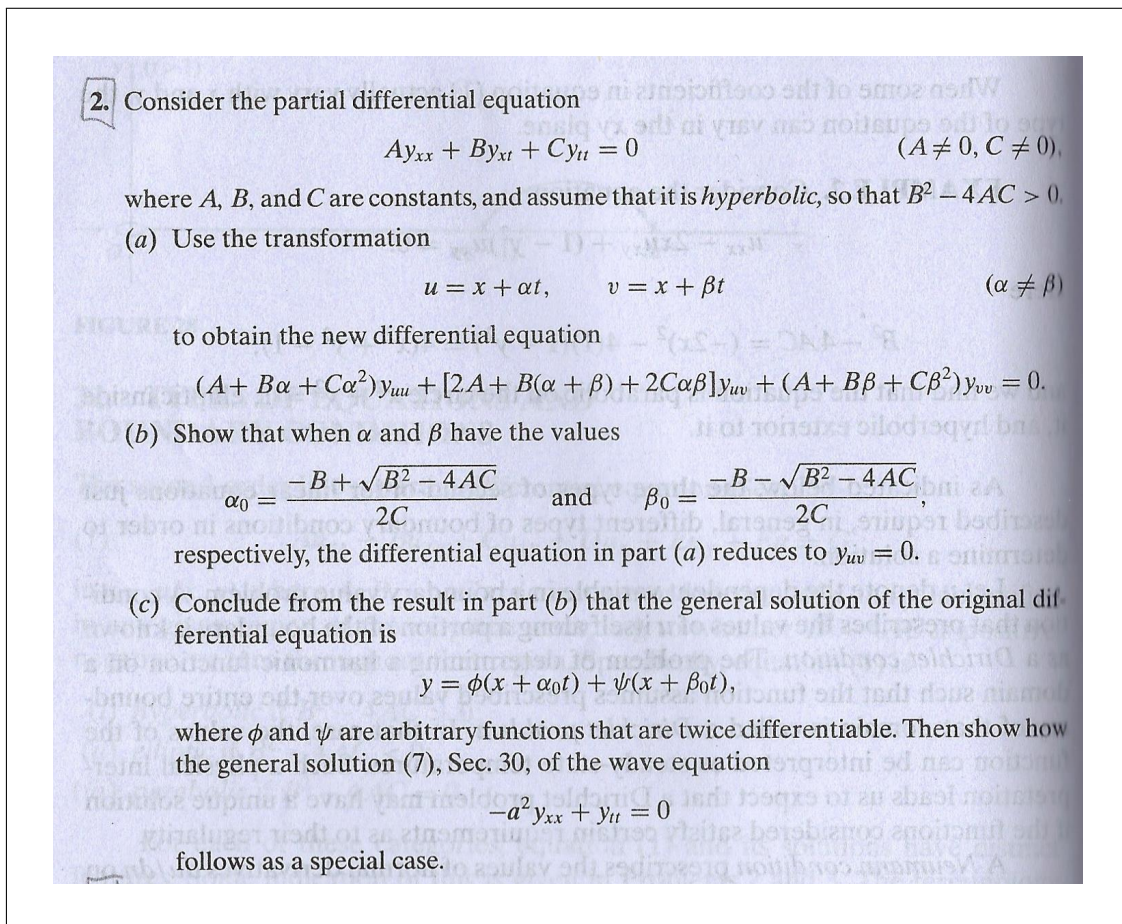


Figure 8: Problem Statement

6.1 Part a

We want to do the transformation from $y(x, t)$ to $y(u, v)$. Therefore

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial y}{\partial v} \frac{\partial v}{\partial x}$$

But $\frac{\partial u}{\partial x} = 1$ and $\frac{\partial v}{\partial x} = 1$, hence the above becomes

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}$$

And

$$\begin{aligned} \frac{\partial^2 y}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{\partial y}{\partial x} \right) \\ &= \frac{\partial}{\partial x} \left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v} \right) \\ &= \frac{\partial}{\partial x} \frac{\partial y}{\partial u} + \frac{\partial}{\partial x} \frac{\partial y}{\partial v} \\ &= \left(\frac{\partial^2 y}{\partial u^2} \frac{\partial u}{\partial x} + \frac{\partial^2 y}{\partial uv} \frac{\partial v}{\partial x} \right) + \left(\frac{\partial^2 y}{\partial v^2} \frac{\partial v}{\partial x} + \frac{\partial^2 y}{\partial vu} \frac{\partial u}{\partial x} \right) \end{aligned}$$

But $\frac{\partial u}{\partial x} = 1$, $\frac{\partial v}{\partial x} = 1$, hence the above becomes

$$\begin{aligned} \frac{\partial^2 y}{\partial x^2} &= \frac{\partial^2 y}{\partial u^2} + 2 \frac{\partial^2 y}{\partial uv} + \frac{\partial^2 y}{\partial v^2} \\ y_{xx} &= y_{uu} + y_{vv} + 2y_{uv} \end{aligned} \tag{1}$$

Similarly,

$$\frac{\partial y}{\partial t} = \frac{\partial y}{\partial u} \frac{\partial u}{\partial t} + \frac{\partial y}{\partial v} \frac{\partial v}{\partial t}$$

But $\frac{\partial u}{\partial t} = \alpha$ and $\frac{\partial v}{\partial t} = \beta$, hence the above becomes

$$\frac{\partial y}{\partial t} = \alpha \frac{\partial y}{\partial u} + \beta \frac{\partial y}{\partial v}$$

And

$$\begin{aligned} \frac{\partial^2 y}{\partial t^2} &= \frac{\partial}{\partial t} \left(\frac{\partial y}{\partial t} \right) \\ &= \frac{\partial}{\partial t} \left(\alpha \frac{\partial y}{\partial u} + \beta \frac{\partial y}{\partial v} \right) \\ &= \alpha \frac{\partial}{\partial t} \left(\frac{\partial y}{\partial u} \right) + \beta \frac{\partial}{\partial t} \left(\frac{\partial y}{\partial v} \right) \\ &= \alpha \left(\frac{\partial^2 y}{\partial u^2} \frac{\partial u}{\partial t} + \frac{\partial^2 y}{\partial uv} \frac{\partial v}{\partial t} \right) + \beta \left(\frac{\partial^2 y}{\partial v^2} \frac{\partial v}{\partial t} + \frac{\partial^2 y}{\partial uv} \frac{\partial u}{\partial t} \right) \end{aligned}$$

But $\frac{\partial u}{\partial t} = \alpha$ and $\frac{\partial v}{\partial t} = \beta$, hence the above becomes

$$\begin{aligned} \frac{\partial^2 y}{\partial t^2} &= \alpha \left(\alpha \frac{\partial^2 y}{\partial u^2} + \beta \frac{\partial^2 y}{\partial uv} \right) + \beta \left(\beta \frac{\partial^2 y}{\partial v^2} + \alpha \frac{\partial^2 y}{\partial uv} \right) \\ &= \alpha^2 \frac{\partial^2 y}{\partial u^2} + \alpha\beta \frac{\partial^2 y}{\partial uv} + \beta^2 \frac{\partial^2 y}{\partial v^2} + \alpha\beta \frac{\partial^2 y}{\partial uv} \\ y_{tt} &= \alpha^2 y_{uu} + \beta^2 y_{vv} + 2\alpha\beta y_{uv} \end{aligned} \tag{2}$$

And to obtain y_{xt} , then starting from above result obtained

$$\frac{\partial y}{\partial t} = \alpha \frac{\partial y}{\partial u} + \beta \frac{\partial y}{\partial v}$$

Now taking partial derivative w.r.t. x gives

$$\begin{aligned} \frac{\partial}{\partial x} \left(\frac{\partial y}{\partial t} \right) &= \frac{\partial}{\partial x} \left(\alpha \frac{\partial y}{\partial u} + \beta \frac{\partial y}{\partial v} \right) \\ &= \alpha \frac{\partial}{\partial x} \left(\frac{\partial y}{\partial u} \right) + \beta \frac{\partial}{\partial x} \left(\frac{\partial y}{\partial v} \right) \\ &= \alpha \left(\frac{\partial^2 y}{\partial u^2} \frac{\partial u}{\partial x} + \frac{\partial^2 y}{\partial uv} \frac{\partial v}{\partial x} \right) + \beta \left(\frac{\partial^2 y}{\partial v^2} \frac{\partial v}{\partial x} + \frac{\partial^2 y}{\partial uv} \frac{\partial u}{\partial x} \right) \end{aligned}$$

But $\frac{\partial u}{\partial x} = 1$, $\frac{\partial v}{\partial x} = 1$, hence the above becomes

$$\begin{aligned} \frac{\partial}{\partial x} \left(\frac{\partial y}{\partial t} \right) &= \alpha \left(\frac{\partial^2 y}{\partial u^2} + \frac{\partial^2 y}{\partial uv} \right) + \beta \left(\frac{\partial^2 y}{\partial v^2} + \frac{\partial^2 y}{\partial uv} \right) \\ y_{xt} &= \alpha y_{uu} + (\alpha + \beta) y_{vu} + \beta y_{vv} \end{aligned} \tag{3}$$

Substituting (1,2,3) into $Ay_{xx} + By_{xt} + Cy_{tt} = 0$ results in

$$A(y_{uu} + y_{vv} + 2y_{uv}) + B(\alpha y_{uu} + (\alpha + \beta) y_{vu} + \beta y_{vv}) + C(\alpha^2 y_{uu} + \beta^2 y_{vv} + 2\alpha\beta y_{uv}) = 0$$

Or

$$y_{uu}(A + B\alpha + C\alpha^2) + y_{uv}(2A + B(\alpha + \beta) + 2C\alpha\beta) + y_{vv}(A + B\beta + C\beta^2) = 0$$

6.2 Part b

Looking at the term above for y_{uu} we see it is $A + B\alpha + C\alpha^2$ which has the root

$$\begin{aligned} \alpha &= -\frac{b}{2a} \pm \frac{1}{2a} \sqrt{b^2 - 4ac} \\ &= -\frac{B}{2C} \pm \frac{1}{2C} \sqrt{B^2 - 4AC} \end{aligned}$$

Hence if we pick the root $\alpha = \alpha_0 = -\frac{B}{2C} + \frac{1}{2C}\sqrt{B^2 - 4AC}$ then the term y_{uu} vanishes. Similarly for the term multiplied by y_{vv} which is $A + B\beta + C\beta^2$. The root is

$$\beta = -\frac{B}{2C} \pm \frac{1}{2C}\sqrt{B^2 - 4AC}$$

And if we pick $\beta = \beta_0 = -\frac{B}{2C} - \frac{1}{2C}\sqrt{B^2 - 4AC}$ then the term y_{vv} vanishes also in the PDE obtained in part (a), and now the PDE becomes

$$y_{uv} (2A + B(\alpha + \beta) + 2C\alpha\beta) = 0$$

Substituting the above selected roots α_0, β_0 into the above in place of α, β since these are the values we picked, then the above becomes

$$y_{uv} \left(2A + B \left(-\frac{B}{2C} + \frac{1}{2C}\sqrt{B^2 - 4AC} - \frac{B}{2C} - \frac{1}{2C}\sqrt{B^2 - 4AC} \right) + 2C\alpha\beta \right) = 0$$

$$y_{uv} \left(2A - \frac{2B^2}{2C} + 2C\alpha\beta \right) = 0$$

And again replacing $\alpha\beta$ above with α_0, β_0 results in

$$y_{uv} \left(2A - \frac{2B^2}{2C} + 2C \left(-\frac{B}{2C} + \frac{1}{2C}\sqrt{B^2 - 4AC} \right) \left(-\frac{B}{2C} - \frac{1}{2C}\sqrt{B^2 - 4AC} \right) \right) = 0$$

$$y_{uv} \left(2A - \frac{2B^2}{2C} + 2C \left(\frac{B^2}{4C^2} + \frac{1}{4C^2} (B^2 - 4AC) \right) \right) = 0$$

$$y_{uv} \left(2A - \frac{2B^2}{2C} + \frac{B^2}{2C} + \frac{1}{2C} (B^2 - 4AC) \right) = 0$$

$$y_{uv} \left(2A - \frac{2B^2}{2C} + \frac{B^2}{2C} + \frac{B^2}{2C} - 2A \right) = 0$$

$$\frac{B^2}{2C} y_{uv} = 0$$

Since $B \neq 0, C \neq 0$ then the above simplifies to

$$y_{uv} = 0$$

6.3 Part c

Since

$$y_{uv} = 0$$

Or

$$\frac{\partial}{\partial v} \left(\frac{\partial y}{\partial u} \right) = 0$$

This implies that

$$\frac{\partial y}{\partial u} = \Phi(u)$$

Integrating w.r.t. u gives

$$y(u, v) = \int \Phi(u) du + \psi(v)$$

Where $\psi(v)$ is the constant of integration which is a function.

Let $\int \Phi(u) du = \phi(u)$ then the above can be written as

$$y(u, v) = \phi(u) + \psi(v)$$

Or in terms of x, t , since $u = x + at$ and $v = x + \beta t$ the above solution becomes

$$y(x, t) = \phi(x + at) + \psi(x + \beta t)$$

Where ϕ, ψ are arbitrary functions twice differentiable. When $\alpha = +a, \beta = -a$, then the above becomes

$$y(x, t) = \phi(x + at) + \psi(x - at)$$

Which is the general solution (7) in section (30). QED

7 Section 31, Problem 3

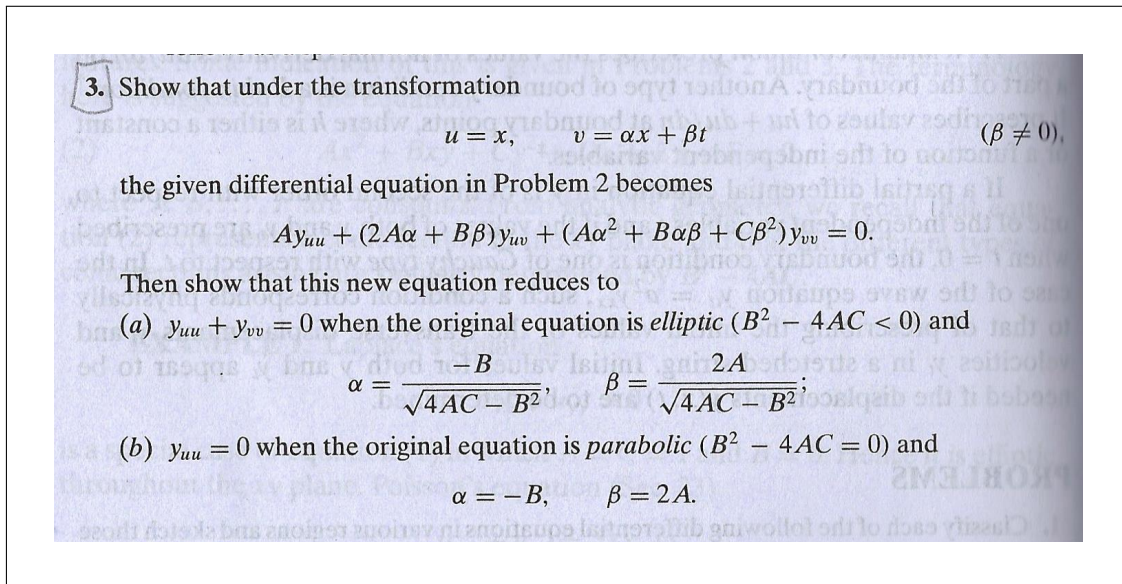


Figure 9: Problem Statement

The differential equation in problem 2 is

$$Ay_{xx} + By_{xt} + Cy_{tt} = 0$$

We want to do the transformation from $y(x, t)$ to $y(u, v)$ with

$$\begin{aligned} u &= x \\ v &= \alpha x + \beta t \end{aligned}$$

Now

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial y}{\partial v} \frac{\partial v}{\partial x}$$

But $\frac{\partial u}{\partial x} = 1$ and $\frac{\partial v}{\partial x} = \alpha$, hence the above becomes

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial u} + \alpha \frac{\partial y}{\partial v}$$

And

$$\frac{\partial y}{\partial t} = \frac{\partial y}{\partial u} \frac{\partial u}{\partial t} + \frac{\partial y}{\partial v} \frac{\partial v}{\partial t}$$

But $\frac{\partial u}{\partial t} = 0$ and $\frac{\partial v}{\partial t} = \beta$, hence the above becomes

$$\frac{\partial y}{\partial t} = \beta \frac{\partial y}{\partial v}$$

Therefore

$$\begin{aligned} \frac{\partial^2 y}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{\partial y}{\partial x} \right) \\ &= \frac{\partial}{\partial x} \left(\frac{\partial y}{\partial u} + \alpha \frac{\partial y}{\partial v} \right) \\ &= \frac{\partial}{\partial x} \left(\frac{\partial y}{\partial u} \right) + \alpha \frac{\partial}{\partial x} \left(\frac{\partial y}{\partial v} \right) \\ &= \left(\frac{\partial^2 y}{\partial u^2} \frac{\partial u}{\partial x} + \frac{\partial^2 y}{\partial uv} \frac{\partial v}{\partial x} \right) + \alpha \left(\frac{\partial^2 y}{\partial v^2} \frac{\partial v}{\partial x} + \frac{\partial^2 y}{\partial vu} \frac{\partial u}{\partial x} \right) \\ &= \left(\frac{\partial^2 y}{\partial u^2} + \alpha \frac{\partial^2 y}{\partial uv} \right) + \alpha \left(\alpha \frac{\partial^2 y}{\partial v^2} + \frac{\partial^2 y}{\partial vu} \right) \\ &= \frac{\partial^2 y}{\partial u^2} + \alpha \frac{\partial^2 y}{\partial uv} + \alpha^2 \frac{\partial^2 y}{\partial v^2} + \alpha \frac{\partial^2 y}{\partial vu} \\ y_{xx} &= y_{uu} + \alpha^2 y_{vv} + 2\alpha y_{uv} \end{aligned} \tag{1}$$

Similarly,

$$\begin{aligned}
\frac{\partial^2 y}{\partial t^2} &= \frac{\partial}{\partial x} \left(\frac{\partial y}{\partial t} \right) \\
&= \frac{\partial}{\partial x} \left(\beta \frac{\partial y}{\partial v} \right) \\
&= \beta \left(\frac{\partial^2 y}{\partial v^2} \frac{\partial v}{\partial t} + \frac{\partial^2 y}{\partial v u} \frac{\partial u}{\partial t} \right) \\
&= \beta \left(\beta \frac{\partial^2 y}{\partial v^2} \right) \\
y_{tt} &= \beta^2 y_{vv} \tag{2}
\end{aligned}$$

And to obtain y_{xt} , then starting from above result obtained

$$\frac{\partial y}{\partial t} = \beta \frac{\partial y}{\partial v}$$

Now taking partial derivative w.r.t. x gives

$$\begin{aligned}
\frac{\partial}{\partial x} \left(\frac{\partial y}{\partial t} \right) &= \frac{\partial}{\partial x} \left(\beta \frac{\partial y}{\partial v} \right) \\
&= \beta \left(\frac{\partial^2 y}{\partial v^2} \frac{\partial v}{\partial x} + \frac{\partial^2 y}{\partial v u} \frac{\partial u}{\partial x} \right) \\
&= \beta \left(\alpha \frac{\partial^2 y}{\partial v^2} + \frac{\partial^2 y}{\partial v u} \right) \\
y_{xt} &= \alpha \beta y_{vv} + \beta y_{vu} \tag{3}
\end{aligned}$$

Substituting (1,2,3) into $Ay_{xx} + By_{xt} + Cy_{tt} = 0$ results in

$$A(y_{uu} + \alpha^2 y_{vv} + 2\alpha y_{uv}) + B(\alpha \beta y_{vv} + \beta y_{vu}) + C(\beta^2 y_{vv}) = 0$$

Or

$$Ay_{uu} + y_{uv}(2A\alpha + B\beta) + y_{vv}(A\alpha^2 + B\alpha\beta + C\beta^2) = 0 \tag{4}$$

Which is what asked to show.

7.1 Part a

Setting $\alpha = \frac{-B}{\sqrt{4AC-B^2}}, \beta = \frac{2A}{\sqrt{4AC-B^2}}$ in (4) above results in

$$\begin{aligned}
Ay_{uu} + y_{uv} \left(2A \left(\frac{-B}{\sqrt{4AC-B^2}} \right) + B \left(\frac{2A}{\sqrt{4AC-B^2}} \right) \right) + y_{vv} (A\alpha^2 + B\alpha\beta + C\beta^2) &= 0 \\
Ay_{uu} + y_{vv} (A\alpha^2 + B\alpha\beta + C\beta^2) &= 0
\end{aligned}$$

And the above now becomes

$$\begin{aligned}
Ay_{uu} + y_{vv} \left(A \left(\frac{-B}{\sqrt{4AC-B^2}} \right)^2 + B \left(\frac{-B}{\sqrt{4AC-B^2}} \right) \left(\frac{2A}{\sqrt{4AC-B^2}} \right) + C \left(\frac{2A}{\sqrt{4AC-B^2}} \right)^2 \right) &= 0 \\
Ay_{uu} + y_{vv} \left(\frac{AB^2}{4AC-B^2} - \frac{2B^2A}{4AC-B^2} + \frac{4CA^2}{4AC-B^2} \right) &= 0 \\
Ay_{uu} + y_{vv} \left(\frac{AB^2 - 2B^2A + 4CA^2}{4AC-B^2} \right) &= 0 \\
Ay_{uu} + Ay_{vv} \left(\frac{-B^2 + 4CA}{4AC-B^2} \right) &= 0 \\
Ay_{uu} + Ay_{vv} &= 0 \\
A(y_{uu} + y_{vv}) &= 0
\end{aligned}$$

Therefore, since $A \neq 0$ the above becomes

$$y_{uu} + y_{vv} = 0$$

7.2 Part b

Setting $\alpha = -B, \beta = 2A$ in (4) above results in

$$Ay_{uu} + y_{uv}(-2AB + 2AB) + y_{vv}(AB^2 - 2B^2A + 4CA^2) = 0$$

$$Ay_{uu} + y_{vv}(4CA^2 - B^2A) = 0$$

$$Ay_{uu} - Ay_{vv}(B^2 - 4CA) = 0$$

But $B^2 - 4CA = 0$, therefore the above becomes

$$y_{uu} = 0$$