HW 4
MATH 4567 Applied Fourier Analysis
Spring 2019
University of Minnesota, Twin Cities

Nasser M. Abbasi

November 2, 2019 Compiled on November 2, 2019 at 9:46pm [public]

Contents

1 Section 27, Problem 8

8. Suppose that temperatures u in a solid hemisphere $r \leq 1, 0 \leq \theta \leq \pi/2$ are independent of the spherical coordinate ϕ, so that $u = u(r, \theta)$, and that the base of the hemisphere is insulated (Fig. 23). Use transformation (13), Sec. 25, which relates spherical and cylindrical coordinates, to show that

$$\frac{\partial u}{\partial \theta} = -\rho \frac{\partial u}{\partial z} + z \frac{\partial u}{\partial \rho}.$$

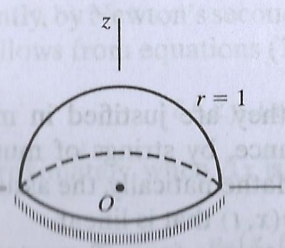


FIGURE 23

Thus show that u must satisfy the boundary condition

$$u_\theta\left(r, \frac{\pi}{2}\right) = 0.$$

Figure 1: Problem statement

<u>Solution</u>

The cylindrical and spherical coordinates are defined as given in the textbook figures shown below

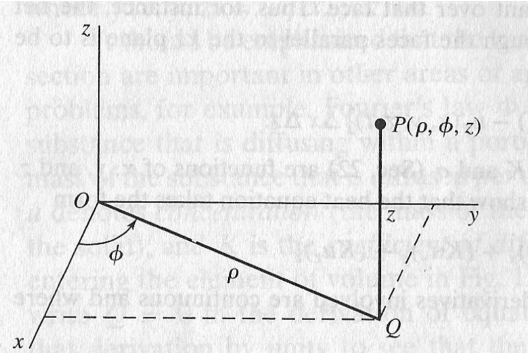


FIGURE 16

Figure 2: Cylinderical coordinates

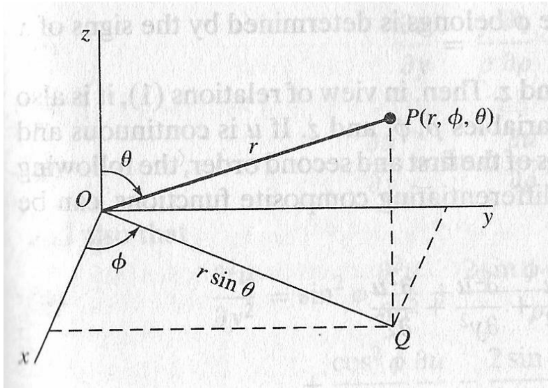


FIGURE 17

Figure 3: Spherical coordinates

The relation between these is given by (13) in the book

$$z = r\cos\theta \tag{1}$$
$$\rho = r\sin\theta \tag{2}$$
$$\phi = \phi \tag{3}$$

To obtain the required formula, we will use the chain rule. Since in spherical we have $u \equiv u(r,\theta)$ and in cylindrical we have $u \equiv u(\rho, z)$, then by chain rule

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial \rho}\frac{\partial \rho}{\partial \theta} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial \theta}$$

But from (2) $\frac{\partial \rho}{\partial \theta} = r\cos\theta$ and from (1) $\frac{\partial z}{\partial \theta} = -r\sin\theta$, hence the above becomes

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial \rho}(r\cos\theta) + \frac{\partial u}{\partial z}(-r\sin\theta)$$

But $r\cos\theta = z$ and $-r\sin\theta = \rho$, hence the above simplifies to

$$\frac{\partial u}{\partial \theta} = z\frac{\partial u}{\partial \rho} - \rho\frac{\partial u}{\partial z} \tag{4}$$

Which is the result required to show. Now we need to show that $\frac{\partial u}{\partial \theta}$ evaluated at boundary $r = 1, \theta = \frac{\pi}{2}$ is zero. But $\theta = \frac{\pi}{2}$ implies that $z = 0$, since $z = r\cos\theta$. Hence (4) now reduces to

$$\frac{\partial u}{\partial \theta} = -\rho\frac{\partial u}{\partial z} \tag{4}$$

Since $\theta = \frac{\pi}{2}$, then $\frac{\partial u}{\partial z}$ is the directional derivative normal to the base surface. But we are told it is insulated. This implies that $\frac{\partial u}{\partial z} = 0$, since by definition this is what insulated means. Therefore $\frac{\partial u}{\partial \theta} = 0$ at $r = 1, \theta = \frac{\pi}{2}$, which is what we are asked to show.

2 Section 28, Problem 1

1. A stretched string, with its ends fixed at the points 0 and $2c$ on the x axis, hangs at rest under its own weight. The y axis is directed vertically upward. Point out how it follows from the nonhomogeneous wave equation (6), Sec. 28, that the static displacements $y(x)$ of points on the string must satisfy the differential equation

$$a^2 y''(x) = g \qquad \left(a^2 = \frac{H}{\delta}\right)$$

on the interval $0 < x < 2c$, in addition to the boundary conditions

$$y(0) = 0, \qquad y(2c) = 0.$$

By solving this boundary value problem, show that the string hangs in the parabolic arc

$$(x-c)^2 = \frac{2a^2}{g}\left(y + \frac{gc^2}{2a^2}\right) \qquad (0 \le x \le 2c)$$

and that the depth of the vertex of the arc varies directly with c^2 and δ and inversely with H.

Figure 4: Problem statement

Eq (6) in section 28 is

$$y_{tt}(x,t) = a^2 y_{xx}(x,t) - g$$

At static displacement, by definition, there is no time dependency, hence $y_{tt} = 0$ and the above becomes

$$0 = a^2 y_{xx}(x,t) - g$$

Therefore now this becomes an ODE instead of a PDE since it does not depend on time, and we can write the above as

$$a^2 y''(x) = g \tag{1}$$

The boundary conditions $y(0,t) = 0$ and $y(2x,t) = 0$ now become $y(0) = 0, y(2x) = 0$. Now we need to solve (1) with these boundary conditions. This is an boundary value ODE.

$$y''(x) = \frac{g}{a^2}$$

The RHS is constant. The solution to the homogeneous ODE $y'' = 0$ is $y_h = Ax + B$. Let the particular solution be $y_p = C_3 x^2$, then $y_p' = 2C_3 x$ and $y_p'' = 2C_3$. Substituting this in the above ODE gives

$$2C_3 = \frac{g}{a^2}$$
$$C_3 = \frac{g}{2a^2}$$

Hence $y_p(x) = \frac{g}{2a^2}x^2$. Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= Ax + B + \frac{g}{2a^2}x^2 \end{aligned} \tag{2}$$

Now we will use the boundary conditions to find A, B above. At $x = 0$, (2) becomes

$$0 = B$$

Hence solution (2) reduces to

$$y(x) = Ax + \frac{g}{2a^2}x^2 \tag{3}$$

At $x = 2c$, the second boundary condition gives

$$\begin{aligned} 0 &= 2cA + \frac{g}{2a^2}\left(4c^2\right) \\ A &= \frac{-g}{2a^2}\frac{\left(4c^2\right)}{2c} \\ &= \frac{-gc}{a^2} \end{aligned}$$

Hence the solution (3) becomes

$$\begin{aligned} y &= \frac{-gc}{a^2}x + \frac{g}{2a^2}x^2 \\ y &= \frac{gx^2 - 2gcx}{2a^2} \end{aligned} \tag{4}$$

To get the result needed, we can manipulate this more as follows. From (4)

$$\begin{aligned} 2a^2y &= gx^2 - 2gcx \\ &= g\left(x^2 - 2cx\right) \\ &= g(x-c)^2 - gc^2 \end{aligned}$$

Hence

$$\begin{aligned} g(x-c)^2 &= 2a^2y + gc^2 \\ (x-c)^2 &= \frac{2a^2y}{g} + c^2 \\ &= \frac{2a^2}{g}\left(y + \frac{gc^2}{2a^2}\right) \end{aligned}$$

Now since $a^2 = \frac{H}{\delta}$ then the above becomes

$$\begin{aligned}\frac{g}{2a^2}(x-c)^2 &= y + \frac{gc^2}{2a^2}\\ y &= \frac{1}{2a^2}\left(g(x-c)^2 - gc^2\right)\\ &= \frac{g}{2\frac{H}{\delta}}\left((x-c)^2 - c^2\right)\\ &= \frac{\delta}{H}\frac{g}{2}\left((x-c)^2 - c^2\right)\end{aligned}$$

We see now that y is directly proportional to δ and c^2 and inversely proportional to H.

3 Section 28, Problem 5

5. A strand of wire 1 ft long, stretched between the origin and the point 1 on the x axis, weighs 0.032 lb ($\delta g = 0.032, g = 32$ ft/s^2) and $H = 10$ lb. At the instant $t = 0$, the strand lies along the x axis but has a velocity of 1 ft/s in the direction of the y axis, perhaps because the supports were in motion and were brought to rest at that instant. Assuming that no external forces act along the wire, state why the displacements $y(x,t)$ should satisfy this boundary value problem:

$$y_{tt}(x,t) = 10^4 y_{xx}(x,t) \qquad (0 < x < 1, t > 0),$$
$$y(0,t) = y(1,t) = 0, \qquad y(x,0) = 0, \qquad y_t(x,0) = 1.$$

Figure 5: Problem statement

<u>solution</u>

The wave PDE in 1D is given by

$$y_{tt}(x,t) = a^2 y_{xx}(x,t) \tag{1}$$

Where

$$a^2 = \frac{H}{\delta}$$

Where H is the tension in the strand and δ is the mass per unit length of the strand. But $weight = (mass)\, g$. hence $\delta = \frac{weight}{g}$. We are given that $weight = 0.032$ lb, and that $g = 32$ ft/s^2. This implies that

$$\delta = \frac{0.032}{32} = \frac{1}{1000}$$

Hence

$$a^2 = \frac{10}{\frac{1}{1000}} = 10^4$$

Therefore (1) becomes

$$y_{tt}(x,t) = 10^4 y_{xx}(x,t) \tag{2}$$

Since at $t = 0$ we are told that strand lies along the $x - axis$, then $y(x,0) = 0$ and problem says $y_t(x,0) = 1$. For boundary conditions, since strand fixed at $x = 0$ and $x = 1$, then this implies $y(0,t) = 0$ and $y(1,t) = 0$. Therefore the PDE is

$$\begin{aligned} y_{tt}(x,t) &= 10^4 y_{xx}(x,t) \qquad 0 < x < 1, t > 0 \\ y(x,0) &= 0 \\ y_t(x,0) &= 1 \\ y(0,t) &= 0 \\ y(1,t) &= 0 \end{aligned}$$

4 Section 30, Problem 3

3. Let $y(x,t)$ represent transverse displacements in a long stretched string one end of which is attached to a ring that can slide along the y axis. The other end is so far out on the positive x axis that it may be considered to be infinitely far from the origin. The ring is initially at the origin and is then moved along the y axis (Fig. 27) so that $y = f(t)$ when $x = 0$ and $t \geq 0$, where f is a prescribed continuous function and $f(0) = 0$. We assume that the string is initially at rest on the x axis; thus $y(x,t) \to 0$ as $x \to \infty$. The boundary value problem for $y(x,t)$ is

$$y_{tt}(x,t) = a^2 y_{xx}(x,t) \qquad (x > 0,\ t > 0),$$

$$y(x,0) = 0, \qquad y_t(x,0) = 0 \qquad (x \geq 0),$$

$$y(0,t) = f(t) \qquad (t \geq 0).$$

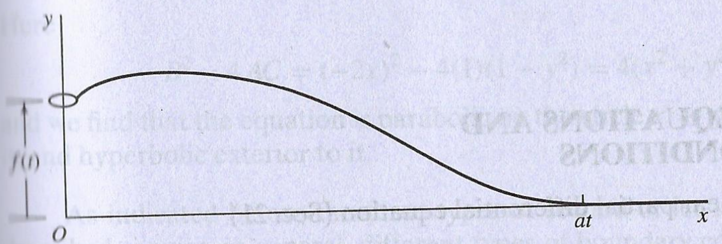


FIGURE 27

(*a*) Apply the first two of these boundary conditions to the general solution (Sec. 30)

$$y(x,\ t) = \phi(x + at) + \psi(x - at)$$

of the one-dimensional wave equation to show that there is a constant C such that

$$\phi(x) = C \qquad \text{and} \qquad \psi(x) = -C \qquad (x \geq 0).$$

Then apply the third boundary condition $y(0,t) = f(t)$ to show that

$$\psi(-x) = f\left(\frac{x}{a}\right) - C \qquad (x \geq 0),$$

where C is the same constant.

(*b*) With the aid of the results in part (*a*), derive the solution

$$y(x,t) = \begin{cases} 0 & \text{when } x \geq at, \\ f\left(t - \dfrac{x}{a}\right) & \text{when } x < at. \end{cases}$$

Note that the part of the string to the right of the point $x = at$ on the x axis is unaffected by the movement of the ring prior to time t, as shown in Fig. 27.

Figure 6: Problem statement

4.1 Part a

Applying the first initial conditions $y(x,0)=0$ to the solution

$$y(x,t)=\phi(x+at)+\psi(x-at) \tag{1}$$

Gives

$$0=\phi(x)+\psi(x) \tag{2}$$

But $y_t=a\phi'-a\psi'$. Hence the second initial conditions at $t=0$ gives

$$0=a\phi'(x)-a\psi'(x) \tag{3}$$

Taking derivative of (2) and multiplying the resulting equation by a gives

$$0=a\phi'(x)+a\psi'(x) \tag{2A}$$

Adding (3,2A) gives

$$\begin{aligned} 2a\phi'(x) &= 0 \\ \phi'(x) &= 0 \end{aligned}$$

Therefore

$$\phi(x)=C \tag{4}$$

Where C is an arbitrary constant. Substituting the above result back in (2) gives

$$\begin{aligned} 0 &= C+\psi(x) \\ \psi(x) &= -C \end{aligned} \tag{5}$$

From (4,5) we see that

$$\begin{aligned} \phi(x) &= C \\ \psi(x) &= -C \end{aligned}$$

Now applying boundary condition $y(0,t)=f(t)$ to (1) gives

$$f(t)=\phi(at)+\psi(-at)$$

But a is the speed of the wave given by $a=\frac{x}{t}$ or $t=\frac{x}{a}$. Hence the above becomes

$$\begin{aligned} f\left(\frac{x}{a}\right) &= \phi(x)+\psi(-x) \\ \psi(-x) &= f\left(\frac{x}{a}\right)-\phi(x) \end{aligned}$$

Since $\phi(x)=C$ from equation (4), then the final result is obtained

$$\psi(-x)=f\left(\frac{x}{a}\right)-C \qquad x\geq 0 \tag{6}$$

4.2 Part b

Since the part to the right of $x=at$ is unaffected by the movement of the right, then

$$y(x,t)=0 \qquad x\geq at \tag{1}$$

So now we need to find the solution for $x < at$ and $x \geq 0$. From

$$y(x,t) = \phi(x+at) + \psi(x-at)$$

And using (6) in part (a), we see that $\psi(x-at) = f\left(\frac{-(x-at)}{a}\right) - C$. Therefore the above becomes

$$y(x,t) = \phi(x+at) + f\left(\frac{-(x-at)}{a}\right) - C$$

But also from part (a) $\phi(x+at) = C$. Hence the above simplifies to

$$\begin{aligned} y(x,t) &= c + f\left(\frac{-(x-at)}{a}\right) - C \\ &= f\left(\frac{-x+at}{a}\right) \\ &= f\left(t - \frac{x}{a}\right) \qquad x < at \end{aligned} \tag{2}$$

Combining (1) and (2) shows that

$$y(x,t) = \begin{cases} 0 & x \geq at \\ f\left(t - \frac{x}{a}\right) & x < at \end{cases}$$

5 Section 30, Problem 4

4. Use the solution obtained in Problem 3 to show that if the ring at the left-hand end of the string in that problem is moved according to the function

$$f(t) = \begin{cases} \sin \pi t & \text{when } 0 \le t \le 1, \\ 0 & \text{when } \quad t > 1, \end{cases}$$

then

$$y(x,t) = \begin{cases} 0 & \text{when } x \le a(t-1) \text{ or } x \ge at, \\ \sin\left[\pi\left(t - \frac{x}{a}\right)\right] & \text{when } a(t-1) < x < at. \end{cases}$$

Observe that the ring is lifted up 1 unit and then returned to the origin, where it remains after time $t = 1$. The expression for $y(x,t)$ here shows that when $t > 1$, the string coincides with the x axis except on an interval of length a, where it forms one arch of a sine curve (Fig. 28). Furthermore, as t increases, the arch moves to the right with speed a.

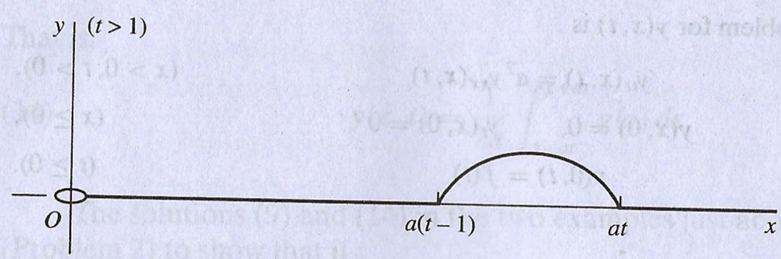


FIGURE 28

Figure 7: Problem statement

This requires just substitution of the function $f(t)$ given into the solution found above which is

$$y(x,t) = \begin{cases} 0 & x \ge at \\ f\left(t - \frac{x}{a}\right) & x < at \end{cases} \tag{1}$$

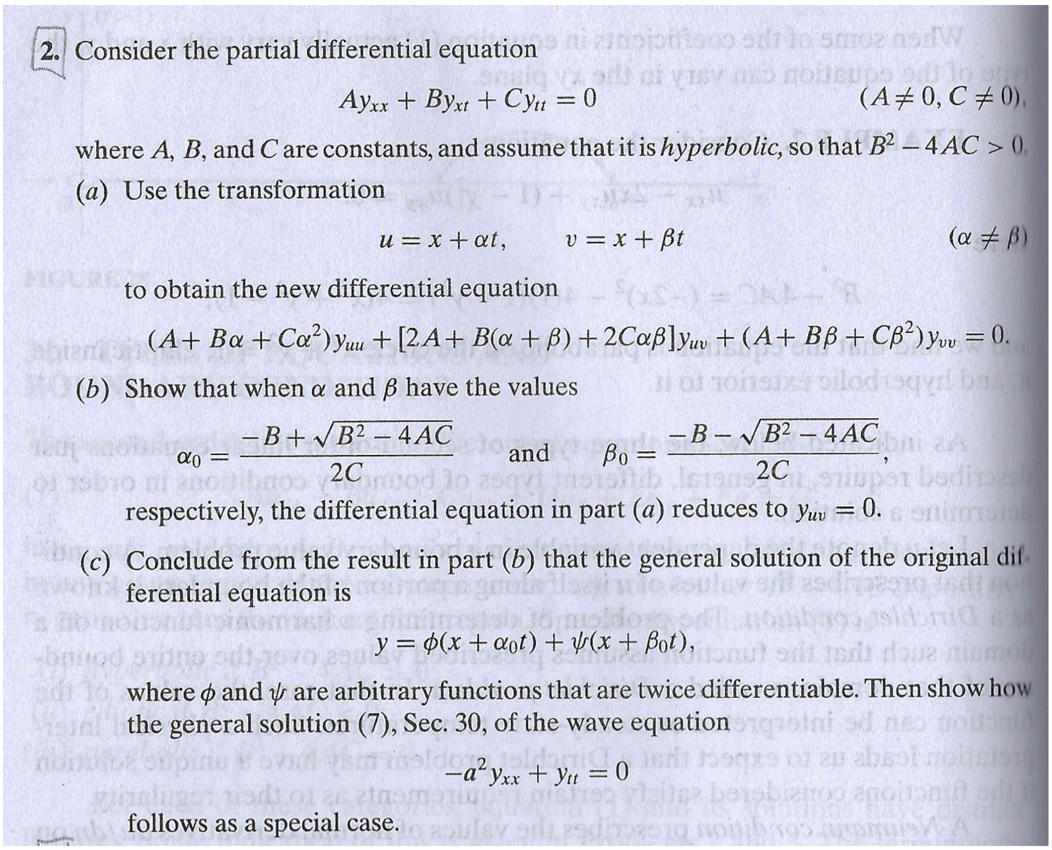
But

$$f(t) = \begin{cases} \sin \pi t & 0 \le t \le 1 \\ 0 & t > 1 \end{cases} \tag{2}$$

Substituting (2) into (1) gives, after replacing each t in (2) by $t - \frac{x}{a}$ the result needed

$$y(x,t) = \begin{cases} 0 & x \ge at \\ \sin\left(\pi\left(t - \frac{x}{a}\right)\right) & a(t-1) < x < at \end{cases}$$

6 Section 31, Problem 2



2. Consider the partial differential equation

$$Ay_{xx} + By_{xt} + Cy_{tt} = 0 \qquad (A \neq 0, C \neq 0),$$

where A, B, and C are constants, and assume that it is *hyperbolic*, so that $B^2 - 4AC > 0$.

(*a*) Use the transformation

$$u = x + \alpha t, \qquad v = x + \beta t \qquad (\alpha \neq \beta)$$

to obtain the new differential equation

$$(A + B\alpha + C\alpha^2)y_{uu} + [2A + B(\alpha + \beta) + 2C\alpha\beta]y_{uv} + (A + B\beta + C\beta^2)y_{vv} = 0.$$

(*b*) Show that when α and β have the values

$$\alpha_0 = \frac{-B + \sqrt{B^2 - 4AC}}{2C} \quad \text{and} \quad \beta_0 = \frac{-B - \sqrt{B^2 - 4AC}}{2C},$$

respectively, the differential equation in part (*a*) reduces to $y_{uv} = 0$.

(*c*) Conclude from the result in part (*b*) that the general solution of the original differential equation is

$$y = \phi(x + \alpha_0 t) + \psi(x + \beta_0 t),$$

where ϕ and ψ are arbitrary functions that are twice differentiable. Then show how the general solution (7), Sec. 30, of the wave equation

$$-a^2 y_{xx} + y_{tt} = 0$$

follows as a special case.

Figure 8: Problem Statement

6.1 Part a

We want to do the transformation from $y(x,t)$ to $y(u,v)$. Therefore

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial x}$$

But $\frac{\partial u}{\partial x} = 1$ and $\frac{\partial v}{\partial x} = 1$, hence the above becomes

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}$$

And

$$\begin{aligned}\frac{\partial^2 y}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right)\\ &= \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\right)\\ &= \frac{\partial}{\partial x}\frac{\partial y}{\partial u} + \frac{\partial}{\partial x}\frac{\partial y}{\partial v}\\ &= \left(\frac{\partial^2 y}{\partial u^2}\frac{\partial u}{\partial x} + \frac{\partial^2 y}{\partial uv}\frac{\partial v}{\partial x}\right) + \left(\frac{\partial^2 y}{\partial v^2}\frac{\partial v}{\partial x} + \frac{\partial^2 y}{\partial vu}\frac{\partial u}{\partial x}\right)\end{aligned}$$

But $\frac{\partial u}{\partial x} = 1, \frac{\partial v}{\partial x} = 1$, hence the above becomes

$$\begin{aligned}\frac{\partial^2 y}{\partial x^2} &= \frac{\partial^2 y}{\partial u^2} + 2\frac{\partial^2 y}{\partial uv} + \frac{\partial^2 y}{\partial v^2}\\ y_{xx} &= y_{uu} + y_{vv} + 2y_{uv} \qquad (1)\end{aligned}$$

Similarly,

$$\frac{\partial y}{\partial t} = \frac{\partial y}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial t}$$

But $\frac{\partial u}{\partial t} = \alpha$ and $\frac{\partial v}{\partial t} = \beta$, hence the above becomes

$$\frac{\partial y}{\partial t} = \alpha\frac{\partial y}{\partial u} + \beta\frac{\partial y}{\partial v}$$

And

$$\begin{aligned}\frac{\partial^2 y}{\partial t^2} &= \frac{\partial}{\partial t}\left(\frac{\partial y}{\partial t}\right)\\ &= \frac{\partial}{\partial t}\left(\alpha\frac{\partial y}{\partial u} + \beta\frac{\partial y}{\partial v}\right)\\ &= \alpha\frac{\partial}{\partial t}\left(\frac{\partial y}{\partial u}\right) + \beta\frac{\partial}{\partial t}\left(\frac{\partial y}{\partial v}\right)\\ &= \alpha\left(\frac{\partial^2 y}{\partial u^2}\frac{\partial u}{\partial t} + \frac{\partial^2 y}{\partial uv}\frac{\partial v}{\partial t}\right) + \beta\left(\frac{\partial^2 y}{\partial v^2}\frac{\partial v}{\partial t} + \frac{\partial^2 y}{\partial uv}\frac{\partial u}{\partial t}\right)\end{aligned}$$

But $\frac{\partial u}{\partial t} = \alpha$ and $\frac{\partial v}{\partial t} = \beta$, hence the above becomes

$$\begin{aligned}\frac{\partial^2 y}{\partial t^2} &= \alpha\left(\alpha\frac{\partial^2 y}{\partial u^2} + \beta\frac{\partial^2 y}{\partial uv}\right) + \beta\left(\beta\frac{\partial^2 y}{\partial v^2} + \alpha\frac{\partial^2 y}{\partial uv}\right)\\ &= \alpha^2\frac{\partial^2 y}{\partial u^2} + \alpha\beta\frac{\partial^2 y}{\partial uv} + \beta^2\frac{\partial^2 y}{\partial v^2} + \alpha\beta\frac{\partial^2 y}{\partial uv}\\ y_{tt} &= \alpha^2 y_{uu} + \beta^2 y_{vv} + 2\alpha\beta y_{uv} \qquad (2)\end{aligned}$$

And to obtain y_{xt}, then starting from above result obtained

$$\frac{\partial y}{\partial t} = \alpha\frac{\partial y}{\partial u} + \beta\frac{\partial y}{\partial v}$$

Now taking partial derivative w.r.t. x gives

$$\begin{aligned}\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial t}\right) &= \frac{\partial}{\partial x}\left(\alpha\frac{\partial y}{\partial u} + \beta\frac{\partial y}{\partial v}\right) \\ &= \alpha\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial u}\right) + \beta\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial v}\right) \\ &= \alpha\left(\frac{\partial^2 y}{\partial u^2}\frac{\partial u}{\partial x} + \frac{\partial^2 y}{\partial uv}\frac{\partial v}{\partial x}\right) + \beta\left(\frac{\partial^2 y}{\partial v^2}\frac{\partial v}{\partial x} + \frac{\partial^2 y}{\partial uv}\frac{\partial u}{\partial x}\right)\end{aligned}$$

But $\frac{\partial u}{\partial x} = 1, \frac{\partial v}{\partial x} = 1$, hence the above becomes

$$\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial t}\right) = \alpha\left(\frac{\partial^2 y}{\partial u^2} + \frac{\partial^2 y}{\partial uv}\right) + \beta\left(\frac{\partial^2 y}{\partial v^2} + \frac{\partial^2 y}{\partial uv}\right)$$

$$y_{xt} = \alpha y_{uu} + \left(\alpha + \beta\right) y_{vu} + \beta y_{vv} \tag{3}$$

Substituting (1,2,3) into $Ay_{xx} + By_{xt} + Cy_{tt} = 0$ results in

$$A\left(y_{uu} + y_{vv} + 2y_{uv}\right) + B\left(\alpha y_{uu} + \left(\alpha + \beta\right) y_{vu} + \beta y_{vv}\right) + C\left(\alpha^2 y_{uu} + \beta^2 y_{vv} + 2\alpha\beta y_{uv}\right) = 0$$

Or

$$y_{uu}\left(A + B\alpha + C\alpha^2\right) + y_{uv}\left(2A + B\left(\alpha + \beta\right) + 2C\alpha\beta\right) + y_{vv}\left(A + B\beta + C\beta^2\right) = 0$$

6.2 Part b

Looking at the term above for y_{uu} we see it is $A + B\alpha + C\alpha^2$ which has the root

$$\begin{aligned}\alpha &= -\frac{b}{2a} \pm \frac{1}{2a}\sqrt{b^2 - 4ac} \\ &= -\frac{B}{2C} \pm \frac{1}{2C}\sqrt{B^2 - 4AC}\end{aligned}$$

Hence if we pick the root $\alpha = \alpha_0 = -\frac{B}{2C} + \frac{1}{2C}\sqrt{B^2 - 4AC}$ then the term y_{uu} vanishes. Similarly for the term multiplied by y_{vv} which is $A + B\beta + C\beta^2$. The root is

$$\beta = -\frac{B}{2C} \pm \frac{1}{2C}\sqrt{B^2 - 4AC}$$

And if we pick $\beta = \beta_0 = -\frac{B}{2C} - \frac{1}{2C}\sqrt{B^2 - 4AC}$ then the term y_{vv} vanishes also in the PDE obtained in part (a), and now the PDE becomes

$$y_{uv}\left(2A + B\left(\alpha + \beta\right) + 2C\alpha\beta\right) = 0$$

Substituting the above selected roots α_0, β_0 into the above in place of α, β since these are the values we picked, then the above becomes

$$y_{uv}\left(2A + B\left(-\frac{B}{2C} + \frac{1}{2C}\sqrt{B^2 - 4AC} - \frac{B}{2C} - \frac{1}{2C}\sqrt{B^2 - 4AC}\right) + 2C\alpha\beta\right) = 0$$

$$y_{uv}\left(2A - \frac{2B^2}{2C} + 2C\alpha\beta\right) = 0$$

And again replacing $\alpha\beta$ above with α_0, β_0 results in

$$y_{uv}\left(2A - \frac{2B^2}{2C} + 2C\left(-\frac{B}{2C} + \frac{1}{2C}\sqrt{B^2 - 4AC}\right)\left(-\frac{B}{2C} - \frac{1}{2C}\sqrt{B^2 - 4AC}\right)\right) = 0$$

$$y_{uv}\left(2A - \frac{2B^2}{2C} + 2C\left(\frac{B^2}{4C^2} + \frac{1}{4C^2}\left(B^2 - 4AC\right)\right)\right) = 0$$

$$y_{uv}\left(2A - \frac{2B^2}{2C} + \frac{B^2}{2C} + \frac{1}{2C}\left(B^2 - 4AC\right)\right) = 0$$

$$y_{uv}\left(2A - \frac{2B^2}{2C} + \frac{B^2}{2C} + \frac{B^2}{2C} - 2A\right) = 0$$

$$\frac{B^2}{2C} y_{uv} = 0$$

Since $B \neq 0, C \neq 0$ then the above simplifies to

$$y_{uv} = 0$$

6.3 Part c

Since

$$y_{uv} = 0$$

Or

$$\frac{\partial}{\partial v}\left(\frac{\partial y}{\partial u}\right) = 0$$

The implies that

$$\frac{\partial y}{\partial u} = \Phi(u)$$

Integrating w.r.t. u gives

$$y(u, v) = \int \Phi(u)\, du + \psi(v)$$

Where $\psi(v)$ is the constant of integration which is a function.

Let $\int \Phi(u)\, du = \phi(u)$ then the above can be written as

$$y(u, v) = \phi(u) + \psi(v)$$

Or in terms of x, t, since $u = x + \alpha t$ and $v = x + \beta t$ the above solution becomes

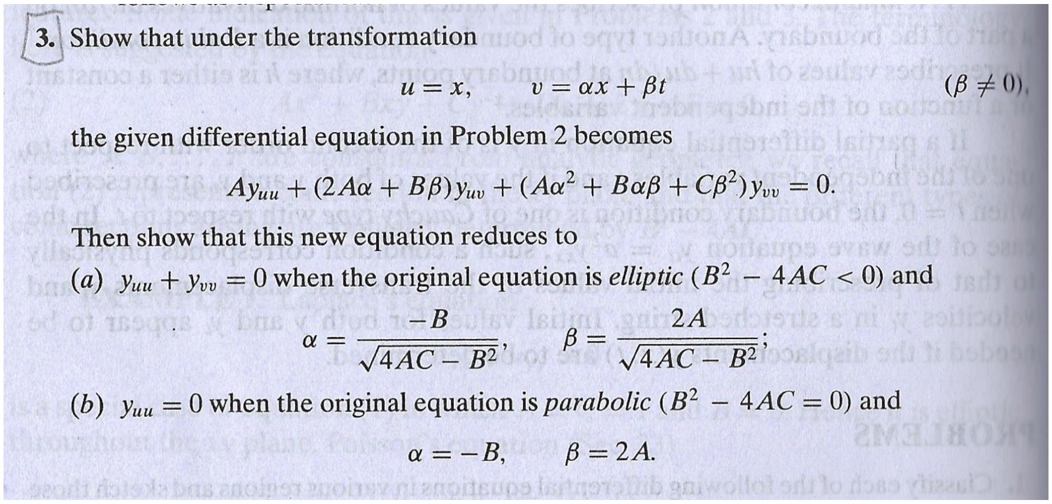
$$y(x,t) = \phi(x + \alpha t) + \psi\left(x + \beta t\right)$$

Where ϕ, ψ are arbitrary functions twice differentiable. When $\alpha = +a, \beta = -a$, then the above becomes

$$y(x,t) = \phi(x + at) + \psi(x - at)$$

Which is the general solution (7) in section (30). QED

7 Section 31, Problem 3



3. Show that under the transformation

$$u = x, \qquad v = \alpha x + \beta t \qquad (\beta \neq 0),$$

the given differential equation in Problem 2 becomes

$$Ay_{uu} + (2A\alpha + B\beta)y_{uv} + (A\alpha^2 + B\alpha\beta + C\beta^2)y_{vv} = 0.$$

Then show that this new equation reduces to

(*a*) $y_{uu} + y_{vv} = 0$ when the original equation is *elliptic* ($B^2 - 4AC < 0$) and

$$\alpha = \frac{-B}{\sqrt{4AC - B^2}}, \qquad \beta = \frac{2A}{\sqrt{4AC - B^2}};$$

(*b*) $y_{uu} = 0$ when the original equation is *parabolic* ($B^2 - 4AC = 0$) and

$$\alpha = -B, \qquad \beta = 2A.$$

Figure 9: Problem Statement

The differential equation in problem 2 is

$$Ay_{xx} + By_{xt} + Cy_{tt} = 0$$

We want to do the transformation from $y(x,t)$ to $y(u,v)$ with

$$u = x$$
$$v = \alpha x + \beta t$$

Now

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial x}$$

But $\frac{\partial u}{\partial x} = 1$ and $\frac{\partial v}{\partial x} = \alpha$, hence the above becomes

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial u} + \alpha\frac{\partial y}{\partial v}$$

And

$$\frac{\partial y}{\partial t} = \frac{\partial y}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial t}$$

But $\frac{\partial u}{\partial t} = 0$ and $\frac{\partial v}{\partial t} = \beta$, hence the above becomes

$$\frac{\partial y}{\partial t} = \beta\frac{\partial y}{\partial v}$$

Therefore

$$\begin{aligned}
\frac{\partial^2 y}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial u} + \alpha\frac{\partial y}{\partial v}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial u}\right) + \alpha\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial v}\right) \\
&= \left(\frac{\partial^2 y}{\partial u^2}\frac{\partial u}{\partial x} + \frac{\partial^2 y}{\partial uv}\frac{\partial v}{\partial x}\right) + \alpha\left(\frac{\partial^2 y}{\partial v^2}\frac{\partial v}{\partial x} + \frac{\partial^2 y}{\partial vu}\frac{\partial u}{\partial x}\right) \\
&= \left(\frac{\partial^2 y}{\partial u^2} + \alpha\frac{\partial^2 y}{\partial uv}\right) + \alpha\left(\alpha\frac{\partial^2 y}{\partial v^2} + \frac{\partial^2 y}{\partial vu}\right) \\
&= \frac{\partial^2 y}{\partial u^2} + \alpha\frac{\partial^2 y}{\partial uv} + \alpha^2\frac{\partial^2 y}{\partial v^2} + \alpha\frac{\partial^2 y}{\partial vu} \\
y_{xx} &= y_{uu} + \alpha^2 y_{vv} + 2\alpha y_{uv} \qquad (1)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{\partial^2 y}{\partial t^2} &= \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial t}\right) \\
&= \frac{\partial}{\partial x}\left(\beta\frac{\partial y}{\partial v}\right) \\
&= \beta\left(\frac{\partial^2 y}{\partial v^2}\frac{\partial v}{\partial t} + \frac{\partial^2 y}{\partial vu}\frac{\partial u}{\partial t}\right) \\
&= \beta\left(\beta\frac{\partial^2 y}{\partial v^2}\right) \\
y_{tt} &= \beta^2 y_{vv} \qquad (2)
\end{aligned}$$

And to obtain y_{xt}, then starting from above result obtained

$$\frac{\partial y}{\partial t} = \beta\frac{\partial y}{\partial v}$$

Now taking partial derivative w.r.t. x gives

$$\begin{aligned}
\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial t}\right) &= \frac{\partial}{\partial x}\left(\beta\frac{\partial y}{\partial v}\right) \\
&= \beta\left(\frac{\partial^2 y}{\partial v^2}\frac{\partial v}{\partial x} + \frac{\partial^2 y}{\partial vu}\frac{\partial u}{\partial x}\right) \\
&= \beta\left(\alpha\frac{\partial^2 y}{\partial v^2} + \frac{\partial^2 y}{\partial vu}\right) \\
y_{xt} &= \alpha\beta y_{vv} + \beta y_{vu} \qquad (3)
\end{aligned}$$

Substituting (1,2,3) into $Ay_{xx} + By_{xt} + Cy_{tt} = 0$ results in

$$A\left(y_{uu} + \alpha^2 y_{vv} + 2\alpha y_{uv}\right) + B\left(\alpha\beta y_{vv} + \beta y_{vu}\right) + C\left(\beta^2 y_{vv}\right) = 0$$

Or

$$Ay_{uu} + y_{uv}\left(2A\alpha + B\beta\right) + y_{vv}\left(A\alpha^2 + B\alpha\beta + C\beta^2\right) = 0 \tag{4}$$

Which is what asked to show.

7.1 Part a

Setting $\alpha = \frac{-B}{\sqrt{4AC-B^2}}, \beta = \frac{2A}{\sqrt{4AC-B^2}}$ in (4) above results in

$$Ay_{uu} + y_{uv}\left(2A\left(\frac{-B}{\sqrt{4AC-B^2}}\right) + B\left(\frac{2A}{\sqrt{4AC-B^2}}\right)\right) + y_{vv}\left(A\alpha^2 + B\alpha\beta + C\beta^2\right) = 0$$

$$Ay_{uu} + y_{vv}\left(A\alpha^2 + B\alpha\beta + C\beta^2\right) = 0$$

And the above now becomes

$$Ay_{uu} + y_{vv}\left(A\left(\frac{-B}{\sqrt{4AC-B^2}}\right)^2 + B\left(\frac{-B}{\sqrt{4AC-B^2}}\right)\left(\frac{2A}{\sqrt{4AC-B^2}}\right) + C\left(\frac{2A}{\sqrt{4AC-B^2}}\right)^2\right) = 0$$

$$Ay_{uu} + y_{vv}\left(\frac{AB^2}{4AC-B^2} - \frac{2B^2A}{4AC-B^2} + \frac{4CA^2}{4AC-B^2}\right) = 0$$

$$Ay_{uu} + y_{vv}\left(\frac{AB^2 - 2B^2A + 4CA^2}{4AC-B^2}\right) = 0$$

$$Ay_{uu} + Ay_{vv}\left(\frac{-B^2 + 4CA}{4AC-B^2}\right) = 0$$

$$Ay_{uu} + Ay_{vv} = 0$$

$$A\left(y_{uu} + y_{vv}\right) = 0$$

Therefore, since $A \neq 0$ the above becomes

$$y_{uu} + y_{vv} = 0$$

7.2 Part b

Setting $\alpha = -B, \beta = 2A$ in (4) above results in

$$Ay_{uu} + y_{uv}\left(-2AB + 2AB\right) + y_{vv}\left(AB^2 - 2B^2A + 4CA^2\right) = 0$$

$$Ay_{uu} + y_{vv}\left(4CA^2 - B^2A\right) = 0$$

$$Ay_{uu} - Ay_{vv}\left(B^2 - 4CA\right) = 0$$

But $B^2 - 4CA = 0$, therefore the above becomes

$$y_{uu} = 0$$