HW 3
MATH 4567 Applied Fourier Analysis
Spring 2019
University of Minnesota, Twin Cities

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Contents

1 Section 20, Problem 1

1. Show that the function

$$f(x)=\begin{cases}0 & \text{when } -\pi\le x\le 0,\\ \sin x & \text{when } \quad 0<x\le\pi\end{cases}$$

satisfies all the conditions in the theorem in Sec. 17. Then, with the aid of the Weierstrass M-test in Sec. 17, verify that the Fourier series

$$\frac{1}{\pi}+\frac{1}{2}\sin x-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{4n^2-1}\qquad(-\pi<x<\pi)$$

for f, found in Problem 7, Sec. 7, converges uniformly on the interval $-\pi\le x\le\pi$, as the theorem in Sec. 17 tells us. Also, state why this series is differentiable in the interval $-\pi<x<\pi$, except at the point $x=0$, and describe graphically the function that is represented by the differentiated series for all x.

Figure 1: Problem statement

The function $f(x)$ is

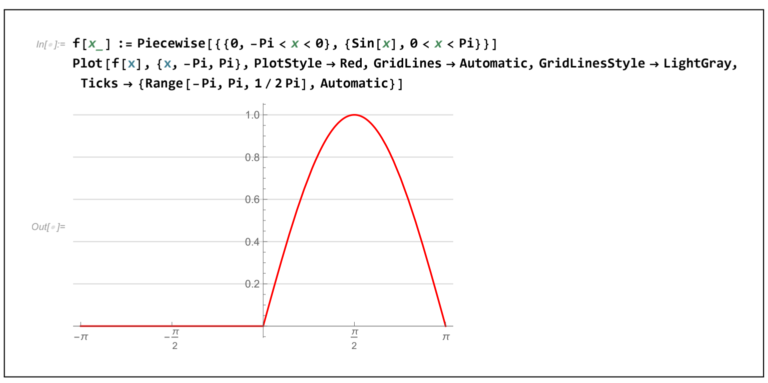


Figure 2: Plot of $f(x)$

The function $f(x)$ is continuous on $-\pi\le x\le\pi$. Also $f(-\pi)=f(\pi)=0$. We now need to show that $f'(x)$ is piecewise continuous. But

$$f'(x)=\begin{cases}0 & -\pi\le x\le 0\\ \cos x & 0<x\le\pi\end{cases}\tag{1}$$

Therefore $f'(x)$ exist and is piecewise continuous on $-\pi<x<\pi$. From the above, we see that $f(x)$ meets the 3 conditions in theorem of section 17, hence we know that the Fourier series of $f(x)$ is absolutely and uniformly convergent. (Here we need to use the M test to confirm this).

The Fourier series of $f(x)$ is

$$\frac{a_0}{2}+\frac{1}{2}\sin x-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2nx)}{4n^2-1}$$

Now, to apply the M test, consider the two series

$$\sum_{n=1}^{\infty}\overbrace{\frac{\cos(2nx)}{4n^2-1}}^{f_n},\sum_{n=1}^{\infty}\overbrace{\frac{1}{4n^2-1}}^{M_n}$$

To show Fourier series is uniformly convergent to $f(x)$, using the M test, then we need to show that $|f_n|\le M_n$ for each n. The series M_n qualifies to use for the Weierstrass series, since each term in it is positive constant and it is convergent series. To show that M_n is convergent, we can compare it to $\sum_{n=1}^{\infty}\frac{1}{n^2}$. Since each term $\frac{1}{4n^2-1}<\frac{1}{n^2}$ and $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is convergent since any $\sum_{n=1}^{\infty}\frac{1}{n^s}$ for $s>1$ is convergent (we can show this if needed using the integral test). Hence we can go ahead and use M_n series. Now we just need to show that

$$\left|\frac{\cos(2nx)}{4n^2-1}\right| \leq \frac{1}{4n^2-1}$$

For each n. But $\cos(2nx) \leq 1$ for each n. Hence the above is true for each n and it follows that the above Fourier series is indeed uniformly convergent to $f(x)$.

From (1), At $x = 0$ we have

$$f'_+(0) = \lim_{x\to 0^+} \frac{f(x) - f(0)}{x} = \lim_{x\to 0^+} \frac{\sin(x)}{x} = 1$$

And

$$f'_-(0) = \lim_{x\to 0^-} \frac{f(x) - f(0)}{x} = \lim_{x\to 0^+} \frac{0}{x} = 0$$

Since $f'_+(0) \neq f'_-(0)$ then $f(x)$ is not differentiable at $x = 0$. This is plot of $f'(x)$ and we see graphically that due to jump discontinuity, that $f'(x)$ is not differentiable at $x = 0$

```
In[◦]:= f[x_] := Piecewise[{{0, -Pi < x < 0}, {Cos[x], 0 < x < Pi}}]
Plot[f[x], {x, -Pi, Pi}, PlotStyle → Red, GridLines → Automatic, GridLinesStyle → LightGray,
 Ticks → {Range[-Pi, Pi, 1/2 Pi], Automatic}]
```

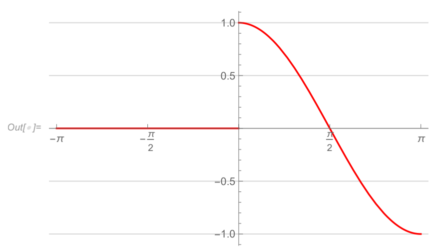
Out[◦]=


Figure 3: Plot of $f'(x)$ shown for one period

```
In[◦]:= Clear[f];
f[x_ /; -Pi < x < Pi] := Piecewise[{{0, -Pi < x < 0}, {Cos[x], 0 < x < Pi}}]
f[x_ /; x > Pi] := f[x - 2 Pi];
f[x_ /; x < -Pi] := f[x + 2 Pi];
Plot[f[x], {x, -4 Pi, 4 Pi}, PlotStyle → {Thick, Red}, Mesh → None,
 GridLines → Automatic, GridLinesStyle → LightGray,
 Ticks → {Range[-4 Pi, 4 Pi, Pi], Automatic},
 Exclusions → {x == -3 Pi, x == -2 Pi, x == -Pi, x == 0, x == Pi, x == 2 Pi, x == 3 Pi},
 Mesh → None, ExclusionsStyle → Dashed]
```

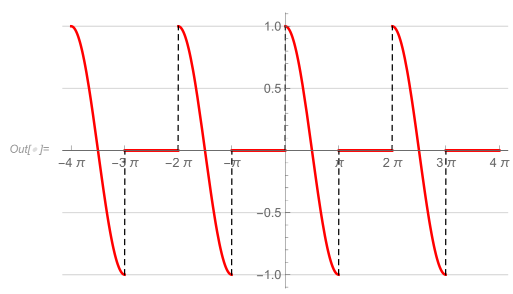
Out[◦]=


Figure 4: Plot of $f'(x)$ for all x, shown for 3 periods

2 Section 20, Problem 2

2. We know from Example 1, Sec. 3, that the series

$$\frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}$$

is the Fourier cosine series for the function $f(x) = x$ on the interval $0 < x < \pi$. Differentiate this series term by term to obtain a representation for the derivative $f'(x) = 1$ on that interval. State why the procedure is reliable here.

Figure 5: Problem statement

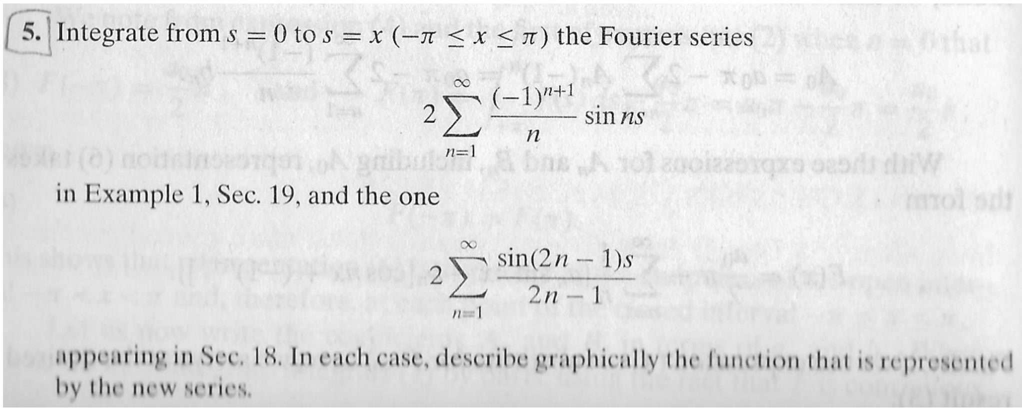
Solution

After doing an even extension of $f(x) = x$ on $0 < x < \pi$ to $-\pi \leq x \leq \pi$, we see that $f(x)$ satisfies the conditions of Theorem section 20 for differentiating the Fourier series term by term. Since

1. $f(x)$ is continuous on the interval $-\pi \leq x \leq \pi$
2. $f(-\pi) = f(\pi)$
3. $f'(x)$ is piecewise continuous on $-\pi < x < \pi$

The only point that $f(x)$ is not differentiable is $x = 0$ which implies $f'(x)$ is piecewise continuous. But that is OK. It is $f(x)$ which must be continuous. Hence differentiating the series term by term to obtain representation of $f(x)$ on $0 < x < \pi$ is reliable.

3 Section 20, Problem 5



5. Integrate from $s = 0$ to $s = x$ $(-\pi \le x \le \pi)$ the Fourier series

$$2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin ns$$

in Example 1, Sec. 19, and the one

$$2\sum_{n=1}^{\infty}\frac{\sin(2n-1)s}{2n-1}$$

appearing in Sec. 18. In each case, describe graphically the function that is represented by the new series.

Figure 6: Problem statement

3.1 Part 1

$$S = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(ns)$$

The above is the Fourier sine series for $f(x) = x$, on $0 < x < \pi$. Integrating gives

$$\int_0^x\left(2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(ns)\right)ds = 2\sum_{n=1}^{\infty}\int_0^x\frac{(-1)^{n+1}}{n}\sin(ns)\,ds$$

We did integration term by term, since that is always allowed (not like with differentiation term by term, where we have to check). Hence the above becomes

$$\begin{aligned}
2\sum_{n=1}^{\infty}\int_0^x\frac{(-1)^{n+1}}{n}\sin(ns)\,ds &= 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\left(\int_0^x\sin(ns)\,ds\right)\\
&= 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\left(-\frac{\cos ns}{n}\right)_0^x\\
&= 2\sum_{n=1}^{\infty}\frac{(-1)^{n+2}}{n^2}(\cos ns)_0^x
\end{aligned}$$

But $(-1)^{n+2} = (-1)^n$ and the above becomes

$$2\sum_{n=1}^{\infty}\int_0^x\frac{(-1)^{n+1}}{n}\sin(ns)\,ds = 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}(\cos nx - 1)$$

But $\int_0^x s\,ds = \frac{1}{2}x^2$. So the above is the Fourier series of $\frac{1}{2}x^2$. A plot of the above is

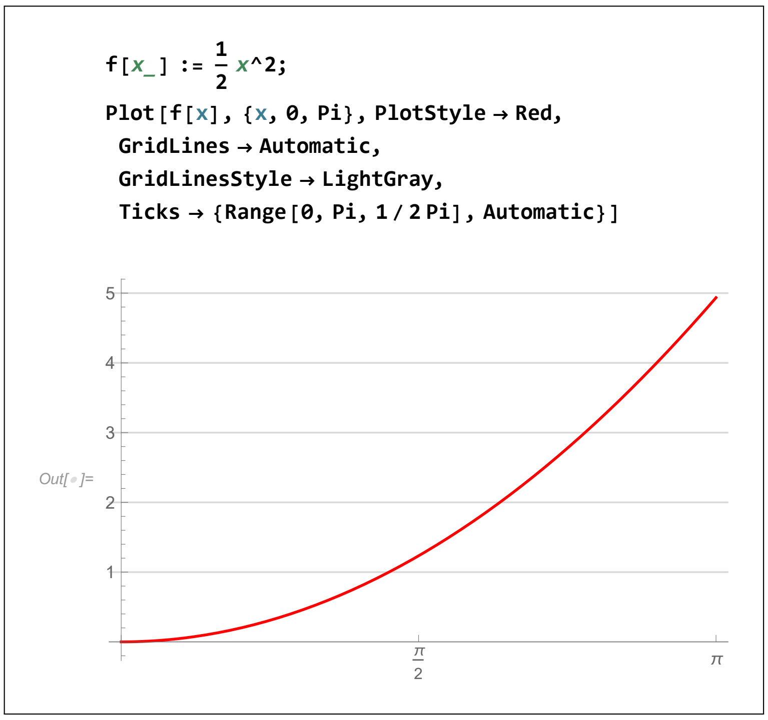


Figure 7: The function represented by the above series $f(x) = \frac{1}{2}x^2$

3.2 Part 2

$$S = 2\sum_{n=1}^{\infty} \frac{\sin((2n-1)s)}{2n-1}$$

The above is the Fourier sine series for $f(x) = \frac{\pi}{2}$, on $0 < x < \pi$. Integrating gives

$$\int_0^x \left(2\sum_{n=1}^{\infty} \frac{1}{2n-1}\sin((2n-1)s)\right) ds = 2\sum_{n=1}^{\infty}\int_0^x \frac{1}{2n-1}\sin((2n-1)s)\, ds$$

We did integration term by term, since that is always allowed (not like with differentiation term by term, where we have to check). Hence the above becomes

$$\begin{aligned} 2\sum_{n=1}^{\infty}\int_0^x \frac{1}{2n-1}\sin((2n-1)s)\, ds &= 2\sum_{n=1}^{\infty} \frac{1}{2n-1}\int_0^x \sin((2n-1)s)\, ds \\ &= 2\sum_{n=1}^{\infty} \frac{1}{2n-1}\left(\frac{-\cos(2n-1)s}{(2n-1)}\right)_0^x \\ &= 2\sum_{n=1}^{\infty} -\frac{(\cos((2n-1)x)-1)}{(2n-1)^2} \end{aligned}$$

Since $\int_0^x \frac{\pi}{2} ds = \frac{\pi}{2}x$, then the above is the representation of this function. Here is a plot to confirm this, showing the above series expansion as more terms are added, showing it converges to $\frac{\pi}{2}x$

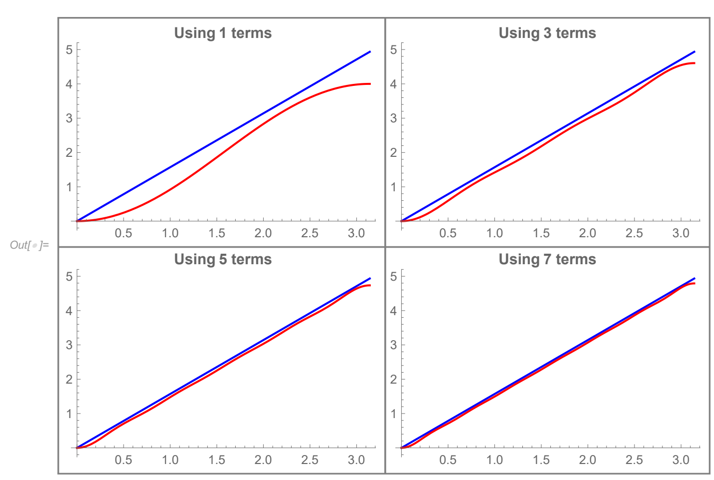


Figure 8: The function represented by the above series $f(x) = \frac{\pi}{2}x$ against its Fourier series

```
fApprox[x_, nTerms_] := 2 Sum[-(Cos[(2 n - 1) x] - 1)/(2 n - 1)^2, {n, 1, nTerms}];
Clear[f];
f[x_ /; 0 < x < Pi] := x * Pi / 2;
Grid[Partition[Table[Plot[{f[x], fApprox[x, n]}, {x, 0, Pi},
     PlotStyle → {Blue, Red},
     PlotLabel → Style[Row[{"Using ", n, " terms"}], Bold],
     ImageSize → 250],
    {n, 1, 10, 2}], 2], Frame → All, FrameStyle → Gray]
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Figure 9: Code used to plot the above

4 Section 27, Problem 1

1. Let $u(x)$ denote the steady-state temperatures in a slab bounded by the planes $x = 0$ and $x = c$ when those faces are kept at fixed temperatures $u = 0$ and $u = u_0$, respectively. Set up the boundary value problem for $u(x)$ and solve it to show that

$$u(x) = \frac{u_0}{c}\, x \qquad \text{and} \qquad \Phi_0 = K\,\frac{u_0}{c},$$

where Φ_0 is the flux of heat to the left across each plane $x = x_0$ $(0 \le x_0 \le c)$.

Figure 10: Problem statement

The heat PDE is $u_t = u_{xx}$. At steady state, $u_t = 0$ leading to $u_{xx} = 0$. So at steady state, the solution depends on x only. This has the solution

$$u(x) = Ax + B \tag{1}$$

With boundary conditions

$$u(0) = 0$$
$$u(c) = u_0$$

When $x = 0$ then $0 = B$. Hence the solution becomes $u(x) = Ax$. To find A, we apply the second boundary conditions. At $x = c$ this gives $u_0 = cA$ or $A = \frac{u_0}{c}$. Hence the solution (1) now becomes

$$u(x) = \frac{u_0}{c} x$$

Now the flux is defined as $\Phi_0 = K\frac{du}{dx}$ at each edge surface. But $\frac{du}{dx} = \frac{u_0}{c}$ from above. Therefore

$$\Phi_0 = K\frac{u_0}{c}$$

5 Section 27, Problem 2

2. A slab occupies the region $0 \leq x \leq c$. There is a constant flux of heat Φ_0 into the slab through the face $x = 0$. The face $x = c$ is kept at temperature $u = 0$. Set up and solve the boundary value problem for the steady-state temperatures $u(x)$ in the slab.

Answer: $u(x) = \dfrac{\Phi_0}{K}(c - x)$.

Figure 11: Problem statement

note: When looking for solution, assume it is a function of x only.

The heat PDE is $u_t = u_{xx}$. At steady state, $u_t = 0$ leading to $u_{xx} = 0$. So at steady state, the solution depends on x only. This has the solution

$$u(x) = Ax + B \tag{1}$$

Since there is constant flux at $x = 0$, then this means $K \left.\frac{du}{dx}\right|_{x=0} = -\Phi_0$. The reason for the minus sign, is that flux is always pointing to the outside of the surface. Hence on the left surface, it will be in the negative x direction and on the right side, it will be on the positive x direction.

Using this, the boundary conditions can be written as

$$\left.\frac{du}{dx}\right|_{x=0} = -K\Phi_0$$
$$u(c) = 0$$

Applying the left boundary condition gives

$$A = -K\Phi_0$$

Hence the solution becomes $u(x) = -K\Phi_0 x + B$.

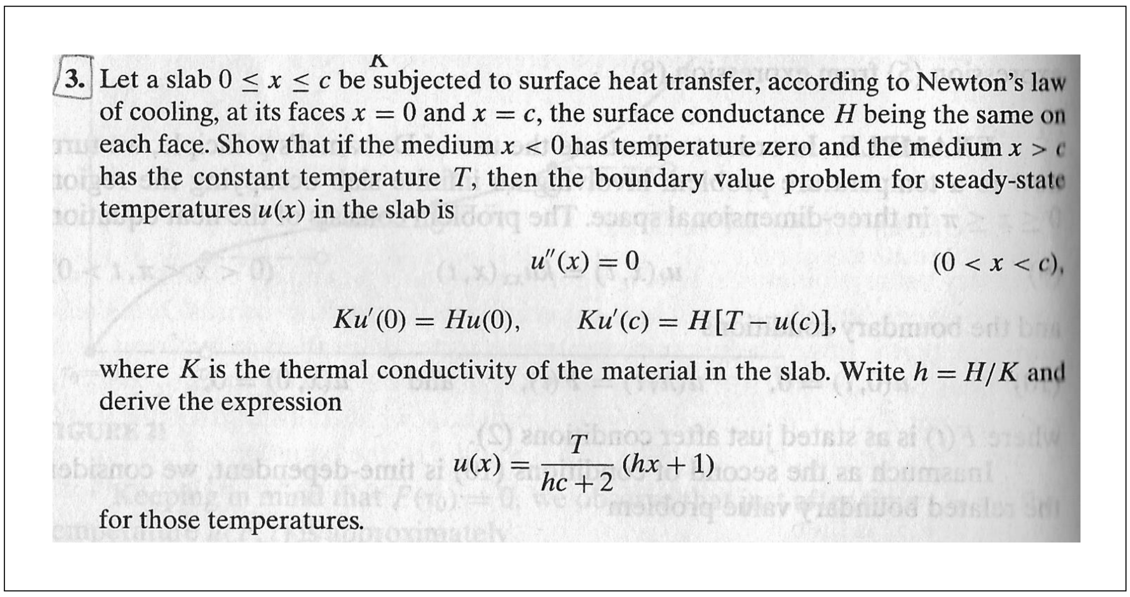
At $x = c$ the second B.C. leads to $0 = -K\Phi_0 c + B$ or

$$B = K\Phi_0 c$$

Hence the solution (1) becomes

$$\begin{aligned} u(x) &= -K\Phi_0 x + K\Phi_0 c \\ &= K\Phi_0 (c - x) \end{aligned}$$

6 Section 27, Problem 3


3. Let a slab $0 \le x \le c$ be subjected to surface heat transfer, according to Newton's law of cooling, at its faces $x = 0$ and $x = c$, the surface conductance H being the same on each face. Show that if the medium $x < 0$ has temperature zero and the medium $x > c$ has the constant temperature T, then the boundary value problem for steady-state temperatures $u(x)$ in the slab is

$$u''(x) = 0 \qquad (0 < x < c),$$

$$Ku'(0) = Hu(0), \qquad Ku'(c) = H[T - u(c)],$$

where K is the thermal conductivity of the material in the slab. Write $h = H/K$ and derive the expression

$$u(x) = \frac{T}{hc+2}(hx+1)$$

for those temperatures.

Figure 12: Problem statement

We start with

$$\Phi = H\left(T_{\text{outside}} - u\right) \tag{1}$$

Where T is the temperature on the outside and u is the temperature on the surface and Φ is the flux at the surface and H is surface conductance. Let us look at the left surface, at $x = 0$. The flux there is negative, since it points to the negative x direction. Therefore

$$\Phi = -K \left.\frac{du}{dx}\right|_{x=0} \tag{2}$$

From (1,2) we obtain

$$-K \left.\frac{du}{dx}\right|_{x=0} = H\left(T_{\text{outside}} - u(0)\right)$$

But $T_{\text{outside}} = 0$ outside the left surface and the above becomes

$$-K \left.\frac{du}{dx}\right|_{x=0} = H\left(0 - u(0)\right)$$

The minus signs cancel, giving

$$\left.\frac{du}{dx}\right|_{x=0} = \frac{H}{K} u(0)$$

$$u'(0) = hu(0) \tag{3}$$

Now, let us look at the right side. There the flux is positive. Hence at $x = c$ we have

$$K \left.\frac{du}{dx}\right|_{x=c} = H\left(T_{\text{outside}} - u(c)\right)$$

But $T_{\text{outside}} = T$ on the right side. Hence the above reduces to

$$\left.\frac{du}{dx}\right|_{x=c} = \frac{H}{K}\left(T - u(c)\right)$$

$$u'(c) = h\left(T - u(c)\right) \tag{4}$$

Now that we found the boundary conditions, we look at the solution. As before, at steady state we have

$$u''(x) = 0$$

$$u(x) = Ax + B \tag{5}$$

Hence $u'(x) = A$. Therefore

$$u'(0) = A = hu(0) \tag{6}$$

$$u'(c) = A = h\left(T - u(c)\right) \tag{7}$$

But we also know that, from (5) that

$$u(0) = B \tag{8}$$

$$u(c) = Ac + B \tag{9}$$

Substituting (8,9) into (6,7) in order to eliminate $u(0), u(c)$ from (6,7) gives

$$A = hB \tag{6A}$$

$$A = h(T - (Ac + B)) \tag{7A}$$

Now from (6A,7A) we solve for A, B. Substituting (7A) into (6A) gives

$$\begin{aligned} hB &= h(T - (hBc + B)) \\ hB &= hT - h^2Bc - hB \\ 2hB + h^2Bc &= hT \\ B &= \frac{hT}{h(2 + hc)} \\ &= \frac{T}{2 + hc} \end{aligned}$$

Hence

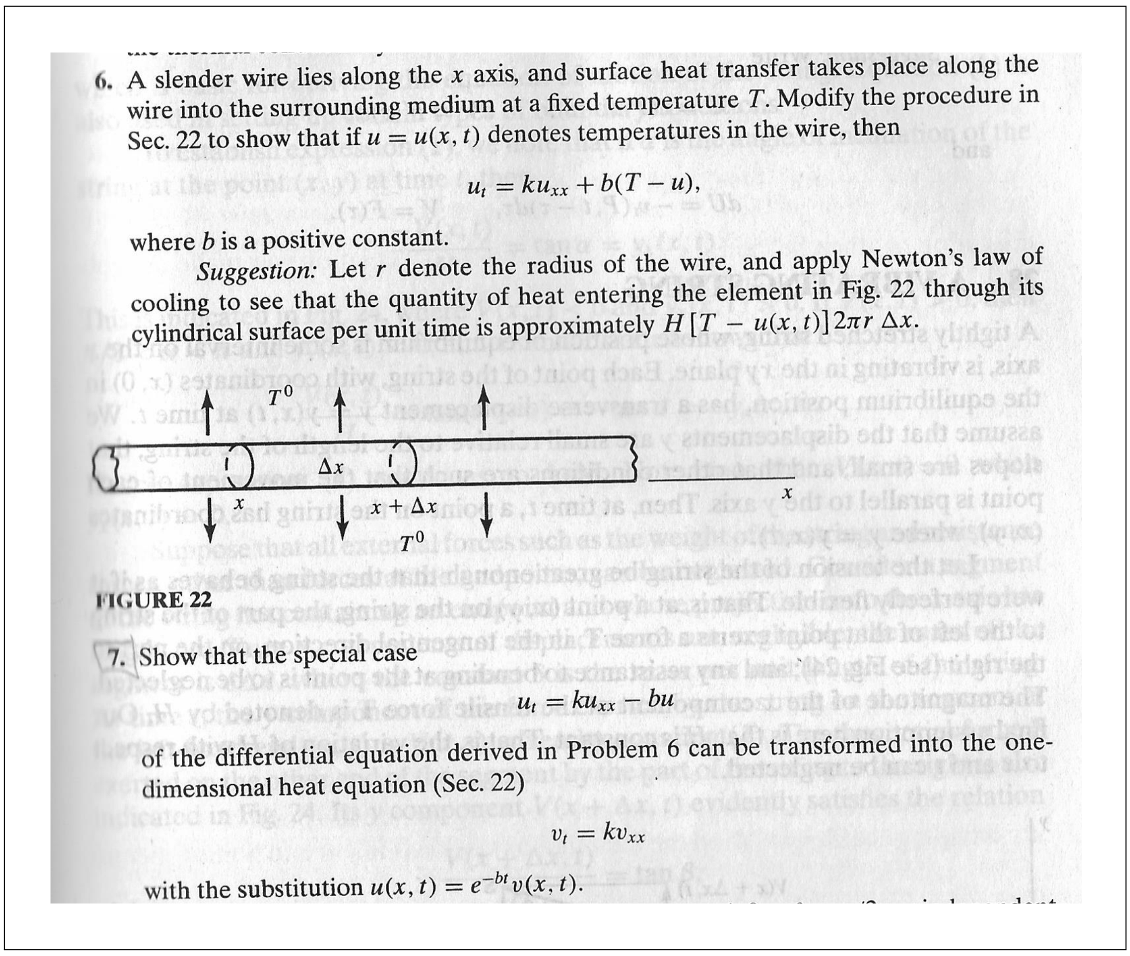
$$\begin{aligned} A &= hB \\ &= \frac{hT}{2 + hc} \end{aligned}$$

Now that we found A, B then since $u(x) = Ax + B$, then

$$\begin{aligned} u(x) &= \frac{hT}{2 + hc}x + \frac{T}{2 + hc} \\ &= \frac{hTx + T}{2 + hc} \\ &= \frac{T}{2 + hc}(1 + hx) \end{aligned}$$

Which is the result we are asked to show.

7 Section 27, Problem 7



6. A slender wire lies along the x axis, and surface heat transfer takes place along the wire into the surrounding medium at a fixed temperature T. Modify the procedure in Sec. 22 to show that if $u = u(x,t)$ denotes temperatures in the wire, then

$$u_t = ku_{xx} + b(T-u),$$

where b is a positive constant.

Suggestion: Let r denote the radius of the wire, and apply Newton's law of cooling to see that the quantity of heat entering the element in Fig. 22 through its cylindrical surface per unit time is approximately $H[T - u(x,t)]2\pi r\,\Delta x$.

FIGURE 22

7. Show that the special case

$$u_t = ku_{xx} - bu$$

of the differential equation derived in Problem 6 can be transformed into the one-dimensional heat equation (Sec. 22)

$$v_t = kv_{xx}$$

with the substitution $u(x,t) = e^{-bt}v(x,t)$.

Figure 13: Problem statement

$$u_t = ku_{xx} - bu \tag{1}$$

Let $u(x,t) = e^{-bt}v(x,t)$ then

$$\begin{aligned} u_t &= -be^{-bt}v + e^{-bt}v_t \\ u_x &= e^{-bt}v_x \\ u_{xx} &= e^{-bt}v_{xx} \end{aligned}$$

Substituting the above back into (1) gives

$$-be^{-bt}v + e^{-bt}v_t = ke^{-bt}v_{xx} - be^{-bt}v$$

Since $e^{-bt} \neq 0$, then the above simplifies to

$$\begin{aligned} -bv + v_t &= kv_{xx} - bv \\ v_t &= kv_{xx} \end{aligned}$$

QED.