HW 11
MATH 4567 Applied Fourier Analysis
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Contents

1 Section 73, Problem 8

8. Find the Fourier constants c_n for the function $f(x) = x$ $(1 < x < b)$ with respect to the normalized eigenfunctions in Problem 6, Sec. 72, and reduce those constants to the form

$$c_n = \sqrt{2\ln b}\,\frac{n\pi[1 + (-1)^{n+1}b]}{(\ln b)^2 + (n\pi)^2} \qquad (n = 1, 2, \ldots).$$

Suggestion: The integration formula

$$\int e^x \sin ax\, dx = \frac{e^x(\sin ax - a\cos ax)}{1 + a^2},$$

derived in calculus, is useful here.

Figure 1: Problem statement

Solution

$$\begin{aligned} c_n &= \langle f(x), \phi_n(x)\rangle \\ &= \int_1^b p(x) f(x) \phi_n(x)\, dx \end{aligned}$$

But $p(x) = \frac{1}{x}$ and $\phi_n(x) = \sqrt{\frac{2}{\ln b}}\sin(\alpha_n \ln x)$ and $f(x) = x$ therefore the above becomes

$$\begin{aligned} c_n &= \int_1^b \frac{1}{x} x \sqrt{\frac{2}{\ln b}} \sin(\alpha_n \ln x)\, dx \\ &= \sqrt{\frac{2}{\ln b}} \int_1^b \sin(\alpha_n \ln x)\, dx \end{aligned}$$

But $\alpha_n = \frac{n\pi}{\ln b}$, therefore

$$c_n = \sqrt{\frac{2}{\ln b}} \int_1^b \sin\left(\frac{n\pi}{\ln b}\ln x\right) dx$$

Let $s = \pi\frac{\ln x}{\ln b}$, hence $\frac{ds}{dx} = \frac{\pi}{\ln b}\frac{1}{x}$. When $x = 1 \rightarrow s = 0$ and when $x = b \rightarrow s = \pi$. The above becomes

$$c_n = \sqrt{\frac{2}{\ln b}} \int_0^\pi \sin(ns)\frac{\ln(b)}{\pi} x ds$$

But $\ln x = \frac{s}{\pi}\ln b$, hence $x = e^{s\frac{\ln b}{\pi}}$, and the above becomes

$$c_n = \frac{\sqrt{2\ln(b)}}{\pi} \int_0^\pi e^{s\frac{\ln b}{\pi}} \sin(ns)\, ds \tag{1}$$

Using

$$\int e^{ax}\sin(bx)\, ds = \frac{e^{ax}}{a^2 + b^2}(a\sin bx - b\cos bx)$$

Where in our case $a = \frac{\ln b}{\pi}$ and $b = n$. Applying the above gives

$$\begin{aligned} \int_0^\pi e^{s\frac{\ln b}{\pi}}\sin(ns)\, ds &= \left[\frac{e^{\frac{\ln b}{\pi}x}}{\left(\frac{\ln b}{\pi}\right)^2 + n^2}\left(\frac{\ln b}{\pi}\sin nx - n\cos nx\right)\right]_0^\pi \\ &= \frac{1}{\left(\frac{\ln b}{\pi}\right)^2 + n^2}\left[e^{\frac{\ln b}{\pi}\pi}\left(\frac{\ln b}{\pi}\sin n\pi - n\cos n\pi\right) - (0 - n)\right] \end{aligned}$$

But $\sin n\pi = 0$ since n integer, giving

$$\int_0^{\pi} e^{s\frac{\ln b}{\pi}} \sin(ns)\, ds = \frac{1}{\left(\frac{\ln b}{\pi}\right)^2 + n^2}\left[-bn\cos n\pi + n\right]$$
$$= \frac{\pi^2}{(\ln b)^2 + \pi^2 n^2}\left[-bn(-1)^n + n\right]$$
$$= \frac{\pi^2\left(bn(-1)^{n+1} + n\right)}{(\ln b)^2 + \pi^2 n^2}$$

Hence (1) becomes

$$c_n = \frac{\sqrt{2\ln(b)}}{\pi}\frac{n\pi^2\left(1 + (-1)^{n+1} b\right)}{(\ln b)^2 + (\pi n)^2}$$
$$= \sqrt{2\ln(b)}\frac{n\pi\left(1 + (-1)^{n+1} b\right)}{(\ln b)^2 + (\pi n)^2}$$

Where $n = 1, 2, 3, \cdots$, which is the result required to show.

2 Section 73, Problem 10

10. Suppose that a function f, defined on the interval $0 < x < c$, is piecewise smooth there.

(*a*) Use the normalized eigenfunctions (Problem 7, Sec. 72)

$$\phi_n(x) = \sqrt{\frac{2}{c}} \sin \alpha_n x \qquad (n = 1, 2, \ldots),$$

where

$$\alpha_n = \frac{(2n-1)\pi}{2c},$$

to show formally that

$$f(x) = \sum_{n=1}^{\infty} B_n \sin \alpha_n x \qquad (0 < x < c),$$

where

$$B_n = \frac{2}{c}\int_0^c f(x) \sin \alpha_n x \, dx \qquad (n = 1, 2, \ldots).$$

(*b*) Note that according to Problem 6, Sec. 15, the series in part (*a*) is actually a Fourier sine series for an extension of f on the interval $0 < x < 2c$. Then, with the aid of Theorem 2 in Sec. 15, state why the representation in part (*a*) is valid for each point x $(0 < x < c)$ at which f is continuous.

Figure 2: Problem statement

Solution

2.1 Part (a)

$$\phi_n(x) = \sqrt{\frac{2}{c}} \sin(\alpha_n x) \qquad n = 1, 2, 3, \cdots$$
$$\alpha_n = \pi \frac{2n-1}{2x}$$

Since $\phi_n(x)$ are complete, then we can represent $f(x)$ using $\phi_n(x)$ as generalized Fourier series using

$$f(x) = \sum_{n=1}^{\infty} B_n \phi_n(x) \qquad 0 < x < c$$

To find B_n, since $\phi_n(x)$ are orthonormal eigenfunctions then

$$\begin{aligned} B_n &= \langle f(x), \phi_n(x) \rangle \\ &= \int_0^c p(x) f(x) \phi_n dx \end{aligned}$$

But problem (7) section 72 is $X'' + \lambda X = 0$ which implies that $p(x) = 1$. Hence the above becomes

$$\begin{aligned} B_n &= \int_0^c f(x) \sqrt{\frac{2}{c}} \sin(\alpha_n x)\, dx \\ &= \sqrt{\frac{2}{c}} \int_0^c f(x) \sin(\alpha_n x)\, dx \end{aligned}$$

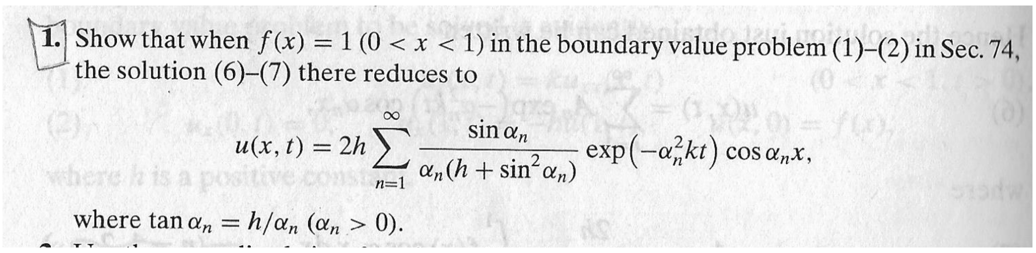
Which is the result required to show.

2.2 Part (b)

Theorem 2 section 15 gives the conditions on $f(x)$ for it to have a Fourier sine series which converges to $f(x)$ where $f(x)$ is continuous and converges to mean value of $f(x)$ where $f(x)$ have a jump discontinuity.

Since $f(x)$ is piecewise continuous in this problem, then for those regions where $f(x)$ is continuous between $0 < x < c$, the series found in part(a) converges to $f(x)$ and is valid Fourier sine series representation of $f(x)$ there.

3 Section 74, Problem 1



1. Show that when $f(x) = 1$ $(0 < x < 1)$ in the boundary value problem (1)–(2) in Sec. 74, the solution (6)–(7) there reduces to

$$u(x,t) = 2h \sum_{n=1}^{\infty} \frac{\sin \alpha_n}{\alpha_n (h + \sin^2 \alpha_n)} \exp\left(-\alpha_n^2 kt\right) \cos \alpha_n x,$$

where $\tan \alpha_n = h/\alpha_n$ $(\alpha_n > 0)$.

Figure 3: Problem statement

Solution

Solution (6) is given by

$$u(x,t) = \sum_{n=1}^{\infty} A_n \exp\left(-\alpha_n^2 kt\right) \cos(\alpha_n x) \tag{6}$$

Where

$$A_n = \frac{2h}{h + \sin^2 \alpha_n} \int_0^1 f(x) \cos(\alpha_n x)\, dx$$

But $f(x) = 1$ which reduces the above to

$$\begin{aligned} A_n &= \frac{2h}{h + \sin^2 \alpha_n} \int_0^1 \cos(\alpha_n x)\, dx \\ &= \frac{2h}{h + \sin^2 \alpha_n} [\sin(\alpha_n x)]_0^1 \\ &= \frac{2h}{h + \sin^2 \alpha_n} \sin(\alpha_n) \end{aligned}$$

Hence (6) becomes

$$u(x,t) = 2h \sum_{n=1}^{\infty} \frac{\sin(\alpha_n)}{h + \sin^2 \alpha_n} \exp\left(-\alpha_n^2 kt\right) \cos(\alpha_n x)$$

But from example 1, section 72 we are given that $\tan(\alpha_n c) = \frac{h}{\alpha_n}$. But $c = 1$ in this problem, hence

$$\tan(\alpha_n) = \frac{h}{\alpha_n}$$

Which is what required to show.

4 Section 74, Problem 4

4. (*a*) Give a physical interpretation of the boundary value problem

$$u_t(x, t) = k u_{xx}(x, t) \qquad (0 < x < 1,$$

$$u(0, t) = 0, \qquad u_x(1, t) = -h u(1, t), \qquad u(x, 0) = f(x),$$

where h is a positive constant. Then derive the solution

$$u(x, t) = \sum_{n=1}^{\infty} B_n \exp\left(-\alpha_n^2 k t\right) \sin \alpha_n x,$$

where $\tan \alpha_n = -\alpha_n / h$ ($\alpha_n > 0$) and

$$B_n = \frac{2h}{h + \cos^2 \alpha_n} \int_0^1 f(x) \sin \alpha_n x \, dx \qquad (n = 1,$$

(*b*) Use an argument similar to the one at the end of Sec. 74 to show that the s found in part (*a*) formally satisfies the boundary value problem (8)–(10) section when the function f there is *odd*, or when

$$f(-x) = -f(x) \qquad (-1 <$$

Figure 4: Problem statement

Solution

4.1 Part (a)

$u(0,t) = 0$ means that the left surface is kept at fixed temperature which is zero. And $u_x(1,t) + hu(1,t) = 0$ means that the surface heat transfer takes place at face $x = 1$ into the medium at temperature zero. To solve the PDE, we first check the boundary conditions by writing them as

$$a_1 u(0,t) + a_2 u_x(0,t) = 0$$
$$b_1 u(1,t) + b_2 u_x(1,t) = 0$$

Then $a_1 = 0, a_2 = 0$. Hence $a_1 a_2 = 0$. And $b_1 = 1, b_2 = h$. Then since it is assumed that $h > 0$ per section 26, then $b_1 b_2 \geq 0$. And since $q(x) = 0$ from the PDE itself, then we know that eigenvalues are $\lambda \geq 0$.

Let $u = X(x) T(t)$ then the PDE becomes

$$T'X = X''T$$
$$\frac{T'}{T} = \frac{X''}{X} = -\lambda$$

Hence the Sturm Liouville problem is

$$X'' + \lambda X = 0$$
$$X(0) = 0$$
$$X'(1) + hX(1) = 0$$

Where $p(x) = 1$.

Case $\lambda = 0$

Solution is

$$X(x) = Ax + B$$

At $x = 0$

$$0 = B$$

Hence solution becomes

$$X(x) = Ax$$

At $x = 1$ the second boundary conditions gives

$$A + hA = 0$$
$$A(1 + h) = 0$$

For non trivial solution $1 + h = 0$ or $h = -1$. But we assumed that $h > 0$. Therefore $\lambda = 0$ is not eigenvalue.

<u>Case $\lambda > 0$</u>

Let $\lambda = \alpha^2, \alpha > 0$. Hence solution is

$$X(x) = A\cos(\alpha x) + B\sin(\alpha x)$$

At $X(0) = 0$

$$0 = A$$

The solution becomes

$$X(x) = B\sin(\alpha x)$$

At $x = 1$ the second boundary conditions gives

$$\begin{aligned} B\alpha\cos(\alpha) + hB\sin(\alpha) &= 0 \\ \alpha\cos(\alpha) + h\sin(\alpha) &= 0 \\ \tan(\alpha) &= -\frac{\alpha}{h} \end{aligned}$$

Therefore the eigenvalues are given by solution to

$$\tan(\alpha_n) = -\frac{\alpha_n}{h} \qquad n = 1, 2, 3, \cdots$$

And eigenfunctions are

$$X_n(x) = \sin(\alpha_n x)$$

The normalized eigenfunctions are

$$\phi_n(x) = \frac{X_n(x)}{\|X_n(x)\|}$$

But

$$\begin{aligned} \|X_n(x)\|^2 &= \int_0^1 p(x)\, X_n^2(x)\, dx \\ &= \int_0^1 \sin^2(\alpha_n x)\, dx \\ &= \frac{1}{2}\int_0^1 1 - \cos(2\alpha_n x)\, dx \\ &= \frac{1}{2}\left(1 - \left[\frac{\sin(2\alpha_n x)}{2\alpha_n}\right]_0^1\right) \\ &= \frac{1}{2}\left(1 - \frac{1}{2\alpha_n}\left[\sin(2\alpha_n x)\right]_0^1\right) \\ &= \frac{1}{2}\left(1 - \frac{\sin(2\alpha_n)}{2\alpha_n}\right) \\ &= \frac{1}{2} - \frac{\sin(2\alpha_n)}{4\alpha_n} \end{aligned}$$

But $\sin(2\alpha_n) = 2\sin\alpha_n\cos\alpha_n$ and $\alpha_n = -h\frac{\sin(\alpha_n)}{\cos(\alpha_n)}$, therefore the above becomes

$$\begin{aligned}\|X_n(x)\|^2 &= \frac{1}{2} + \frac{2\sin\alpha_n\cos\alpha_n}{4h\frac{\sin(\alpha_n)}{\cos(\alpha_n)}}\\ &= \frac{1}{2} + \frac{\cos^2\alpha_n}{2h}\\ &= \frac{h+\cos^2\alpha_n}{2h}\end{aligned}$$

Hence

$$\begin{aligned}\phi_n(x) &= \frac{X_n(x)}{\sqrt{\frac{h+\cos^2\alpha_n}{2h}}}\\ &= \sqrt{\frac{2h}{h+\cos^2\alpha_n}}\sin(\alpha_n x)\end{aligned}$$

Now we use generalized Fourier series to find the solution. Let

$$u(x,t) = \sum_{n=1}^{\infty} B_n(t)\,\phi_n(x) \tag{1}$$

Substituting this back into the PDE gives

$$\sum_{n=1}^{\infty} B_n'(t)\,\phi_n(x) = k\sum_{n=1}^{\infty} B_n(t)\,\phi_n''(x)$$

But $\phi_n''(x) = -\lambda_n\phi_n(x) = -\alpha_n^2\phi_n(x)$. The above becomes

$$\begin{aligned}\sum_{n=1}^{\infty} B_n'(t)\,\phi_n(x) &= -k\sum_{n=1}^{\infty} B_n(t)\,\alpha_n^2\phi_n(x)\\ B_n'(t) + k\alpha_n^2 B_n(t) &= 0\end{aligned}$$

The solution is

$$B_n(t) = B_n(0)\,e^{-k\alpha_n^2 t}$$

Hence (1) becomes

$$u(x,t) = \sum_{n=1}^{\infty} B_n(0)\,e^{-k\alpha_n^2 t}\phi_n(x)$$

At $t=0$ the above becomes

$$f(x) = \sum_{n=1}^{\infty} B_n(0)\,\phi_n(x)$$

Therefore

$$\begin{aligned}B_n(0) &= \langle f(x), \phi_n(x)\rangle\\ &= \int_0^1 p(x)\,f(x)\,\phi_n(x)\,dx\\ &= \sqrt{\frac{2h}{h+\cos^2\alpha_n}}\int_0^1 f(x)\sin(\alpha_n x)\,dx\end{aligned}$$

Therefore

$$\begin{aligned}B_n(t) &= B_n(0)\,e^{-k\alpha_n^2 t}\\ &= \left(\sqrt{\frac{2h}{h+\cos^2\alpha_n}}\int_0^1 f(x)\sin(\alpha_n x)\,dx\right)e^{-k\alpha_n^2 t}\end{aligned}$$

and solution (1) becomes

$$\begin{aligned}u(x,t) &= \sum_{n=1}^{\infty}\sqrt{\frac{2h}{h+\cos^2\alpha_n}}\left(\int_0^1 f(x)\sin(\alpha_n x)\,dx\right)e^{-k\alpha_n^2 t}\sqrt{\frac{2h}{h+\cos^2\alpha_n}}\sin(\alpha_n x)\\ &= \frac{2h}{h+\cos^2\alpha_n}\sum_{n=1}^{\infty}\left(\int_0^1 f(x)\sin(\alpha_n x)\,dx\right)e^{-k\alpha_n^2 t}\sin(\alpha_n x)\end{aligned}$$

Which is what required to show.

4.2 Part (b)

We need to show that the solution found in part (a) also satisfies the PDE when $-1 < x < 1$

$$u_t = ku_{xx} \qquad -1 < x < 1, t > 0$$

With boundary conditions (9)

$$\begin{aligned} u_x(-1,t) &= hu(-1,t) \\ u_x(1,t) &= -hu(1,t) \end{aligned}$$

And initial conditions (10)

$$u(x,0) = f(x)$$

When $f(x)$ is odd.

The solution found in a already satisfies the above PDE with the second boundary conditions in (9). Since sine is odd then the solution in part(a) is also odd. Then its partial derivative is even in x, hence the first boundary conditions in (9) is also satisfied

$$u_x(-1,t) = hu(-1,t) = -u_x(1,t) = hu(1,t)$$

Finally we know that $u(x,0) = f(x)$ for $0 < x < 1$. Furthermore when $-1 < x < 0$ the fact that u and $f(x)$ are odd enables us to write

$$u(-x,0) = -u(x,0) = f(-x) = -f(x)$$

5 Section 77, Problem 2

where $\alpha_n = (2n-1)\pi/2$.

2. Heat transfer takes place at the surface $x=0$ of a slab $0 \le x \le 1$ into a medium at temperature zero, according to the linear law of surface heat transfer, so that (Sec. 26)

$$u_x(0,t) = hu(0,t) \qquad (h>0).$$

The other boundary conditions are as indicated in Fig. 61, and the unit of time is chosen so that $k=1$ in the heat equation. By proceeding as in Sec. 77, derive the temperature formula

$$u(x,t) = \frac{hx+1}{h+1} - 2h\sum_{n=1}^{\infty} \frac{\sin\alpha_n(1-x)}{\alpha_n(h+\cos^2\alpha_n)}\exp(-\alpha_n^2 t),$$

where $\tan\alpha_n = -\alpha_n/h$ $(\alpha_n > 0)$.

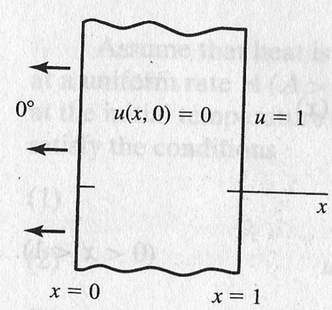


FIGURE 61

Suggestion: In simplifying the expression for the Fourier constants that arise, it is useful to note that

$$-\frac{h\sin\alpha_n}{\alpha_n^2} = \frac{\cos\alpha_n}{\alpha_n}.$$

Figure 5: Problem statement

<u>Solution</u>

Solve

$$u_t = u_{xx} \qquad 0<x<1, t>0$$

With boundary conditions

$$u_x(0,t) - hu(0,t) = 0$$
$$u(1,t) = 1$$

With $h>0$. And initial conditions $u(x,0) = f(x)$.

Because the second B.C. is not zero, we need to introduce a reference function $r(x)$ which satisfies the nonhomogeneous boundary conditions.

Let $r(x) = Ax + B$. When $x=0$ then the first BC gives

$$A - hB = 0$$

And the second BC gives

$$A + B = 1$$

From the first equation $A = hB$. Substituting in the second equation give $hB + B = 1$ or

$B(1+h)=1$ or $B=\frac{1}{1+h}$. Hence $A=\frac{h}{1+h}$. Therefore

$$\begin{aligned} r(x) &= Ax+B \\ &= \frac{h}{1+h}x+\frac{1}{1+h} \\ &= \frac{hx+1}{1+h} \end{aligned} \tag{1}$$

To verify. $r_x=\frac{h}{1+h}$. When $x=0$ then $r(0)=\frac{1}{1+h}$. Hence $r_x(0)-hr(0)=\frac{h}{1+h}-h\frac{1}{1+h}=0$ as expected. And when $x=1$ then $r(1)=1$ as expected. Now that we found $r(x)$ then we write

$$u(x,t)=v(x,t)+r(x)$$

Where $v(x,t)$ is the solution to the homogenous PDE

$$v_t=v_{xx} \qquad 0<x<1, t>0$$

With boundary conditions

$$\begin{aligned} v_x(0,t)-hv(0,t) &= 0 \\ v(1,t) &= 0 \end{aligned}$$

We can now solve for $v(x,t)$ using separation of variables since boundary conditions are homogenous. Separation of variables gives

$$\begin{aligned} X''+\lambda X &= 0 \\ X'(0)-hX(0) &= 0 \\ X(1) &= 0 \end{aligned}$$

Using problem 5 section 72, the eigenfunctions and eigenvalues for the above are

$$\begin{aligned} \phi_n(x) &= \sqrt{\frac{2h}{h+\cos^2\alpha_n}}\sin(\alpha_n(1-x)) \qquad n=1,2,\cdots \\ \tan(\alpha_n) &= \frac{-\alpha_n}{h} \end{aligned}$$

With $\alpha_n>0$. Hence the solution $v(x,t)$ using generalized Fourier series is

$$v(x,t)=\sum_{n=1}^{\infty}B_n(t)\phi_n(x) \tag{2}$$

Substituting into the PDE $v_t=v_{xx}$ gives

$$\begin{aligned} \sum_{n=1}^{\infty}B_n'(t)\phi_n(x) &= \sum_{n=1}^{\infty}B_n(t)\phi_n''(x) \\ &= -\sum_{n=1}^{\infty}B_n(t)\alpha_n^2\phi_n(x) \end{aligned}$$

Therefore the ODE is

$$B_n'(t)+\alpha_n^2B_n(t)=0$$

The solution is

$$B_n(t)=B_n(0)e^{-\alpha_n^2t}$$

Hence (2) becomes

$$v(x,t)=\sum_{n=1}^{\infty}B_n(0)e^{-\alpha_n^2t}\phi_n(x)$$

And since $u(x,t)=v(x,t)+r(x)$ then

$$u(x,t)=\sum_{n=1}^{\infty}B_n(0)e^{-\alpha_n^2t}\phi_n(x)+\frac{hx+1}{1+h}$$

Now we find $B_n(0)$ from initial conditions. At $t=0$ the above becomes

$$\begin{aligned} 0 &= \sum_{n=1}^{\infty}B_n(0)\phi_n(x)+\frac{hx+1}{1+h} \\ -\frac{hx+1}{1+h} &= \sum_{n=1}^{\infty}B_n(0)\phi_n(x) \end{aligned}$$

Hence

$$
\begin{aligned}
B_n(0) &= \left\langle -\frac{hx+1}{1+h}, \phi_n(x) \right\rangle \\
&= -\int_0^1 p(x)\frac{hx+1}{1+h}\phi_n(x)\,dx \\
&= -\int_0^1 \frac{hx+1}{1+h}\sqrt{\frac{2h}{h+\cos^2\alpha_n}}\sin(\alpha_n(1-x))\,dx \\
&= -\frac{1}{1+h}\sqrt{\frac{2h}{h+\cos^2\alpha_n}}\int_0^1 (hx+1)\sin(\alpha_n(1-x))\,dx \qquad (3)
\end{aligned}
$$

But

$$
\begin{aligned}
\int_0^1 (hx+1)\sin(\alpha_n(1-x))\,dx &= \int_0^1 \sin(\alpha_n(1-x))\,dx + h\int_0^1 x\sin(\alpha_n(1-x))\,dx \\
&= \left[\frac{\cos(\alpha_n(1-x))}{\alpha_n}\right]_0^1 + h\left[\frac{\alpha_n x\cos(\alpha_n(1-x)) + \sin(\alpha_n(1-x))}{\alpha_n^2}\right]_0^1 \\
&= \frac{1-\cos(\alpha_n)}{\alpha_n} + \frac{h}{\alpha_n^2}\left[\alpha_n x\cos(\alpha_n(1-x)) + \sin(\alpha_n(1-x))\right]_0^1 \\
&= \frac{1-\cos(\alpha_n)}{\alpha_n} + \frac{h}{\alpha_n^2}\left[\alpha_n - \sin\alpha_n\right] \\
&= \frac{\alpha_n - \alpha_n\cos(\alpha_n) + h\alpha_n - h\sin\alpha_n}{\alpha_n^2}
\end{aligned}
$$

But $\frac{\sin(\alpha_n)}{\cos(\alpha_n)} = -\frac{\alpha_n}{h}$ or $h\sin(\alpha_n) = -\alpha_n\cos(\alpha_n)$ or $-h\sin\alpha_n = \alpha_n\cos(\alpha_n)$, hence the above simplifies to

$$
\begin{aligned}
\int_0^1 (hx+1)\sin(\alpha_n(1-x))\,dx &= \frac{\alpha_n + h\alpha_n}{\alpha_n^2} \\
&= \frac{1+h}{\alpha_n}
\end{aligned}
$$

Therefore (3) becomes

$$
\begin{aligned}
B_n(0) &= \frac{-1}{1+h}\sqrt{\frac{2h}{h+\cos^2\alpha_n}}\left(\frac{1+h}{\alpha_n}\right) \\
&= -\frac{1}{\alpha_n}\sqrt{\frac{2h}{h+\cos^2\alpha_n}}
\end{aligned}
$$

Hence final solution becomes

$$
\begin{aligned}
u(x,t) &= \frac{hx+1}{1+h} + \sum_{n=1}^{\infty} B_n(0)\,e^{-\alpha_n^2 t}\phi_n(x) \\
&= \frac{hx+1}{1+h} + \sum_{n=1}^{\infty} B_n(0)\exp\left(-\alpha_n^2 t\right)\sqrt{\frac{2h}{h+\cos^2\alpha_n}}\sin(\alpha_n(1-x)) \\
&= \frac{hx+1}{1+h} + \sum_{n=1}^{\infty} -\frac{1}{\alpha_n}\sqrt{\frac{2h}{h+\cos^2\alpha_n}}\exp\left(-\alpha_n^2 t\right)\sqrt{\frac{2h}{h+\cos^2\alpha_n}}\sin(\alpha_n(1-x)) \\
&= \frac{hx+1}{1+h} - 2h\sum_{n=1}^{\infty}\frac{\sin(\alpha_n(1-x))}{\alpha_n\left(h+\cos^2\alpha_n\right)}\exp\left(-\alpha_n^2 t\right)
\end{aligned}
$$

Which is what required to show.