Homework 9 – Solutions

These solutions demonstrate one way to approach each of the homework problems. In many cases, there are other correct solutions. If you would like to discuss alternative solutions or the grading of your assignment, please see me during office hours or send me an email.

Textbook Problems:

7.1.8 Our original differential equations are

$$\begin{aligned} x'' + 3x' + 4x - 2y &= 0 \\ y'' + 2y' - 3x + y &= \cos t \end{aligned}$$

We define new functions $x_1 = x, x_2 = x', x_3 = y, x_4 = y'$. Since x'' and y'' are the highest derivatives we have, we can stop defining new functions at the first derivatives. We make the appropriate substitutions, and add additional equations to explicate the relationships between our newly defined variables.

$$\begin{aligned} x_2' + 3x_2 + 4x_1 - 2x_3 &= 0 \\ x_4' + 2x_4 - 3x_1 + x_3 &= \cos t \\ x_3' &= x_4 \\ x_1' &= x_2 \end{aligned}$$

Note that the first derivative of each of our functions appears exactly once, which indicates we have enough equations.

7.2.1 We have $A(t) = \begin{bmatrix} t & 2t-1 \\ t^3 & \frac{1}{t} \end{bmatrix}$ and $B(t) = \begin{bmatrix} 1-t & 1+t \\ 3t^2 & 4t^3 \end{bmatrix}$. We compute

$$\begin{aligned} \frac{d}{dt}(AB) &= \frac{d}{dt}\begin{bmatrix} t(1-t) + (2t-1)3t^2 & t(1+t) + (2t-1)4t^3 \\ t^3(1-t) + 3t & t^3(1+t) + 4t^2 \end{bmatrix} \\ &= \frac{d}{dt}\begin{bmatrix} t - 4t^2 + 6t^3 & t + t^2 + 8t^4 - 4t^3 \\ t^3 - t^4 + 3t & t^3 + t^4 + 4t^2 \end{bmatrix} \\ &= \begin{bmatrix} 1 - 8t + 18t^2 & 1 + 2t + 32t^3 - 12t^2 \\ 3t^2 - 4t^3 + 3 & 3t^2 + 4t^3 + 8t \end{bmatrix} \end{aligned}$$

On the other hand, we have

$$\begin{aligned} A'B + AB' &= \begin{bmatrix} 1 & 2 \\ 3t^2 & -t^{-2} \end{bmatrix}\begin{bmatrix} 1-t & 1+t \\ 3t^2 & 4t^3 \end{bmatrix} + \begin{bmatrix} t & 2t-1 \\ t^3 & \frac{1}{t} \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 6t & 12t^2 \end{bmatrix} \\ &= \begin{bmatrix} 1 - t + 6t^2 & 1 + t + 8t^3 \\ 3t^2 - 3t^3 - 3 & 3t^2 + 3t^3 - 4t \end{bmatrix} + \begin{bmatrix} -7t + 12t^2 & t + 24t^3 - 12t^2 \\ -t^3 + 6 & t^3 + 12t \end{bmatrix} \\ &= \begin{bmatrix} 1 - 8t + 18t^2 & 1 + 2t - 12t^2 + 32t^3 \\ 3 + 3t^2 - 4t^3 & 8t + 3t^2 + 4t^3 \end{bmatrix} \end{aligned}$$

A few of the entries need rearranging, but the resulting matrices are indeed equal.

7.2.5 We can rewrite the system as

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 2 & 4 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3e^t \\ -t^2 \end{bmatrix}$$

7.2.9 We can rewrite the system as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}' = \begin{bmatrix} 3 & -4 & 1 \\ 1 & 0 & -3 \\ 0 & 6 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} t \\ t^2 \\ t^3 \end{bmatrix}$$

7.2.15 $\vec{x}_1$ is a solution, since

$$2e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = e^{2t} \begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = e^{2t} \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$\vec{x}_2$ is a solution, since

$$-2e^{-2t} \begin{bmatrix} 1 \\ 5 \end{bmatrix} = e^{-2t} \begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \end{bmatrix} = e^{-2t} \begin{bmatrix} -2 \\ -10 \end{bmatrix}$$

These solutions are linearly independent, since the Wronskian is

$$e^{2t} e^{-2t} \det \begin{bmatrix} 1 & 1 \\ 1 & 5 \end{bmatrix} = 4$$

The general solution is

$$\vec{x}(t) = c_1 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

7.2.19 $\vec{x}_1$ is a solution, since

$$2e^{2t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = e^{2t} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = e^{2t} \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

$\vec{x}_2$ is a solution, since

$$-e^{-t} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = e^{2t} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = e^{-t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\vec{x}_3$ is a solution, since

$$-e^{-t} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = e^{2t} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = e^{-t} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

These solutions are linearly independent, since the Wronskian is

$$e^{2t}e^{-t}e^{-t}\det\begin{bmatrix}1 & 1 & 0\\ 1 & 0 & 1\\ 1 & -1 & -1\end{bmatrix} = \det\begin{bmatrix}0 & 1\\ -1 & -1\end{bmatrix} - \det\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$$
$$= 1-(-2)$$
$$= 3$$

The general solution is

$$\vec{x}(t) = c_1 e^{2t}\begin{bmatrix}1\\1\\1\end{bmatrix} + c_2 e^{-t}\begin{bmatrix}1\\0\\-1\end{bmatrix} + c_3 e^{-t}\begin{bmatrix}0\\1\\-1\end{bmatrix}$$

7.2.24 Using the general solution we wrote down in problem 15,

$$\vec{x}(0) = c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\5\end{bmatrix} = \begin{bmatrix}c_1+c_2\\ c_1+5c_2\end{bmatrix}$$

Given that $\vec{x}(0) = \begin{bmatrix}0\\5\end{bmatrix}$, we have the linear system

$$\begin{bmatrix}1 & 1\\ 1 & 5\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}0\\5\end{bmatrix}$$

We row reduce

$$\begin{bmatrix}1 & 1 & 0\\ 1 & 5 & 5\end{bmatrix} \xrightarrow{-R_1+R_2} \begin{bmatrix}1 & 1 & 0\\ 0 & 4 & 5\end{bmatrix}$$

Now we can solve to get $c_2 = 5/4$ and $c_1 = -5/4$. The particular solution is thus $\vec{x} = -\frac{5}{4}\vec{x}_1 + \frac{5}{4}\vec{x}_2$

7.2.28

Using the general solution we wrote down in problem 19,

$$\vec{x}(0) = c_1\begin{bmatrix}1\\1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\0\\-1\end{bmatrix} + c_3\begin{bmatrix}0\\1\\-1\end{bmatrix}$$

Given the initial conditions, we have the linear system

$$\begin{bmatrix}1 & 1 & 0\\ 1 & 0 & 1\\ 1 & -1 & -1\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}10\\12\\-1\end{bmatrix}$$

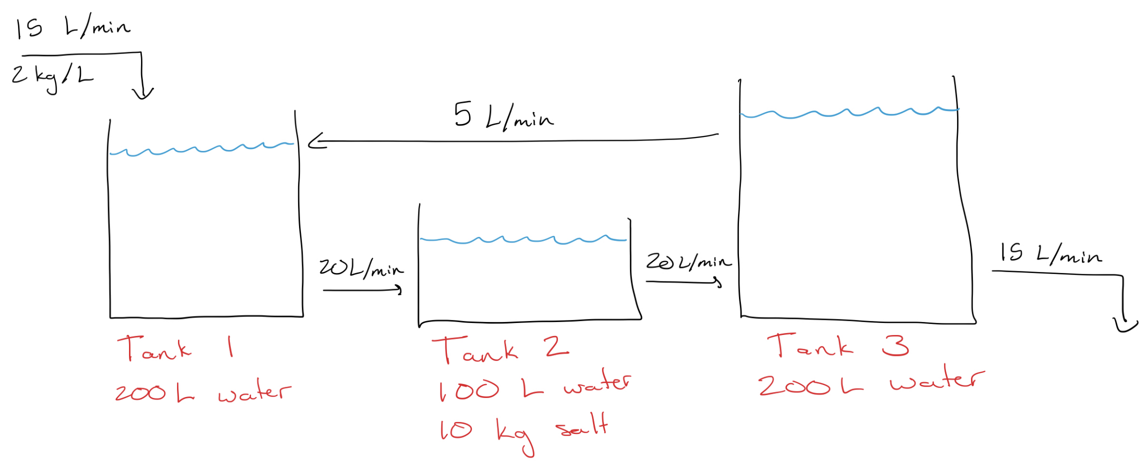
We row reduce

$$\begin{bmatrix}1&1&0&10\\1&0&1&12\\1&-1&-1&-1\end{bmatrix}\xrightarrow[-R_1+R_3]{-R_1+R_2}\begin{bmatrix}1&1&0&10\\0&-1&1&2\\0&-2&-1&-11\end{bmatrix}$$
$$\xrightarrow{-2R_2+R_3}\begin{bmatrix}1&1&0&10\\0&-1&1&2\\0&0&-3&-15\end{bmatrix}$$

We now solve to get $c_3 = 5$, $c_2 = 3$, and $c_1 = 7$. The particular solution is thus $\vec{x} = 7\vec{x}_1 + 3\vec{x}_2 + 5\vec{x}_3$

Additional Problems:

1. The picture below illustrates this system:



Note that each tank has 20 L of water flowing in and 20 L of water flowing out, so all volumes are constant. We calculate the change in salt as (rate in) ∗ (concentration in) − (rate out) ∗ (concentration out). This gives us the system of differential equations

$$x_1' = 30 + 5\frac{x_3}{200} - 20\frac{x_1}{200} = -\frac{x_1}{10} + \frac{x_3}{40} + 30$$

$$x_2' = 20\frac{x_1}{200} - 20\frac{x_2}{100} = \frac{x_1}{10} - \frac{x_2}{5}$$

$$x_3' = 20\frac{x_2}{100} - 5\frac{x_3}{200} - 15\frac{x_3}{200} = \frac{x_2}{5} - \frac{x_3}{10}$$

The matrix version of this system is

$$\vec{x}' = \begin{bmatrix}-1/10&0&1/40\\1/10&-1/5&0\\0&1/5&-1/10\end{bmatrix}\vec{x} + \begin{bmatrix}30\\0\\0\end{bmatrix}$$

Our initial condition is

$$\vec{x}(0) = \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix}$$