HW 9

Math 2243
Linear Algebra and Differential Equations

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University of Minnesota, Twin Cities

Nasser M. Abbasi

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Contents

1 Problem 8, section 7.1

Transform the given differential equations into an equivalent system of first-order differential equations.

$$
\begin{aligned}
x'' + 3x' + 4x - 2y &= 0 \\
y'' + 2y' - 3x + y &= \cos t
\end{aligned}
$$

Solution

There are two second order ODE's, hence we needs 4 states variables x_1, x_2, x_3, x_4 where (it is better and more standard to use x_i notation for all state variables. The book uses x_i, y_i which is not optimal. x_i will be used here for all state variables)

$$
\begin{aligned}
x_1 &= x \\
x_2 &= x' \\
x_3 &= y \\
x_4 &= y'
\end{aligned} \tag{1}
$$

Taking derivatives w.r.t time t gives

$$
\begin{aligned}
x_1' &= x' \\
&= x_2 \\
x_2' &= x'' \\
&= -\left(3x' + 4x - 2y\right) \\
&= -4x_1 - 3x_2 + 2x_3 \\
x_3' &= y' \\
&= x_4 \\
x_4' &= y'' \\
&= -\left(2y' - 3x + y\right) + \cos t \\
&= 3x_1 - x_3 - 2x_4 + \cos t
\end{aligned}
$$

Or in Matrix form (if needed)

$$
\begin{aligned}
\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -4 & -3 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 3 & 0 & -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cos t \end{bmatrix} \\
\vec{x}' &= A\vec{x} + \vec{f}(t)
\end{aligned}
$$

2 Problem 1 section 7.2

Verify the product law for differentiation $(AB)' = A'B + AB'$

$$A(t) = \begin{bmatrix} t & 2t-1 \\ t^3 & \frac{1}{t} \end{bmatrix}, B(t) = \begin{bmatrix} 1-t & 1+t \\ 3t^2 & 4t^3 \end{bmatrix}$$

<u>Solution</u>

$$\begin{aligned} AB &= \begin{bmatrix} t & 2t-1 \\ t^3 & t^{-1} \end{bmatrix} \begin{bmatrix} 1-t & 1+t \\ 3t^2 & 4t^3 \end{bmatrix} \\ &= \begin{bmatrix} t\left(6t^2-4t+1\right) & t\left(8t^3-4t^2+t+1\right) \\ t\left(-t^3+t^2+3\right) & t^2\left(t^2+t+4\right) \end{bmatrix} \\ &= \begin{bmatrix} 6t^3-4t^2+t & 8t^4-4t^3+t^2+t \\ -t^4+t^3+3t & t^4+t^3+4t^2 \end{bmatrix} \end{aligned}$$

Taking derivative of each entry w.r.t t gives

$$(AB)' = \begin{bmatrix} 18t^2-8t+1 & 32t^3-12t^2+2t+1 \\ -4t^3+3t^2+3 & 4t^3+3t^2+8t \end{bmatrix} \tag{1}$$

Now

$$\begin{aligned} A'(t) &= \frac{d}{dt}\begin{bmatrix} t & 2t-1 \\ t^3 & \frac{1}{t} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 \\ 3t^2 & -t^{-2} \end{bmatrix} \end{aligned}$$

Hence

$$\begin{aligned} A'(t)B(t) &= \begin{bmatrix} 1 & 2 \\ 3t^2 & -t^{-2} \end{bmatrix} \begin{bmatrix} 1-t & 1+t \\ 3t^2 & 4t^3 \end{bmatrix} \\ &= \begin{bmatrix} 6t^2-t+1 & 8t^3+t+1 \\ -3t^3+3t^2-3 & t\left(3t^2+3t-4\right) \end{bmatrix} \\ &= \begin{bmatrix} 6t^2-t+1 & 8t^3+t+1 \\ -3t^3+3t^2-3 & 3t^3+3t^2-4t \end{bmatrix} \end{aligned} \tag{2}$$

And

$$\begin{aligned} B'(t) &= \frac{d}{dt}\begin{bmatrix} 1-t & 1+t \\ 3t^2 & 4t^3 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 1 \\ 6t & 12t^2 \end{bmatrix} \end{aligned}$$

Hence

$$\begin{aligned}
A(t)B'(t) &= \begin{bmatrix} t & 2t-1 \\ t^3 & \frac{1}{t} \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 6t & 12t^2 \end{bmatrix} \\
&= \begin{bmatrix} t(12t-7) & t\left(24t^2-12t+1\right) \\ 6-t^3 & t^3+12t \end{bmatrix} \\
&= \begin{bmatrix} 12t^2-7t & 24t^3-12t^2+t \\ 6-t^3 & t^3+12t \end{bmatrix} \qquad (3)
\end{aligned}$$

Therefore, from (2,3)

$$\begin{aligned}
A'B + AB' &= \begin{bmatrix} 6t^2-t+1 & 8t^3+t+1 \\ -3t^3+3t^2-3 & 3t^3+3t^2-4t \end{bmatrix} + \begin{bmatrix} 12t^2-7t & 24t^3-12t^2+t \\ 6-t^3 & t^3+12t \end{bmatrix} \\
&= \begin{bmatrix} \left(6t^2-t+1\right)+\left(12t^2-7t\right) & \left(8t^3+t+1\right)+\left(24t^3-12t^2+t\right) \\ \left(-3t^3+3t^2-3\right)+\left(6-t^3\right) & \left(3t^3+3t^2-4t\right)+\left(t^3+12t\right) \end{bmatrix} \\
&= \begin{bmatrix} 18t^2-8t+1 & 32t^3-12t^2+2t+1 \\ -4t^3+3t^2+3 & 4t^3+3t^2+8t \end{bmatrix} \qquad (4)
\end{aligned}$$

Comparing (1) and (4) shows they are the same. Therefore $(AB)' = A'B + AB'$ has been verified.

3 Problem 5 section 7.2

Write the given system in the form $\vec{x}' = P(t)\vec{x} + \vec{f}(t)$

$$
\begin{aligned}
x' &= 2x + 4y + 3e^t \\
y' &= 5x - y - t^2
\end{aligned}
$$

Solution

There are two first order ODE's, hence we needs 2 states variables x_1, x_2. Let

$$
\begin{aligned}
x_1 &= x \\
x_2 &= y
\end{aligned} \tag{1}
$$

Taking derivatives w.r.t time t gives

$$
\begin{aligned}
x_1' &= x' \\
&= 2x + 4y + 3e^t \\
&= 2x_1 + 4x_2 + 3e^t \\
x_2' &= y' \\
&= 5x - y - t^2 \\
&= 5x_1 - x_2 - t^2
\end{aligned}
$$

Or in Matrix form

$$
\begin{aligned}
\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} &= \overbrace{\begin{bmatrix} 2 & 4 \\ 5 & -1 \end{bmatrix}}^{P(t)} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \overbrace{\begin{bmatrix} 3e^t \\ -t^2 \end{bmatrix}}^{\vec{f}(t)} \\
\vec{x}' &= P\vec{x} + \vec{f}(t)
\end{aligned}
$$

Or using book notation

$$
\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3e^t \\ -t^2 \end{bmatrix}
$$

4 Problem 9 section 7.2

Write the given system in the form $\vec{x}' = P(t)\vec{x} + \vec{f}(t)$

$$\begin{aligned} x' &= 3x - 4y + z + t \\ y' &= x - 3z + t^2 \\ z' &= 6y - 7z + t^3 \end{aligned}$$

Solution

There are three first order ODE's, hence we needs 3 states variables x_1, x_2, x_3. Let

$$\begin{aligned} x_1 &= x \\ x_2 &= y \\ x_3 &= z \end{aligned} \tag{1}$$

Taking derivatives w.r.t time t gives

$$\begin{aligned} x_1' &= x' \\ &= 3x - 4y + z + t \\ &= 3x_1 - 4x_2 + x_3 + t \\ x_2' &= y' \\ &= x - 3z + t^2 \\ &= x_1 - 3x_3 + t^2 \\ x_3' &= 6y - 7z + t^3 \\ &= 6x_2 - 7x_3 + t^3 \end{aligned}$$

Or in Matrix form

$$\begin{aligned} \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} &= \overbrace{\begin{bmatrix} 3 & -4 & 1 \\ 1 & 0 & -3 \\ 0 & 6 & -7 \end{bmatrix}}^{P(t)} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \overbrace{\begin{bmatrix} t \\ t^2 \\ t^3 \end{bmatrix}}^{\vec{f}(t)} \\ \vec{x}' &= P\vec{x} + \vec{f}(t) \end{aligned}$$

Or using book notation

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 3 & -4 & 1 \\ 1 & 0 & -3 \\ 0 & 6 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} t \\ t^2 \\ t^3 \end{bmatrix}$$

5 Problem 15 section 7.2

First verify that the given vectors are solutions of the given system. Then use the Wronskian to show that they are linearly independent. Finally, write the general solution of the system

$$\vec{x}' = \begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix} \vec{x}$$
$$\vec{x}_1 = e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$\vec{x}_2 = e^{-2t} \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

Solution

The system is

$$\vec{x}'(t) = A\vec{x} \tag{1}$$

To verify each vector solution, we will check if the LHS is the same as the RHS. The LHS of (1) is

$$\begin{aligned} \frac{d}{dt}\vec{x}_1(t) &= \frac{d}{dt} e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= 2e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= 2\vec{x}_1(t) \end{aligned} \tag{2}$$

The RHS of (1) is

$$\begin{aligned} A\vec{x}_1 &= \begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix} \vec{x}_1 \\ &= \begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix} e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= e^{2t} \begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= e^{2t} \begin{bmatrix} 2 \\ 2 \end{bmatrix} \\ &= 2e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= 2\vec{x}_1(t) \end{aligned} \tag{3}$$

Comparing (1,2) shows they are the same. Now we do the same for the second vector solution. The LHS of (1) is

$$\begin{aligned}
\frac{d}{dt}\vec{x}_2(t) &= \frac{d}{dt}e^{-2t}\begin{bmatrix}1\\5\end{bmatrix}\\
&= -2e^{-2t}\begin{bmatrix}1\\5\end{bmatrix}\\
&= -2\vec{x}_2(t) \qquad (4)
\end{aligned}$$

The RHS of (1) is

$$\begin{aligned}
A\vec{x}_2 &= \begin{bmatrix}3 & -1\\5 & -3\end{bmatrix}\vec{x}_2\\
&= \begin{bmatrix}3 & -1\\5 & -3\end{bmatrix}e^{-2t}\begin{bmatrix}1\\5\end{bmatrix}\\
&= e^{-2t}\begin{bmatrix}3 & -1\\5 & -3\end{bmatrix}\begin{bmatrix}1\\5\end{bmatrix}\\
&= e^{-2t}\begin{bmatrix}-2\\-10\end{bmatrix}\\
&= -2e^{-2t}\begin{bmatrix}1\\5\end{bmatrix}\\
&= -2\vec{x}_2(t) \qquad (5)
\end{aligned}$$

Comparing (4,5) shows they are the same. Both solution vectors verified. The Wronskian is the determinant of the matrix whose columns are the vectors $\vec{x}_1(t), \vec{x}_2(t)$. Hence

$$\begin{aligned}
\begin{vmatrix}e^{2t} & e^{-2t}\\e^{2t} & 5e^{-2t}\end{vmatrix} &= 5\left(e^{2t}e^{-2t}\right) - e^{2t}e^{-2t}\\
&= 5-1\\
&= 4
\end{aligned}$$

Since the determinant is not zero (anywhere), then $\vec{x}_1(t), \vec{x}_2(t)$ are linearly independent.

The general solution is linear combination of the basis vector solutions. Therefore

$$\begin{aligned}
\vec{x}(t) &= c_1\vec{x}_1(t) + c_2\vec{x}_2(t)\\
&= c_1e^{2t}\begin{bmatrix}1\\1\end{bmatrix} + c_2e^{-2t}\begin{bmatrix}1\\5\end{bmatrix}\\
&= \begin{bmatrix}c_1e^{2t} + c_2e^{-2t}\\c_1e^{2t} + 5c_2e^{-2t}\end{bmatrix}
\end{aligned}$$

6 Problem 19 section 7.2

First verify that the given vectors are solutions of the given system. Then use the Wronskian to show that they are linearly independent. Finally, write the general solution of the system

$$\vec{x}' = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \vec{x}$$

$$\vec{x}_1 = e^{2t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{x}_2 = e^{-t} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\vec{x}_2 = e^{-t} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

<u>Solution</u>

The system is

$$\vec{x}'(t) = A\vec{x} \tag{1}$$

To verify each vector solution, we will check if the LHS is the same as the RHS. The LHS of (1) is

$$\begin{aligned} \frac{d}{dt}\vec{x}_1(t) &= \frac{d}{dt} e^{2t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ &= 2e^{2t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ &= 2\vec{x}_1(t) \end{aligned} \tag{2}$$

The RHS of (1) is

$$
\begin{aligned}
A\vec{x}_1 \overset{\rightarrow}{=} & \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \vec{x}_1 \\
&= e^{2t} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\
&= e^{2t} \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \\
&= 2e^{2t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\
&= 2\vec{x}_1(t) \qquad (3)
\end{aligned}
$$

Comparing (1,2) shows they are the same. Now we do the same for $\vec{x}_2(t)$. The LHS of (1) is

$$
\begin{aligned}
\frac{d}{dt}\vec{x}_2(t) &= \frac{d}{dt} e^{-t} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \\
&= -e^{-t} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \\
&= -\vec{x}_2(t) \qquad (4)
\end{aligned}
$$

The RHS of (1) is

$$
\begin{aligned}
A\vec{x}_2 &= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \vec{x}_2 \\
&= e^{-t} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \\
&= e^{-t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \\
&= -e^{-t} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \\
&= -\vec{x}_2(t) \qquad (5)
\end{aligned}
$$

Comparing (4,5) shows they are the same. Now we do the same for $\vec{x}_3(t)$. The LHS of (1) is

$$
\begin{aligned}
\frac{d}{dt}\vec{x}_3(t) &= \frac{d}{dt} e^{-t} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \\
&= -e^{-t} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \\
&= -\vec{x}_3(t) \qquad (6)
\end{aligned}
$$

The RHS of (1) is

$$
\begin{aligned}
A\vec{x}_3 &= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \vec{x}_3 \\
&= e^{-t} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \\
&= e^{-t} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \\
&= -e^{-t} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \\
&= -\vec{x}_3(t) \qquad (7)
\end{aligned}
$$

Comparing (6,7) shows they are the same. All three vectors solutions verified. The Wronskian is the determinant of the matrix whose columns are the vectors $\vec{x}_1(t), \vec{x}_2(t), \vec{x}_3(t)$. Hence

$$
\begin{aligned}
\begin{vmatrix} e^{2t} & e^{-t} & 0 \\ e^{2t} & 0 & e^{-t} \\ e^{2t} & -e^{-t} & -e^{-t} \end{vmatrix} &= e^{2t} \begin{vmatrix} 0 & e^{-t} \\ -e^{-t} & -e^{-t} \end{vmatrix} - e^{-t} \begin{vmatrix} e^{2t} & e^{-t} \\ e^{2t} & -e^{-t} \end{vmatrix} \\
&= e^{2t}\left(e^{-2t}\right) - e^{-t}\left(e^{t} - e^{t}\right) \\
&= 1 - e^{-t}(0) \\
&= 1
\end{aligned}
$$

Since the determinant is not zero (anywhere), then $\vec{x}_1(t), \vec{x}_2(t), \vec{x}_3(t)$ are linearly independent. The general solution is linear combination of the basis vector solutions. Therefore

$$
\begin{aligned}
\vec{x}(t) &= c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t) \\
&= c_1 e^{2t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_3 e^{-t} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} c_1 e^{2t} + c_2 e^{-t} \\ c_1 e^{2t} + c_3 e^{-t} \\ c_1 e^{2t} + c_2 e^{-t} + c_3 e^{-t} \end{bmatrix}
\end{aligned}
$$

7 Problem 24 section 7.2

Find the particular solution of the indicated linear system that satisfies the given initial conditions. The system of problem 15.

$$\vec{x}' = \begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix} \vec{x}$$
$$\vec{x}_1 = e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$\vec{x}_2 = e^{-2t} \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

$$x_1(0) = 0, x_2(0) = 5$$

<u>Solution</u>

The general solution is

$$\vec{x}(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$
$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ 5 \end{bmatrix} \tag{1}$$

At $t = 0$, the above becomes

$$\begin{bmatrix} 0 \\ 5 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$
$$\begin{bmatrix} 0 \\ 5 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ c_1 + 5c_2 \end{bmatrix}$$

Two equations with two unknown. From first equation $c_1 = -c_2$. Substituting in the second equation gives $5 = -c_2 + 5c_2$ or $4c_2 = 5$. Hence $c_2 = \frac{5}{4}$. Therefore $c_1 = -\frac{5}{4}$.Therefore the solution that satisfies the initial conditions is, from (1)

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = -\frac{5}{4} e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{5}{4} e^{-2t} \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$
$$= \begin{bmatrix} -\frac{5}{4} e^{2t} + \frac{5}{4} e^{-2t} \\ -\frac{5}{4} e^{2t} + \frac{25}{4} e^{-2t} \end{bmatrix}$$

Or

$$\vec{x}(t) = \frac{5}{4} \left(-\vec{x}_1(t) + \vec{x}_2(t) \right)$$

8 Problem 28 section 7.2

Find the particular solution of the indicated linear system that satisfies the given initial conditions. The system of problem 19.

$$\vec{x}' = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \vec{x}$$

$$\vec{x}_1 = e^{2t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{x}_2 = e^{-t} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\vec{x}_2 = e^{-t} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$x_1(0) = 10, x_2(0) = 12, x_3(0) = -1$$

Solution

The general solution is

$$\vec{x}(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 e^{2t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_3 e^{-t} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \tag{1}$$

At $t = 0$, the above becomes

$$\begin{bmatrix} 10 \\ 12 \\ -1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 10 \\ 12 \\ -1 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ c_1 + c_3 \\ c_1 - c_2 - c_3 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 12 \\ -1 \end{bmatrix} \tag{2}$$

The augmented system is

$$\begin{bmatrix} 1 & 1 & 0 & 10 \\ 1 & 0 & 1 & 12 \\ 1 & -1 & -1 & -1 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$$\begin{bmatrix} 1 & 1 & 0 & 10 \\ 0 & -1 & 1 & 2 \\ 1 & -1 & -1 & -1 \end{bmatrix}$$

$R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & 1 & 0 & 10 \\ 0 & -1 & 1 & 2 \\ 0 & -2 & -1 & -11 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\begin{bmatrix} 1 & 1 & 0 & 10 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & -3 & -15 \end{bmatrix}$$

Therefore the system (2) now is

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 2 \\ -15 \end{bmatrix} \tag{3}$$

Last row gives $c_3 = 5$. Second row gives $-c_2 + c_3 = 2$. Hence $-c_2 = 2 - 5 = -3$ or $c_2 = 3$. First row gives $c_1 + c_2 = 10$. Hence $c_1 = 10 - 3 = 7$. Therefore

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 5 \end{bmatrix}$$

Substituting these in (1) gives

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = 7e^{2t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 3e^{-t} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 5e^{-t} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

or

$$\vec{x}(t) = 7\vec{x}_1(t) + 3\vec{x}_2(t) + 5\vec{x}_3(t)$$

9 Additional problem 1

There is a system of three brine tanks. Tanks 1 and 3 begin with 200 L of fresh water each and tank 2 begins with 100 L of water and 10 kg of salt.

Water containing 2 kg of salt per liter is pumped into tank 1 at a rate of 15 L/min. The well-mixed solution is pumped from tank 1 to tank 2 at a rate of 20 L/min, from tank 2 to tank 3 at a rate of 20 L/min, and from tank 3 to tank 1 at a rate of 5 L/min. The well-mixed solution is pumped out of tank 3 at a rate of 15 L/min.

(a) Draw and label a picture that illustrates this situation. (b) Let $x_1(t), x_2(t), x_3(t)$ denote the amount of salt (in kilograms) in tanks 1, 2, and 3 respectively after t minutes. Write down differential equations for $x_1'(t), x_2'(t), x_3'(t)$.(c) Write the system of differential equations in (b) as a matrix equation $\vec{x}' = P\vec{x} + \vec{f}(t)$. What are the initial conditions $\vec{x}(0)$?

Solution

9.1 Part (a)

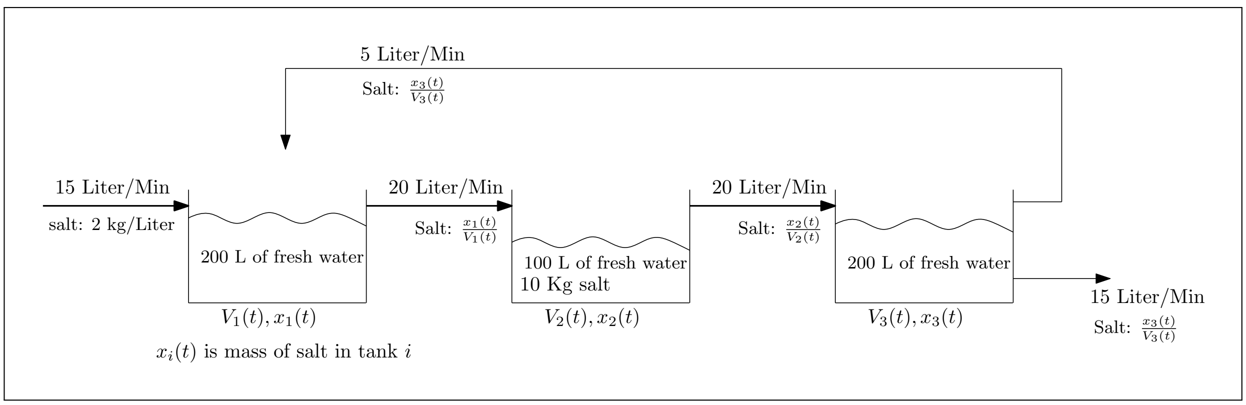

Figure 1: Diagram description of the problem

9.2 Part (b)

$$\begin{aligned} x_1'(t) &= \text{rate of flow in} - \text{ rate of flow out} \\ &= \left(15\left(\frac{\text{L}}{\text{min}}\right)2\left(\frac{\text{kg}}{\text{L}}\right)\right) + \left(5\left(\frac{\text{L}}{\text{min}}\right)\frac{x_3(t)}{V_3(t)}\left(\frac{\text{kg}}{\text{L}}\right)\right) - \left(20\left(\frac{\text{L}}{\text{min}}\right)\frac{x_1(t)}{V_1(t)}\left(\frac{\text{kg}}{\text{L}}\right)\right) \end{aligned} \tag{1}$$

And

$$\begin{aligned} x_2'(t) &= \text{rate of flow in} - \text{ rate of flow out} \\ &= \left(20\left(\frac{\text{L}}{\text{min}}\right)\frac{x_1(t)}{V_1(t)}\left(\frac{\text{kg}}{\text{L}}\right)\right) - \left(20\left(\frac{\text{L}}{\text{min}}\right)\frac{x_2(t)}{V_2(t)}\left(\frac{\text{kg}}{\text{L}}\right)\right) \end{aligned} \tag{2}$$

And

$$x_3'(t) = \text{rate of flow in} - \text{ rate of flow out}$$
$$= \left(20\left(\frac{\text{L}}{\text{min}}\right)\frac{x_2(t)}{V_2(t)}\left(\frac{\text{kg}}{\text{L}}\right)\right) - \left(15\left(\frac{\text{L}}{\text{min}}\right)\frac{x_3(t)}{V_3(t)}\left(\frac{\text{kg}}{\text{L}}\right)\right) - \left(5\left(\frac{\text{L}}{\text{min}}\right)\frac{x_3(t)}{V_3(t)}\left(\frac{\text{kg}}{\text{L}}\right)\right) \tag{3}$$

The volume of water at time t is found as follows. $V(t) = V(0) + (\text{rate in-rate out})t$. Therefore

$$\begin{aligned} V_1(t) &= V_1(0)(\text{L}) + (15 + 5 - 20)\left(\frac{\text{L}}{\text{min}}\right)t \\ &= 200 + 0t \\ &= 200 \end{aligned} \tag{4}$$

And

$$\begin{aligned} V_2(t) &= V_2(0)(\text{L}) + (20 - 20)\left(\frac{\text{L}}{\text{min}}\right)t \\ &= 100 + 0t \\ &= 100 \end{aligned} \tag{5}$$

And

$$\begin{aligned} V_3(t) &= V_3(0)(\text{L}) + (20 - 5 - 15)\left(\frac{\text{L}}{\text{min}}\right)t \\ &= 200 + 0t \\ &= 200 \end{aligned} \tag{6}$$

We see from the above, that the volume of water in each tank is constant over time. Now, substituting (4,5,6) into (1,2,3) gives the equations needed.

$$\begin{aligned} x_1'(t) &= 30 + \frac{5}{200}x_3 - \frac{20}{200}x_1 \\ &= 30 + \frac{1}{40}x_3 - \frac{1}{10}x_1 \end{aligned} \tag{7}$$

And

$$\begin{aligned} x_2'(t) &= \frac{20}{200}x_1 - \frac{20}{100}x_2 \\ &= \frac{1}{10}x_1 - \frac{2}{10}x_2 \end{aligned} \tag{8}$$

And

$$\begin{aligned} x_3'(t) &= \frac{20}{100}x_2 - \frac{15}{200}x_3 - \frac{5}{200}x_3 \\ &= \frac{2}{10}x_2 - \frac{1}{10}x_3 \end{aligned} \tag{9}$$

In summary, the differential equations are

$$
\begin{aligned}
x_1'(t) &= 30 + \frac{1}{40}x_3(t) - \frac{1}{10}x_1(t) \\
x_2'(t) &= \frac{1}{10}x_1(t) - \frac{2}{10}x_2(t) \\
x_3'(t) &= \frac{2}{10}x_2(t) - \frac{1}{10}x_3(t)
\end{aligned}
$$

9.3 Part (c)

In Matrix form, the solution found in part b is

$$
\begin{aligned}
\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} &= \begin{bmatrix} -\frac{1}{10} & 0 & \frac{1}{40} \\ \frac{1}{10} & -\frac{2}{10} & 0 \\ 0 & \frac{2}{10} & -\frac{1}{10} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 30 \\ 0 \\ 0 \end{bmatrix} \\
&= \frac{1}{10} \begin{bmatrix} -1 & 0 & \frac{1}{4} \\ 1 & -2 & 0 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 30 \\ 0 \\ 0 \end{bmatrix} \\
\vec{x}' &= P\vec{x} + \vec{f}(t)
\end{aligned}
$$

The initial conditions are

$$
\begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix} \text{(kg)}
$$