Homework 8 - Due November 5

Homework instructions: Complete the assigned problems on your own paper. Once you are finished, scan or photograph your work and upload it to Gradescope. When prompted, tell Gradescope where to find each problem.

You are allowed (and in fact encouraged) to work with other students on homework assignments. If you do that, please indicate on each problem who you worked with. If you use sources other than your notes, the textbook, and any resources on Canvas for your homework, you must indicate the source on each problem. You are not permitted to view, request, or look for solutions to any of the homework problems from solutions manuals, homework help websites, online forums, other students, or any other sources.

Textbook Problems:

- §6.2: 7, 15, 19
- §6.3: 7, 13, 25

Additional Problems:

1. For $n \times n$ matrices A and B, we say that **A is similar to B** if there is an invertible matrix P so that $A = PBP^{-1}$. So, in order to show that A is similar to B, you need to (1) say what the matrix P is in that case (2) check that your choice of P is invertible and (3) explain why the equation $A = PBP^{-1}$ is true.

   (a) Let A be any $n \times n$ matrix. Show that A is similar to itself.

   (b) Let A, B be $n \times n$ matrices. Suppose that A is similar to B. Show that B is similar to A.

   (c) Let A, B, C be $n \times n$ matrices. Suppose that A is similar to B and that B is similar to C. Show that A is similar to C.

   Cultural Aside: A matrix is **diagonalizable** if it is similar to a diagonal matrix. For a matrix that isn't diagonalizable, it may still be useful to find a nice matrix that it is similar to (even if we can't get to something quite as nice as a diagonal matrix). If you are interested in these not-quite-diagonal nice matrices, you should look up "Jordan normal form."

   Further Cultural Aside: These three properties are called (a) reflexivity, (b) symmetry, and (c) transitivity. A relation like "is similar to" that satisfies all three properties is called an *equivalence relation.* Equivalence relations behave a lot like "is equal to" and are nearly as useful in mathematics. In fact, we have already seen another example of an equivalence relation in this class: "is row equivalent to".

2. The **Fibonacci sequence** begins as

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144 \ldots$$

It can be defined recursively by $f_0 = 0$ and $f_1 = 1$ and $f_{n+1} = f_n + f_{n-1}$. That is, we begin the sequence with $1, 1$ and then each successive term is the sum of the two previous terms. (This sequence is usually interpreted as the number of pairs of rabbits in a population.)

Notice that we can write

$$\begin{bmatrix} f_{n+1} \\ f_n \end{bmatrix} = \begin{bmatrix} f_n + f_{n-1} \\ f_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} f_n \\ f_{n-1} \end{bmatrix}$$

Giving some labels to things, let $\vec{x}_n = \begin{bmatrix} f_{n+1} \\ f_n \end{bmatrix}$ and let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Then we have $\vec{x}_n = A\vec{x}_{n-1}$. So, $\vec{x}_n = A^n \vec{x}_0$ where $\vec{x}_0 = \begin{bmatrix} f_1 \\ f_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(a) Find the eigenvalues and corresponding eigenvectors of A. (I recommend using the quadratic formula to find the roots of the characteristic polynomial.)

(b) Find a diagonalization $A = PDP^{-1}$ of A. Use this write down a formula for A^n.

(c) Use the fact that $\vec{x}_n = A^n \vec{x}_0$ to write down a formula for f_n.

Hint: It may be useful to denote the number $\frac{1+\sqrt{5}}{2} \approx 1.61803$ by φ. This number is called the **golden ratio** and satisfies

$$\frac{1-\sqrt{5}}{2} = -\frac{1}{\varphi} = 1 - \varphi \approx -0.61803$$