Homework 8 – Solutions

These solutions demonstrate one way to approach each of the homework problems. In many cases, there are other correct solutions. If you would like to discuss alternative solutions or the grading of your assignment, please see me during office hours or send me an email.

Textbook Problems:

6.2.7 We have $A = \begin{bmatrix} 6 & -10 \\ 2 & -3 \end{bmatrix}$. First, we compute eigenvalues.

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} 6-\lambda & -10 \\ 2 & -3-\lambda \end{bmatrix} \\ &= (6-\lambda)(-3-\lambda) + 20 \\ &= \lambda^2 - 3\lambda + 2 \\ &= (\lambda - 1)(\lambda - 2) \end{aligned}$$

For $\lambda_1 = 1$, we have

$$A - I = \begin{bmatrix} 5 & -10 \\ 2 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

So we have eigenvector $\vec{v}_1 = (2, 1)$.

For $\lambda_2 = 2$, we have

$$A - 2I = \begin{bmatrix} 4 & -10 \\ 2 & -5 \end{bmatrix} \to \begin{bmatrix} 2 & -5 \\ 0 & 0 \end{bmatrix}$$

So we have eigenvector $\vec{v}_2 = (5, 2)$.

We have 2 distinct eigenvalues, so A is diagonalizable. The diagonalization is

$$A = \begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix}^{-1}$$

6.2.15 First, we compute eigenvalues.

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} 3-\lambda & -3 & 1 \\ 2 & -2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix} \\ &= (1-\lambda) \det \begin{bmatrix} 3-\lambda & -3 \\ 2 & -2-\lambda \end{bmatrix} \\ &= (1-\lambda)\left[(3-\lambda)(-2-\lambda) + 6\right] \\ &= (1-\lambda)(\lambda^2 - \lambda) = -\lambda(\lambda - 1)^2 \end{aligned}$$

For $\lambda_1 = 0$, we have

$$A - 0I = \begin{bmatrix} 3 & -3 & 1 \\ 2 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So we have eigenvector $\vec{v}_1 = (1, 1, 0)$.

For $\lambda_2 = 1$, we have

$$A - I = \begin{bmatrix} 2 & -3 & 1 \\ 2 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & -3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So we have basis for the eigenspace $\{(3, 2, 0), (1, 0, -2)\}$.

We have one dimension 1 eigenspace and one dimension 2 eigenspace, so A is diagonalizable. The diagonalization is

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}^{-1}$$

6.2.19 First, we compute eigenvalues.

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} 1-\lambda & 1 & -1 \\ -2 & 4-\lambda & -1 \\ -4 & 4 & 1-\lambda \end{bmatrix} \\ &= (1-\lambda)\det \begin{bmatrix} 4-\lambda & -1 \\ 4 & 1-\lambda \end{bmatrix} - \det \begin{bmatrix} -2 & -1 \\ -4 & 1-\lambda \end{bmatrix} - \det \begin{bmatrix} -2 & 4-\lambda \\ -4 & 4 \end{bmatrix} \\ &= (1-\lambda)\left[(4-\lambda)(1-\lambda) + 4\right] - (-2 + 2\lambda - 4) - (-8 + 16 - 4\lambda) \\ &= (1-\lambda)(\lambda^2 - 5\lambda + 8) + 2\lambda - 2 \\ &= (1-\lambda)(\lambda^2 - 5\lambda + 8 - 2) \\ &= (1-\lambda)(\lambda^2 - 5\lambda + 6) \\ &= (1-\lambda)(\lambda - 2)(\lambda - 3) \end{aligned}$$

For $\lambda_1 = 1$, we have

$$A - I = \begin{bmatrix} 0 & 1 & -1 \\ -2 & 3 & -1 \\ -4 & 4 & 0 \end{bmatrix} \to \begin{bmatrix} -2 & 3 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

So we have eigenvector $\vec{v}_1 = (1, 1, 1)$.

For $\lambda_2 = 2$, we have

$$A - 2I = \begin{bmatrix} -1 & 1 & -1 \\ -2 & 2 & -1 \\ -4 & 4 & -1 \end{bmatrix} \to \begin{bmatrix} -1 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So we have eigenvector $\vec{v}_2 = (1, 1, 0)$

For $\lambda_3 = 3$, we have

$$A - 3I = \begin{bmatrix} -2 & 1 & -1 \\ -2 & 1 & -1 \\ -4 & 4 & -2 \end{bmatrix} \to \begin{bmatrix} -2 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So we have eigenvector $\vec{v}_3 = (1, 0, -2)$.

We have 3 distinct eigenvalues, so A is diagonalizable. The diagonalization is

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & -2 \end{bmatrix}^{-1}$$

6.3.7 We need to diagonalize first. So, we compute eigenvalues:

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 3 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{bmatrix} = (1-\lambda)(2-\lambda)(2-\lambda)$$

For $\lambda_1 = 1$, we have

$$A - I = \begin{bmatrix} 0 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So we have eigenvector $\vec{v}_1 = (1, 0, 0)$.

For $\lambda_2 = 2$, we have

$$A - 2I = \begin{bmatrix} -1 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

A basis for the eigenspace is $\{(3, 1, 0), (0, 0, 1)\}$.

So we can diagonalize A as

$$A = PDP^{-1} = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1}$$

We compute the inverse of P:

$$\begin{bmatrix} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{-3R_2+R_1} \begin{bmatrix} 1 & 0 & 0 & 1 & -3 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Now $A^5 = PD^5P^{-1}$, so

$$\begin{aligned} A^5 &= \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}^5 \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 32 & 0 \\ 0 & 0 & 32 \end{bmatrix} \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 96 & 0 \\ 0 & 32 & 0 \\ 0 & 0 & 32 \end{bmatrix} \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 93 & 0 \\ 0 & 32 & 0 \\ 0 & 0 & 32 \end{bmatrix} \end{aligned}$$

6.3.13 To diagonalize, first we compute eigenvalues.

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} 1-\lambda & -1 & 1 \\ 2 & -2-\lambda & 1 \\ 4 & -4 & 1-\lambda \end{bmatrix} \\ &= (1-\lambda)\det \begin{bmatrix} -2-\lambda & 1 \\ -4 & 1-\lambda \end{bmatrix} + \det \begin{bmatrix} 2 & 1 \\ 4 & 1-\lambda \end{bmatrix} + \det \begin{bmatrix} 2 & -2-\lambda \\ 4 & -4 \end{bmatrix} \\ &= (1-\lambda)\left[(-2-\lambda)(1-\lambda)+4\right] + 2 - 2\lambda - 4 + -8 + 8 + 4\lambda \\ &= (1-\lambda)(\lambda^2+\lambda+2) + 2\lambda - 2 \\ &= (1-\lambda)(\lambda^2+\lambda+2-2) \\ &= \lambda(1-\lambda)(\lambda+1) \end{aligned}$$

For $\lambda_1 = 0$, we have

$$A - 0I = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -2 & 1 \\ 4 & -4 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So we have eigenvector $\vec{v}_1 = (1, 1, 0)$.

For $\lambda_2 = -1$, we have

$$A + I = \begin{bmatrix} 2 & -1 & 1 \\ 2 & -1 & 1 \\ 4 & -4 & 2 \end{bmatrix} \to \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So we have eigenvector $\vec{v}_2 = (1, 0, -2)$.

For $\lambda_3 = 1$, we have

$$A - I = \begin{bmatrix} 0 & -1 & 1 \\ 2 & -3 & 1 \\ 4 & -4 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & -3 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So we have eigenvector $\vec{v}_3 = (1, 1, 1)$.

So we can diagonalize A as

$$A = PDP^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & -2 & 1 \end{bmatrix}^{-1}$$

We compute the inverse of P:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{-R_1+R_2} \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[R_2+R_1]{-2R_2+R_3} \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{bmatrix}$$

$$\xrightarrow[(-1)R_2]{-R_3+R_1} \begin{bmatrix} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{bmatrix}$$

Now $A^{10} = PD^{10}P^{-1}$, so

$$\begin{aligned} A^{10} &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{10} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & -2 & 1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 0 \\ 2 & -2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} -2 & 3 & -1 \\ 1 & -1 & 0 \\ 2 & -2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & -3 & 1 \\ 2 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

6.3.25 We need to diagonalize $A = \begin{bmatrix} .9 & .1 \\ .1 & .9 \end{bmatrix}$. First, we find eigenvalues:

$$\begin{aligned}
\det(A - \lambda I) &= \det \begin{bmatrix} .9-\lambda & .1 \\ .1 & .9-\lambda \end{bmatrix} \\
&= (.9-\lambda)(.9-\lambda) - .01 \\
&= \lambda^2 - 1.8\lambda + .8 \\
&= (\lambda - 1)(\lambda - .8)
\end{aligned}$$

For $\lambda_1 = 1$, we have

$$A - I = \begin{bmatrix} -.1 & .1 \\ .1 & -.1 \end{bmatrix} \to \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$$

So we have eigenvector $\vec{v}_1 = (1, 1)$.

For $\lambda_2 = .8$, we have

$$A - .8I = \begin{bmatrix} .1 & .1 \\ .1 & .1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

So we have eigenvector $\vec{v}_2 = (-1, 1)$.

Our diagonalization is thus

$$\begin{aligned}
A &= PDP^{-1} \\
&= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 4/5 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1}
\end{aligned}$$

Using our 2×2 inverse formula, we get $P^{-1} = \frac{1}{1+1} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, so we have

$$\begin{aligned}
A^k &= PD^kP^{-1} \\
&= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & (4/5)^k \end{bmatrix} \left(\frac{1}{2}\right) \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} 1 & -(4/5)^k \\ 1 & (4/5)^k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} 1 + (4/5)^k & 1 - (4/5)^k \\ 1 - (4/5)^k & 1 + (4/5)^k \end{bmatrix}
\end{aligned}$$

Now, we can use this to tell us what the population in the city and the suburbs is at any given time from a starting population of C_0 in the city and S_0 in the suburbs.

$$\begin{aligned}
\begin{bmatrix} C_k \\ S_k \end{bmatrix} &= A^k \begin{bmatrix} C_0 \\ S_0 \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} 1 + (4/5)^k & 1 - (4/5)^k \\ 1 - (4/5)^k & 1 + (4/5)^k \end{bmatrix} \begin{bmatrix} C_0 \\ S_0 \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} C_0 + C_0(4/5)^k + S_0 - S_0(4/5)^k \\ C_0 - C_0(4/5)^k + S_0 + S_0(4/5)^k \end{bmatrix}
\end{aligned}$$

As we let $k \to \infty$, we have $(4/5)^k \to 0$. So in the long run,

$$\begin{bmatrix} C_k \\ S_k \end{bmatrix} \approx \frac{1}{2} \begin{bmatrix} C_0 + S_0 \\ C_0 + S_0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} (C_0 + S_0)$$

So in the long run, 50% of the total population will live in the city and 50% will live in the suburbs.

Additional Problems:

1. (a) Let A be an $n \times n$ matrix. For our matrix P, we take the $n \times n$ identity matrix I. Note that I is invertible and $I^{-1} = I$. We have $IAI^{-1} = IAI = A$, so A is similar to A.

   (b) Suppose that A is similar to B. So there is an invertible matrix P with $A = PBP^{-1}$. Multiplying this equation on the left by P^{-1} and on the right by P, we have $P^{-1}AP = B$. Since P is invertible, P^{-1} is invertible as well with inverse P. So, $B = (P^{-1})A(P^{-1})^{-1}$. Thus B is similar to A.

   (c) Suppose that A is similar to B and B is similar to C. So there are invertible matrices P and Q with $A = PBP^{-1}$ and $B = QCQ^{-1}$.

   Substituting this expression for B into the first equation, we get $A = PQCQ^{-1}P^{-1}$. Since both P and Q are invertible, PQ is invertible with inverse $Q^{-1}P^{-1}$. So, we have $A = (PQ)C(PQ)^{-1}$ and thus A is similar to C.

2. (a) We compute the characteristic polynomial

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix} \\ &= (1-\lambda)(-\lambda) - 1 \\ &= -\lambda + \lambda^2 - 1 \\ &= \lambda^2 - \lambda - 1 \end{aligned}$$

   Applying the quadratic formula, we have

$$\lambda = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

   To avoid getting lost in square roots, we will write $\varphi = \frac{1+\sqrt{5}}{2}$. Notice then that $1 - \varphi = -\frac{1}{\varphi} = \frac{1-\sqrt{5}}{2}$ which will make a lot of our calculations easier. For $\lambda_1 = \varphi$, we have the matrix

$$A - \varphi I = \begin{bmatrix} 1-\varphi & 1 \\ 1 & -\varphi \end{bmatrix} \to \begin{bmatrix} 1-\varphi & 1 \\ 0 & 0 \end{bmatrix}$$

   So we have the eigenvector $\vec{v}_1 = (1, \varphi - 1)$. For $\lambda_2 = 1 - \varphi$, we have the matrix

$$A - (1-\varphi)I = \begin{bmatrix} \varphi & 1 \\ 1 & \varphi - 1 \end{bmatrix} \to \begin{bmatrix} \varphi & 1 \\ 0 & 0 \end{bmatrix}$$

   So we have the eigenvector $\vec{v}_2 = (1, -\varphi)$.

(b) Using what we found in (a), we can write the diagonalization of A:

$$\begin{aligned} A &= PDP^{-1} \\ &= \begin{bmatrix} 1 & 1 \\ \varphi - 1 & -\varphi \end{bmatrix} \begin{bmatrix} \varphi & 0 \\ 0 & 1-\varphi \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \varphi - 1 & -\varphi \end{bmatrix}^{-1} \end{aligned}$$

Using our 2×2 matrix inverse formula, we can write

$$\begin{aligned} \begin{bmatrix} 1 & 1 \\ \varphi - 1 & -\varphi \end{bmatrix}^{-1} &= \frac{1}{-\varphi - (-1+\varphi)} \begin{bmatrix} -\varphi & -1 \\ 1-\varphi & 1 \end{bmatrix} \\ &= \frac{1}{1-2\varphi} \begin{bmatrix} -\varphi & -1 \\ 1-\varphi & 1 \end{bmatrix} \end{aligned}$$

At this point, it is worthwhile to remember that we have set $\varphi = \frac{1+\sqrt{5}}{2}$. Using this, we can simplify $\frac{1}{1-2\varphi} = -\frac{1}{\sqrt{5}}$. So the inverse matrix is more simply

$$\begin{bmatrix} 1 & 1 \\ \varphi - 1 & -\varphi \end{bmatrix}^{-1} = \frac{1}{\sqrt{5}} \begin{bmatrix} \varphi & 1 \\ \varphi - 1 & -1 \end{bmatrix}$$

Since $A^n = PD^nP^{-1}$, we compute

$$\begin{aligned} A^n &= \begin{bmatrix} 1 & 1 \\ \varphi - 1 & -\varphi \end{bmatrix} \begin{bmatrix} \varphi & 0 \\ 0 & 1-\varphi \end{bmatrix}^n \begin{bmatrix} 1 & 1 \\ \varphi - 1 & -\varphi \end{bmatrix}^{-1} \\ &= \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 1 \\ \varphi - 1 & -\varphi \end{bmatrix} \begin{bmatrix} \varphi^n & 0 \\ 0 & (1-\varphi)^n \end{bmatrix} \begin{bmatrix} \varphi & 1 \\ \varphi - 1 & -1 \end{bmatrix} \\ &= \frac{1}{\sqrt{5}} \begin{bmatrix} \varphi^n & (1-\varphi)^n \\ \varphi^n(\varphi-1) & (-\varphi)(1-\varphi)^n \end{bmatrix} \begin{bmatrix} \varphi & 1 \\ \varphi - 1 & -1 \end{bmatrix} \\ &= \frac{1}{\sqrt{5}} \begin{bmatrix} \varphi^{n+1} + (1-\varphi)^n(\varphi-1) & \varphi^n - (1-\varphi)^n \\ \varphi^{n+1}(\varphi-1) - \varphi(1-\varphi)^n(\varphi-1) & \varphi^n(\varphi-1) + \varphi(1-\varphi)^n \end{bmatrix} \\ &= \frac{1}{\sqrt{5}} \begin{bmatrix} \varphi^{n+1} - (1-\varphi)^{n+1} & \varphi^n - (1-\varphi)^n \\ (\varphi-1)(\varphi^{n+1} - \varphi(1-\varphi)^n) & \varphi^n(\varphi-1) + \varphi(1-\varphi)^n \end{bmatrix} \end{aligned}$$

At this point, it is useful to remember that $1 - \varphi = -\frac{1}{\varphi}$ and $\varphi - 1 = \frac{1}{\varphi}$, so we can simplify a bit further to get

$$A^n = \frac{1}{\sqrt{5}} \begin{bmatrix} \varphi^{n+1} - (1-\varphi)^{n+1} & \varphi^n - (1-\varphi)^n \\ \varphi^n - (1-\varphi)^n & \varphi^{n-1} - (1-\varphi)^{n-1} \end{bmatrix}$$

(c) We know that $\vec{x}_n = A^n \vec{x}_0 = A^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, so

$$\begin{aligned}
\vec{x}_n &= A^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \frac{1}{\sqrt{5}} \begin{bmatrix} \varphi^{n+1} - (1-\varphi)^{n+1} & \varphi^n - (1-\varphi)^n \\ \varphi^n - (1-\varphi)^n & \varphi^{n-1} - (1-\varphi)^{n-1} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \frac{1}{\sqrt{5}} \begin{bmatrix} \varphi^{n+1} - (1-\varphi)^{n+1} \\ \varphi^n - (1-\varphi)^n \end{bmatrix}
\end{aligned}$$

Now, we know that $\vec{x}_n = \begin{bmatrix} f_{n+1} \\ f_n \end{bmatrix}$, so this computation tells us that

$$\begin{aligned}
f_n &= \frac{1}{\sqrt{5}}(\varphi^n - (1-\varphi)^n) \\
&= \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right]
\end{aligned}$$

Aside: This is a remarkable formula. For one thing, it should be surprising that this expression *ever* yields an integer, let alone that it is an integer for every nonnegative integer n. The golden ratio φ is an irrational number, meaning that it cannot be expressed as a fraction of two integers. It is quite unusual for expressions involving irrational numbers to produce rational numbers. It is even more unusual for them to produce integers.

We should also be surprised that we can compute the 100th Fibonacci number without computing the 99 before it. Ostensibly, I only know that $f_{100} = f_{99} + f_{98}$ and I have to use the rule $f_{n+1} = f_n + f_{n-1}$ many many times before I can apply the initial values $f_0 = 0$ and $f_1 = 1$. But with this formula, I can use any calculator to immediately compute $f_{100} = 354,224,848,179,261,915,075$.